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Generic Fréchet Differentiability of the Pressure in Certain Lattice Systems

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Abstract. This note has two goals. The first is to give an explicit description (Theorem 1) of the duals of certain weighted products $\mathscr{B}_h$ of a countable family of Banach spaces. These products include the usual spaces of interactions which arise in statistical mechanics. The second goal is to use this description to prove that if the factor spaces are finite dimensional and the weight function h satisfies a certain growth condition, then the pressure is Fréchet differentiable wherever it is Gateaux differentiable (hence is Fréchet differentiable in a dense G_δ subset).

1. Notation and Definitions

The set-up described below is a slight generalization of a standard one for lattice systems. Let $\mathbb{Z}^\nu$ denote the ν-dimensional integer lattice. Throughout the paper the letters X and Y will denote non-empty finite subsets of $\mathbb{Z}^\nu$. We assume that to each X there corresponds a real Banach space $\mathfrak{A}_X$ containing an element 1_X of norm 1. For each ν-tuple i in $\mathbb{Z}^\nu$ we let τ_i be an isometry from $\mathfrak{A}_X$ onto $\mathfrak{A}_{X+i}$, with $\tau_i 1_X = 1_{X+i}$ and $\tau_i\tau_j = \tau_{i+j}$.

Definition. Let h be a positive function on the non-empty finite subsets X of $\mathbb{Z}^\nu$ satisfying $h(X+i) = h(X)$ for each such X and each i in $\mathbb{Z}^\nu$. Let $\mathscr{B}_h$ denote the Banach space of all functions Φ on the non-empty finite subsets X of $\mathbb{Z}^\nu$ which satisfy

$$\Phi(X) \in \mathfrak{A}_X \quad \text{(for each such } X\text{)},$$
$$\tau_i\Phi(X) = \Phi(X+i) \quad \text{(each } X\text{, each } i \text{ in } \mathbb{Z}^\nu\text{)},$$

and

$$\sum_{0\in X} h(X)\|\Phi(X)\| < +\infty\,,$$

with norm

$$\|\Phi\| = \sum_{0\in X} h(X)\|\Phi(X)\|\,.$$

The basic example of such a space $\mathscr{B}_h$ uses $h(X)=|X|^{-1}$, where $|X|$ is the number of points in the set X. The space $\mathscr{B}_h$ in this particular case will be denoted simply by $\mathscr{B}$. Although our representation theorem for $\mathscr{B}_h^*$ is valid with no restriction on h, the only spaces $\mathscr{B}_h$ of interest are those which are naturally included in $\mathscr{B}$. This is the case, obviously, if $h(X)\geqq|X|^{-1}$; for instance, if $h(X)\equiv 1$ (this corresponds to the space $\tilde{\mathscr{B}}$ of [4]) or if $h(X)=|X|$. Our differentiability theorem (Theorem 2) applies to the case

$$h(X)=|X|^{-1}(1+\operatorname{diam}X)^r,$$

where $r>0$. (The diameter can be defined using the supremum metric on $\mathbb{Z}^\nu$, say.)

2. A Representation Theorem

Before proving Theorem 1, we prove a useful lemma which exhibits an absolutely convergent basis-like decomposition of the members of $\mathscr{B}_h$. To this end, we introduce certain elements of $\mathscr{B}_h$ which correspond to the interactions Ψ_A^X defined by Israel [4, p. 7].

Definition. If $X\subseteqq\mathbb{Z}^\nu$ and $A\in\mathfrak{A}_X$, let $[A;X]$ be that element of $\mathscr{B}_h$ such that $[A;X](Y)=\tau_i A$ if $Y=X+i$ for some $i\in\mathbb{Z}^\nu$, while $[A;X](Y)=0$ otherwise.

Note that $[A;X]$ *is in* $\mathscr{B}_h$ *for any* h; indeed,

$$\begin{aligned}\|[A;X]\| &= \sum_{0\in Y} h(Y)\|[A;X](Y)\| \\ &= \sum_{0\in X+i} h(X+i)\|\tau_i A\| = \sum_{-i\in X} h(X)\|A\| \\ &= |X|\cdot h(X)\cdot\|A\|.\end{aligned}$$

