Generalized Internal Long Waves Equations: Construction, Hamiltonian Structure, and Conservation Laws

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Abstract. A general class of the ILW type equations is constructed. We introduce a Hamiltonian structure and construct an infinite number of conservation laws.

0. Introduction

Recent studies [1–4] have shown that the following equation

$$u_t + \delta^{-1}u_x + 2uu_x + T[u_{xx}] = 0,\tag{1}$$

where $T[u](x) = \int_{-\infty}^{+\infty} (1/2\delta)\coth(\pi(y-x)/(2\delta))u(y)dy$ is of mathematical and physical interest. This equation has many important mathematical features similar to those of the Korteweg de Vries equation (KdV). Physically it represents internal long waves (ILW) in a stratified fluid of finite depth characterized by the real parameter δ [5–6]. The limiting cases of ILW are: the KdV ($\delta \to 0$) and the Benjamin-Ono (BO) ($\delta \to \infty$) equations [3].

For a long time the outward similarity of the ILW (BO) equation with the KdV equation, the existence of an infinite number of conservation laws, a Bäcklund transformation and so on have suggested that there should be a general theory where ILW (BO) would be the simplest example (like KdV for the general Lax equations). As early as October 1978 at the Leningrad Soliton Conference L. D. Faddeev emphasized the importance of studying the BO equation by making use of group-theoretical methods, as with KdV.

In this paper we consider some aspects of the theory of the ILW type equations. Let us state the main results of this paper.

The first result is the construction of a general class of the ILW type equations by means of the formal Zakharov-Shabat "dressing" method [7] (Zakharov-Shabat's technique for ILW was discovered in [4]). Let L_0 be the symbol of a skew Hermitian differential operator with constant coefficients, and let K be the symbol of a Volterra operator with the coefficients holomorphic in the strip,

$-\delta < \operatorname{Im} Z < \delta$, satisfying some additional conditions (see Sect. 3 below). It is required that K should satisfy the following condition: the symbol $L = (1 + K^-) L_0 (1 + K^+)^{-1}$ is purely differential. $\Big[$Here $K^{\pm}$ denote the boundary values of the symbol K, i.e. if $K = \sum\limits_{j=-\infty}^{-1} K_j(z)(i\xi)^j$, then $K^{\pm} = \sum\limits_{j=-\infty}^{-1} K_j(x \pm i\delta)(i\xi)^j$.

The definitions of symbols, symbol multiplication and so on are given below.$\Big]$

Dress another skew Hermitian operator M_0 with constant coefficients using the symbol K and denote the differential part of $(1 + K)M_0(1 + K)^{-1}$ by M. Then the equation

$$L_t = LM^+ - M^- L \tag{2}$$

is well defined, i.e. the number of unknown coefficients of the operator L is equal to the number of nonlinear equations. (For the ILW equation $L_0 = -\xi + 1/(2i\delta)$, $M_0 = i\xi^2$, $u = -i(K^-_{-1} - K^+_{-1})$ [4].) Of course one can apply the "dressing" method to arbitrary L_0 and M_0. This yields equations for complex valued unknown functions. Choosing skew Hermitian symbols is one way to construct equations with real valued solutions.

The second result of our paper is that the Eq. (2) are Hamiltonian in the so-called second Hamiltonian structure [8–12]. The first non-trivial question in this field is how to construct a suitable space of functionals, on which the Hamiltonian structure should be defined. As far as we know, none of the previous works which study the Hamiltonian formalism for ILW and BO bother about this matter. We give a rigorous construction of a space of functionals with the usual Poisson bracket in Sect. 5.

Our method of forming Hamiltonians is rather unusual: we reconstruct them from known gradients. This saves us the trouble of computing the variational derivatives of the Hamiltonians. Then we show that the generating function for these Hamiltonians satisfies an analog of the Riccati equation. In the ILW case this last result coincides with the results of [3].

Our third result is that the Hamiltonians are in involution.

We do not discuss the limits $\delta \to 0$ and $\delta \to \infty$ of (2) as was done in [3] because of the brevity of our note.

Finally we would like to point out that our work has been inspired by our thinking out the relation between Zakharov-Shabat's technique for ILW discovered in [4] and the group-theoretical methods of the works [13, 14].

