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Exponential Bounds and Semi-Finiteness of Point Spectrum for N-Body Schrödinger Operators

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Abstract. For a large class of N-body Schrödinger operators H, we prove that eigenvalues of H cannot accumulate from above at any threshold of H. Our proof relies on L^2 exponential upper bounds for eigenfunctions of H with explicit constants obtained by modifying methods of Froese and Herbst.

In this note we study the point spectrum of certain N-body Schrödinger operators. To specify them, let $m_i>0$ and $x_i \in \mathbb{R}^\nu$, $1 \leqq i \leqq N$, denote the mass and position of the i^{th} particle, let $x \in \mathbb{R}^{N\nu}$ be given by $x=(x_1,\ldots,x_n)$, and let

$$X=\left\{x\in \mathbb{R}^{N\nu}: \sum_{i=1}^{N} m_i x_i = 0\right\}$$

with norm

$$|x|^2=\sum_{i=1}^{N} 2m_i x_i \cdot x_i,$$

where $\cdot$ is the usual inner product on $\mathbb{R}^\nu$. We consider operators H on $L^2(X, dv)$ (with volume measure determined by the norm $|\cdot|$) of the form

$$H=-\Delta_X+\sum_{1\leqq i<j\leqq N} V_{ij}(x_i-x_j),$$

where $-\Delta_X$ is the Laplace-Beltrami operator on X and the $V_{ij}(y)$ are real-valued, measurable functions on $\mathbb{R}^\nu$. Throughout, we assume that

$$V_{ij}(-\Delta+1)^{-1} \quad \text{and} \quad (-\Delta+1)^{-1}(y\cdot \nabla V_{ij})(-\Delta+1)^{-1}$$
$$\text{are compact as operators on } L^2(\mathbb{R}^\nu, d^\nu y),\ 1\leqq i, j\leqq N. \tag{1}$$

Here $-\Delta$ is the Laplacian on $L^2(\mathbb{R}^\nu, d^\nu y)$ and ∇V_{ij} is the distributional gradient of V_{ij}. Under these assumptions, H is well-defined as an operator perturbation of $-\Delta_X$ and the crucial "Mourre estimate" holds [1, 4, 6].

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We will study point spectrum of H embedded in its continuous spectrum. To state known results, we recall that the set of thresholds of H, denoted $\mathcal{T}(H)$, is defined as follows. Let $C=\{i_1,\ldots,i_k\}\subset\{1,\ldots,N\}$, let $X_C=\{x\in X: x_i=0 \text{ if } i\notin C\}$, and denote by $H(C)$ the operator

$$H(C)=-\Delta_{X_C}+\sum_{i<j:i,j\in C} V_{ij}(x_i-x_j)$$

on $L^2(X_C)$ if $k\geqq 2$, or the zero operator on $\mathbb{C}$ if $k=1$. $H(C)$ is the cluster Hamiltonian for the cluster C, and if $|C|$ denotes the number of elements in C, $H(C)$ is a $|C|$-body Schrödinger operator. A real number E is a threshold of H if there is a partition of $\{1,\ldots,N\}$ into clusters $\{C_1,\ldots,C_\ell\}$, $\ell\geqq 2$, such that $E=E_1+\ldots+E_\ell$ and E_i is an eigenvalue of $H(C_i)$, $1\leqq i\leqq \ell$. $\mathcal{T}(H)$ is the set of all such E.

For N-body Schrödinger operators with two-body potentials V_{ij} obeying hypothesis (1), $\mathcal{T}(H)$ is a closed, countable set and eigenvalues of H can accumulate only at points of $\mathcal{T}(H)$ [4, 6]. Recent work of Froese and Herbst [2] has revealed the close connection between $\mathcal{T}(H)$ and the spatial decay of eigenfunctions of H. To state a slight extension of one of their results (cf. [5]) which we will use below, define for $E\in\mathbb{R}$ the function

$$\Lambda(E)=\inf\{\lambda\in\mathcal{T}(H):\lambda>E\},$$

where we set $\Lambda(E)=+\infty$ if no such λ exists. Then we have:

Theorem 1 [2, 5]. *Let H be an N-body Schrödinger operator with V_{ij} satisfying hypothesis* (1), *and let $H\psi=E\psi$ with $\psi\in L^2(X)$. Then:*

(i) $\sup\{\alpha^2+E:\exp(\alpha|x|)\psi\in L^2(X)\}$ *is either $+\infty$ or an element of $\mathcal{T}(H)$.*

(ii) *The estimate $\|\exp(\alpha|x|)\psi\|\leqq C\|\psi\|$ holds, where C can be chosen uniform in ψ and in those E,α such that $|E|$, α are uniformly bounded and $\Lambda(E)-(E+\alpha^2)$ is bounded below by a fixed, positive constant.*

Conclusion (i) is due to Froese and Herbst [2] who proved their result by contradiction. Using their ideas and a slight rearrangement of their proof, one recovers the uniform estimate (ii) (cf. [5]).

Using Theorem 1, we can give the following characterization ("semi-finiteness") of point spectrum of H embedded in its continuous spectrum.

Theorem 2. *Let H be an N-body Schrödinger operator with two-body potentials V_{ij} obeying hypothesis* (1). *Then:*

(i) *Points of $\mathcal{T}(H)$ are "isolated from above", i.e., if $E_0\in\mathcal{T}(H)$ and $E_1=\Lambda(E_0)$, then $E_1>E_0$ (strict inequality).*

(ii) *Let E_0, E_1 be defined as in* (i). *Then for any $E'\in(E_0,E_1)$, point spectrum of H in (E_0,E') is finite and of finite multiplicity.*

Conclusion (ii) shows that point spectrum of H is "semi-finite": eigenvalues of H may accumulate from *below* at thresholds, but never from *above*. Conclusion (i) shows that the statement of conclusion (ii) is never vacuous.

Proof of Theorem 2. We proceed by induction on N. For $N=1$, H is the zero operator on $\mathbb{C}$, $\mathcal{T}(H)$ is empty, and the point spectrum of H is $\{0\}$, so (i) and (ii) are trivial. Given an N-body Schrödinger operator H, we assume inductively that all $H(C)$, $1\leqq|C|\leqq N-1$, obey (i) and (ii).

We claim H obeys (i). If not, there is a point $E_\infty \in \mathscr{T}(H)$ and a decreasing sequence $\{E_n\}$ from $\mathscr{T}(H)$ with $E_n \downarrow E_\infty$ as $n \to \infty$. Hence there is a cluster Hamiltonian $H(C)$ and a decreasing sequence $\{E'_n\}$ of its eigenvalues converging to a limit E'_∞ which therefore lies in $\mathscr{T}(H(C))$. This contradicts (ii) for $H(C)$: hence (i) holds for H.

We now show (ii) holds for H. Let $E_0 \in \mathscr{T}(H)$, let $E_1 = \Lambda(E_0)$, and let $E' \in (E_0, E_1)$. On the one hand, for any fixed α with $\alpha^2 < E_1 - E'$ and any L^2 eigenvector ψ of H with eigenvalue $E \in (E_0, E')$, the estimate

$$\|\exp(\alpha|x|)\psi\| \leqq C\|\psi\|$$

holds, with C uniform in such E, by Theorem 1(ii). On the other, the equivalence of the $\mathscr{D}(H)$ and $\mathscr{D}(-\Delta_X)$ graph norms [$\mathscr{D}(A)$ denotes the operator domain of A] implies that for a fixed C' and any eigenvector ψ with eigenvalue $E \in (E_0, E')$,

$$\|(-\Delta_X + 1)\psi\| \leqq C'\|\psi\| .$$

Hence the set of all such normalized ψ is compact, hence finite-dimensional, so (ii) holds. □

Remark. After this work was done, Ira Herbst informed me that one can also prove Theorem 2 by combining estimates of [2] with an argument involving weakly converging sequences. Hence, one can prove Theorem 2 without the use of Theorem 1(ii) [3].

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