Lemma. *If* $\Phi\in\mathscr{B}_h$, *then*

$$\|\Phi\| = \sum_{0\in X}|X|^{-1}\|[\phi(X);X]\|, \tag{1}$$

and

$$\Phi = \sum_{0\in X}|X|^{-1}[\Phi(X);X]. \tag{2}$$

Proof. From the norm computation just completed above, the right side of (1) is

$$\sum_{0\in X}|X|^{-1}|X|h(X)\|\Phi(X)\|,$$

as required. This shows that the infinite vector sum in (2) is absolutely convergent. We prove (2) by showing that for each $Y\subseteqq\mathbb{Z}^\nu$,

$$\Phi(Y)=\Big(\sum_{0\in X}|X|^{-1}[\Phi(X);X]\Big)(Y).$$

If X is not a translate of Y, then by definition $[\Phi(X);X](Y)=0$, so we need only consider the above sum over those sets X which contain 0 and are a translate of Y, say $X=Y-i$. There are $|Y|$ such sets and for each one we have, of course, $|X|=|Y|$

as well as

$$[\Phi(X);X](Y)=[\Phi(Y-i);Y-i](Y)=\tau_i\Phi(Y-i)=\Phi(Y).$$

The factor $|X|^{-1}$ now guarantees that the sum of these $|Y|$-many copies of $\Phi(Y)$ is equal to $\Phi(Y)$.

Theorem 1. *The dual $\mathscr{B}_h^*$ of $\mathscr{B}_h$ is linearly isometric to the Banach space $\mathscr{A}_h$ of all elements $\alpha=(\alpha^X)$ satisfying*

(i) $\alpha^X\in\mathfrak{A}_X^*$ *for each X,*

(ii) $\alpha^{X+i}\circ\tau_i=\alpha^X$ *for each X and each $i\in\mathbb{Z}^\nu$,*

and

(iii) $$\|\alpha\|=\sup_{X\ni 0}|X|^{-1}h(X)^{-1}\|\alpha^X\|<\infty.$$

The pairing between an element $\alpha=(\alpha^X)\in\mathscr{A}_h$ and an interaction $\Phi\in\mathscr{B}_h$ is given by

(1) $$\langle\alpha,\Phi\rangle=-\sum_{0\in X}|X|^{-1}\alpha^X[\Phi(X)].$$

Proof. It is straightforward to verify that $\mathscr{A}_h$ is a Banach space. We first show that, given $\alpha\in\mathscr{A}_h$, the pairing in (1) defines a continuous linear functional L_α on $\mathscr{B}_h$ for which $\|L_\alpha\|=\|\alpha\|$. Indeed, defining $L_\alpha(\Phi)=\langle\alpha,\Phi\rangle$ for each $\Phi\in\mathscr{B}_h$, we have

$$\begin{aligned}|L_\alpha(\Phi)|&\leqq\sum_{0\in X}|X|^{-1}\|\alpha^X\|\cdot\|\Phi(X)\|\\&=\sum_{0\in X}|X|^{-1}h(X)^{-1}\|\alpha^X\|\cdot h(X)\|\Phi(X)\|\\&\leqq\Big(\sum_{0\in X}h(X)\|\Phi(X)\|\Big)\cdot\sup_{0\in X}|X|^{-1}h(X)^{-1}\|\alpha^X\|\\&=\|\alpha\|\cdot\|\Phi\|,\end{aligned}$$

so that L_α is continuous, with $\|L_\alpha\|\leqq\|\alpha\|$. On the other hand, if $A\in\mathfrak{A}_X$, then $[A;X]\in\mathscr{B}_h$ so from (1) we have

$$\begin{aligned}L_\alpha([A;X])&=-\sum_{0\in X+i}|X+i|^{-1}\alpha^{X+i}(\tau_iA)\\&=-\sum_{0\in X+i}|X|^{-1}\alpha^X(A)=-\alpha^X(A).\end{aligned}$$

Thus, for $A\in\mathfrak{A}_X$,

(2) $$|\alpha^X(A)|=|L_\alpha([A;X])|\leqq\|L_\alpha\|\cdot\|[A;X]\|=\|L_\alpha\|\cdot|X|\cdot h(X)\cdot\|A\|,$$

which implies that $|X|^{-1}h(X)^{-1}\|\alpha^X\|\leqq\|L_\alpha\|$; in turn, this implies that $\|\alpha\|=\|L_\alpha\|$.

We must now show that the map $\alpha\to L_\alpha$ (which is clearly linear) takes $\mathscr{A}_h$ *onto* $\mathscr{B}_h^*$; that is, given $L\in\mathscr{B}_h^*$, it is necessarily of the form L_α for some α. To this end, define, for each X and each $A\in\mathfrak{A}_X$,

(3) $$\alpha^X(A)=-L([A;X]).$$

The inequality in (2) shows that $\alpha^X\in\mathfrak{A}_h^*$ for each X and property (ii) follows easily from (3), so (α^X) defines an element α of $\mathscr{A}_h$. To prove that $L=L_\alpha$ is equivalent to proving that for each $\Phi \in \mathscr{B}_h$,

$$L(\Phi) = \langle \alpha, \Phi \rangle = \sum_{0 \in X} |X|^{-1} \alpha^X[\Phi(X)]$$
$$\equiv \sum_{0 \in X} |X|^{-1} L([\Phi(X); X]).$$

This is immediate from the lemma and the continuity of L.