1. Symbol Algebras

Let $\mathscr{B}$ be some differential ring of complex valued functions from the Schwarz space (smooth functions rapidly vanishing at $\pm \infty$ with their derivatives). Such a ring is equipped with a derivation $\partial_x : \mathscr{B} \to \mathscr{B}$. We denote by $\mathscr{B}((\xi^{-1}))$ the ring of formal Laurent series $X = \sum\limits_{j=-\infty}^{N} X_j(i\xi)^j$ over $\mathscr{B}$ with a finite number of positive

terms. There are two derivations in $\mathscr{B}((\xi^{-1})):\partial_\xi=\partial/\partial_\xi$ and $\partial_x:\sum_j X_j(i\xi)^j$ $\mapsto \sum_j \partial_x X_j(i\xi)^j$. Using them one may define a new associative multiplication on $\mathscr{B}((\xi^{-1}))$ which is called symbol multiplication:

$$X \circ Y = \sum_{\alpha \geq 0} 1/\alpha! X_\xi^{(\alpha)} Y_x^{(\alpha)},$$

where $X_\xi^{(\alpha)}=\partial_\xi^\alpha X$, $Y_x^{(\alpha)}=\partial_x^\alpha Y$. Usually we will omit the sign $\circ$. The inverse of X (if it exists) is defined with respect to the symbol multiplication.

Using the symbol multiplication one can make $\mathscr{B}((\xi^{-1}))$ into a complex Lie algebra $\mathfrak{g}$ with bracket $a \circ b - b \circ a$. Our algebra can be split into a direct sum of two complex subspaces $\mathfrak{g}=\mathfrak{g}_+ \dot{+} \mathfrak{g}_-$, where $\mathfrak{g}_+ = \left\{ \sum_{j=0}^{N} X_j(i\xi)^j \right\}$ and $\mathfrak{g}_- = \left\{ \sum_{j=-\infty}^{-1} X_j(i\xi)^j \right\}$. We call $\mathfrak{g}_+$ ($\mathfrak{g}_-$) the Lie algebra of differential (Volterra) operators. By X_+ (X_-) we will denote the projection of $X \in \mathfrak{g}$ onto $\mathfrak{g}_+$ ($\mathfrak{g}_-$); X_+ (X_-) will be referred to as the differential (integral) part of X.

Let $\mathrm{res}\left(\sum_j X_j(i\xi)^j\right) = X_{-1}$. Then, as usual, for any $X \in \mathscr{B}((\xi^{-1}))$ we set

$$\mathrm{tr} X = \int_{-\infty}^{\infty} \mathrm{res} X dx.$$

The main property of the trace is the equality $\mathrm{tr}[X,Y]=0$, so there is the invariant, non-degenerate scalar product $\langle X, Y \rangle = \mathrm{tr}(XY)$ on the Lie algebra $\mathfrak{g}$ (see [13–15] for further details). Identify $\mathfrak{g}_+$ with the dual space to $\mathfrak{g}_-$ via the scalar product.

There are two operations on $\mathfrak{g}$, transposition and complex conjugation: for any $X = \sum_{j=-\infty}^{N} X_j(i\xi)^j$ we define $^tX = \sum_{j=-\infty}^{N} (-1)^j(i\xi)^j X_j$ and $X^* = \sum_{j=-\infty}^{N} (-1)^j \cdot X_j^*(i\xi)^j$. An element $X \in \mathfrak{g}$ is called skew Hermitian if $^tX^*=-X$. The skew Hermitian symbols form a real Lie subalgebra $\mathfrak{ug}$ in $\mathfrak{g}$. The restriction of $\langle,\rangle$ to $\mathfrak{ug}$ gives an invariant, real, non-degenerate scalar product on $\mathfrak{ug}$. The subalgebra $\mathfrak{ug}_+ = \mathfrak{ug} \cap \mathfrak{g}_+$ may be regarded as the dual space to $\mathfrak{ug}_-$ via this scalar product. (The reality of the restriction of $\langle,\rangle$ means that $\langle X, Y \rangle^* = \langle X, Y \rangle$ for any X, Y.)

Any $X \in \mathfrak{ug}$ can be written in the form $X = i \sum_{j=-\infty}^{N} X_j(i\xi)^j$. From the relation $^tX^*=-X$ it is easy to show that $\mathrm{Re} X_j$ can be chosen arbitrarily, and $\mathrm{Im} X_j$ can be expressed as a linear function of $\mathrm{Re} X_k$ with $k>j$:

$$\mathrm{Im} X_j = \Lambda_{k>j}(\mathrm{Re} X_k) = \sum_{k,\alpha} \lambda_{k,\alpha}(\mathrm{Re} X_k)_x^{(\alpha)}, \qquad \lambda_{k,\alpha} \in \mathbb{R}.$$

2. Analytical Properties of the Operator T

Denote by $\mathscr{S}_\mathbb{C}(\mathscr{S}_\mathbb{R})$ the Schwarz space of smooth complex (real) valued functions on the real axis which are rapidly vanishing at infinity with their derivatives. (This

space will be denoted by $\mathscr{S}$ if the index $\mathbb{C}$ or $\mathbb{R}$ is not important.) The Fourier transform converts T into the operator of multiplication by the function $-i\coth(k\delta)$. This enables us to compute the asymptotic behaviour of $Tu(x)$ as $|x|\to\infty$:

$$Tu(x) \sim -(1/(2\delta))\operatorname{sign}(x)\int_{-\infty}^{\infty} u(y)dy.$$

So the operator T maps the space $\mathscr{S}$ into the space of $\mathscr{C}^{\infty}$ functions approaching constants at $\pm\infty$. One can also show that the operator Td/dx maps $\mathscr{S}$ into $\mathscr{S}$.