Henceforth, we will write α in place of L whenever we consider an element of $\mathscr{B}_h^*$. There is a simple but useful consequence of the relationship $\alpha^X(A) = -\alpha([A; X])$ $(A \in \mathfrak{A}_X)$, namely, *if* (α_γ) *is a net in* $\mathscr{B}_h^*$ *with* $\alpha_\gamma \to 0$ *weak*, then* $\alpha_\gamma^X \to 0$ *weak* in* $\mathfrak{A}_X^*$ *for each* X.

3. Differentiability of the Pressure

We will shortly need a few specific properties of the pressure, but first we look at an arbitrary convex continuous function P on a Banach space $\mathscr{A}$. Recall that if $\Phi \in \mathscr{A}$, then *the subdifferential* $\partial P(\Phi)$ *of* P *at* Φ is the non-empty weak* compact convex subset of $\mathscr{A}^*$ consisting of all those $\alpha \in \mathscr{A}^*$ satisfying

$$\alpha(\Psi) - \alpha(\Phi) \leqq P(\Psi) - P(\Phi), \qquad \Psi \in \mathscr{A}.$$

The differentiability of P (both Gateaux and Fréchet) can be characterized in terms of the subdifferential as follows. (These will be our definitions.) For details see, for instance, [3] or [6].

The continuous convex function P *is Gateaux differentiable at* Φ *if and only if* $\partial P(\Phi)$ *consists of a single point.* If this be the case, then the (set-valued) subdifferential map ∂P *is norm-weak* upper semicontinuous at* Φ, that is, if $\partial P(\Phi) = \{\alpha\}$, say, if $\|\Phi_n - \Phi\| \to 0$ and if $\alpha_n \in \partial P(\Phi_n)$, then $\alpha_n \to \alpha$ weak*.

The continuous convex function P *is Fréchet differentiable at* Φ *if and only if it is Gateaux differentiable at* Φ [with $\partial P(\Phi) = \{\alpha\}$, say] *and* ∂P *is norm-norm upper semicontinuous at* Φ, that is

$$\|\alpha_n - \alpha\| \to 0 \quad \text{whenever} \quad \|\Phi_n - \Phi\| \to 0 \quad \text{and} \quad \alpha_n \in \partial P(\Phi_n).$$

In order to define the pressure one must, of course, be much more specific about the spaces $\mathfrak{A}_X$. For our purposes, however, it suffices to assume that we are given a continuous convex function P, defined on the space $\mathscr{B}$ (hence on each $\mathscr{B}_h$ contained in $\mathscr{B}$), which satisfies the following two conditions:

(a) $P([t \cdot 1_X; X]) = -t$ for each X and each real t.

(b) $|P(\Phi) - P(\Psi)| \leqq \|\Phi - \Psi\|$ for each Φ, Ψ in $\mathscr{B}$, the norm on the right being that of $\mathscr{B}$.

(These are well-known properties of the pressure as defined in [4], say. Property (a) follows from the definition in [4, p. 35], keeping in mind that $[A; X]$ corresponds to Ψ_A^X.)

In the proof of the next proposition, which uses a well-known technique, we will simply write t in place of $t \cdot 1_X$, since no ambiguity seems likely.

Proposition. *If* $\alpha = (\alpha^X) \in \mathscr{B}_h^*$ *is in* $\partial P(\Phi)$ *for some* $\Phi \in \mathscr{B}_h$, *then for each* X,

$$\|\alpha^X\| = 1 = \alpha^X(1_X)$$

$\left(\text{and hence } \|\alpha\| = \sup_{0 \in X} |X|^{-1} h(X)^{-1}\right)$.

Proof. If $\alpha \in \partial P(\Phi)$, then in particular, for any X and any real number t we have

$$-\alpha^X(t) = \alpha([t;X]) \leqq P([t;X]) - P(\Phi) + \alpha(\Phi) = -t - P(\Phi) + \alpha(\Phi).$$

Taking $t > 0$, we can divide both sides by t and let $t \to \infty$ to conclude that $\alpha^X(1) \geqq 1$. A similar argument with $t \to -\infty$ yields $\alpha^X(1) = 1$, which in turn implies that $\|\alpha^X\| \geqq 1$. If $t > 0$ and $A \in \mathfrak{A}_X$, then from the definition of the subdifferential and property (b) we conclude that

$$\alpha([tA;X]) - \alpha(\Phi) \leqq P([tA;X]) - P(\Phi) \leqq \|[tA;X] - \Phi\| \leqq t\|A\| + \|\Phi\|.$$

The same device as before shows that $\alpha([A;X]) \leqq \|A\|$, so that $\|\alpha^X\| \leqq 1$ and the proof is complete.

In order to prove the differentiability result we seek, we need to impose a growth condition on the weight function h; more precisely, on the function $|X|h(X)$.

Definition. We say that *h satisfies condition* (G) provided the following holds:

(G) For any $\varepsilon > 0$, we have $|X|h(X) \geqq \varepsilon^{-1}$ for almost all X containing 0, that is, this inequality holds for all but finitely many non-empty finite subsets X for which $0 \in X \subseteqq \mathbb{Z}^\nu$.

Condition (G) is satisfied if $|X|h(X) \to \infty$ as $\operatorname{diam} X \to \infty$; for instance, this is the case when $h(X) = |X|^{-1}(1 + \operatorname{diam} X)$. It is clear that if h satisfies condition (G), then $\mathscr{B}_h \subseteqq \mathscr{B}$.

The following theorem is now quite easy.

Theorem 2. *Suppose that the weight function h satisfies property* (G) *and that each of the spaces $\mathfrak{A}_X$ is finite dimensional. Then the pressure P is Fréchet differentiable at any point of $\mathscr{B}_h$ where it is Gateaux differentiable; in particular, then, it is Fréchet differentiable on a dense G_δ subset of $\mathscr{B}_h$.*

Proof. Suppose that P is Gateaux differentiable at $\Phi \in \mathscr{B}_h$, with $\partial P(\Phi) = \{\alpha\}$, say. Suppose further that $\{\Phi_n\} \subseteqq \mathscr{B}_h$ with $\|\Phi_n - \Phi\| \to 0$ and that $\alpha_n \in \partial P(\Phi_n)$, $n = 1, 2, \ldots$. By the Gateaux differentiability hypothesis, $\alpha_n \to \alpha$ weak*, and hence $\alpha_n^X \to \alpha^X$ weak*, for each X. Since $\mathfrak{A}_X^*$ is finite dimensional, this means that $\|\alpha_n^X - \alpha^X\| \to 0$ for each X. Moreover, from the proposition, we know that $\|\alpha_n^X - \alpha^X\| \leqq 2$ for all X. Since, by Theorem 1,

$$\|\alpha_n - \alpha\| = \sup_{0 \in X} \{|X|^{-1} h(X)^{-1} \|\alpha_n^X - \alpha^X\|\},$$

it follows easily from property (G) that $\|\alpha_n - \alpha\| \to 0$ and hence P is Fréchet differentiable at Φ. That such points form a dense G_δ subset of the separable space $\mathscr{B}_h$ is guaranteed by Mazur's theorem.

An example where the weight function $h(X) = |X|^{-1}(1 + \operatorname{diam} X)$ is used may be found in Chap. 5 of Ruelle [8], which is devoted to one-dimensional lattice systems. Corollary 5.6 of [8] shows that the pressure for such systems is everywhere Gateaux differentiable; Theorem 2 shows that it is therefore Fréchet differentiable. (This is, of course, also a consequence of Dobrushin's (Gateaux) analyticity theorem. For a recent proof of the latter, see Cassandro and Olivieri [1].)

Daniëls and van Enter [2] have shown that for the space $\mathscr{B}$ of [4], the pressure is nowhere Fréchet differentiable (see also [5]), even when the spaces $\mathfrak{A}_X$ are finite dimensional. Thus, some restriction on h is needed in order that the conclusion of Theorem 2 be valid. That $h(X)$ should depend on the diameter of X is indicated by some specific examples. For instance, in a private communication, Israel has pointed out that the construction which Daniëls and van Enter use in their Theorem 1 can be applied to the interaction Φ for a two-dimensional nearest-neighbor antiferromagnet at low temperature to show that the pressure is Gateaux – but not Fréchet – differentiable at Φ. The interactions involved in this example have finite range, hence lie in every $\mathscr{B}_h$, while the proof itself is valid provided h satisfies

$$\sup\{h(X):|X|=2\}<\infty.$$

Thus, if $h(X)$ does not grow somehow with the diameter of X, the conclusion of Theorem 2 can fail. Daniëls and van Enter's Theorem 3 can be modified to show that if the pressure is *twice* continuously Fréchet differentiable in a neighborhood in $\mathscr{B}_h$ of the interaction they consider, then

$$\liminf\{h(X)^2/\mathrm{diam}X:|X|=2, \mathrm{diam}X\to\infty\}>0.$$

[Their inequality (16) can be replaced by a similar one derived from the Taylor formula for a C^2 function.]

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