Proposition 1 [4]. a) *For any function* $u(x)\in\mathscr{S}$ *there exists a unique function* $U(z)$ *with the following properties:*

(i) $U(z)$ *is holomorphic, bounded and continuous up to the boundary in the strip* $-\delta<\operatorname{Im}z<\delta$;

(ii) $(U^{-}-U^{+})(x)=iu(x)$;

(iii) $Tu(x)=(U^{-}+U^{+})(x)$. *Here* $U^{\pm}$ *denotes the boundary values* $U(x\pm i\delta)$ *of* $U(z)$.

b) *The function* $U(z)$ *is given by*

$$U(z-i\delta)=\tfrac{1}{2}\int (1/(2\delta))\coth(\pi(y-z)/(2\delta))u(y)dy. \tag{3}$$

c) *If* $W(z)$ *has the properties* (i), (ii), *then* $W(z)-U(z)=\mathrm{const}$.

d) *If* $W(z)$ *has the property* (i) *and* $(W^{-}-W^{+})(x)\in\mathscr{S}$, *then*

$$T(W^{-}-W^{+})=i(W^{-}+W^{+})+\mathrm{const}. \tag{4}$$

Remark. The constant in (4) can be computed by comparing the asymptotic behaviour of the right and left hand sides in Eq. (4). For example if $U(z)$ is constructed from $u(x)\in\mathscr{S}$ [as given by Eq. (3)] and $V(z)$ from $v(x)\in\mathscr{S}$, then $U(z)V(z)$ satisfies conditions d) of Proposition 1. So

$$T(U^{-}V^{-}-U^{+}V^{+})=i(U^{-}V^{-}+U^{+}V^{+})-(i/(8\delta^{2}))\int_{-\infty}^{\infty} udy\int_{-\infty}^{\infty} vdy.$$

From this one can obtain the formula:

$$TuTv-uv=T(uTv+vTu)+(1/(4\delta^{2}))\int_{-\infty}^{\infty} udy\int_{-\infty}^{\infty} vdy.$$

In conclusion let us point out the following obvious properties of the operator T:

a) $Td/dx-d/dxT=0$; b) $\int_{-\infty}^{\infty} u_{1}Tu_{2}dy=-\int_{-\infty}^{\infty}(Tu_{1})u_{2}dy$ for any $u_{1},u_{2}\in\mathscr{S}$; c) $(Tu)^{*}=T(u^{*})$.

3. The Formal Version of the Zakharov-Shabat "Dressing" Method

Proposition 2. *Let* L_{0} *be the symbol of a skew Hermitian differential operator with constant coefficients:* $L_{0}=i\sum_{k=0}^{n} c_{k}(i\xi)^{k}$, $c_{k}\in\mathbb{R}[\delta,\delta^{-1}]$, $c_{n}=1$. *Let* $l_{j}(x)$ $(j=0,...,n-1)$ *be a set of* n *functions from the Schwarz space* $\mathscr{S}_{\mathbb{C}}$ *such that the symbol*

$L_1 = i \sum_{j=0}^{n-1} l_j(i\xi)^j$ is skew Hermitian. Then

    a) *there exists a unique symbol* $K = \sum_{j=-\infty}^{-1} K_j(z)(i\xi)^j$ *satisfying the conditions;*

    (i) *the coefficients* $K_j(z)$ *are holomorphic, bounded and continuous up to the boundary in the strip* $-\delta < \mathrm{Im}\, z < \delta$;

    (ii) $(1+K^-)L_0(1+K^+)^{-1} = L$ *is a differential skew Hermitian symbol;*

    (iii) $L - L_0 = L_1$;

    (iv) $T(K^- - K^+) \equiv \sum_{j=-\infty}^{-1} T(K_j^- - K_j^+)(i\xi)^j = i(K^- + K^+).$

    b) *The symbol* K *satisfies the conditions*

$$ {}^t(1+K^+)^* = (1+K^-)^{-1}, \qquad {}^t(1+K^-)^* = (1+K^+)^{-1}, $$

which we will call the reality conditions.

    c) *The coefficients* $K_j^\pm$ *of the symbols* $K^\pm$ *lie in the ring generated by* l_j *and the operators* T *and* d/dx *(see Sect. 4 below).*

Let $W = (1+K)M_0(1+K)^{-1}$, $M = W_+$, $M_0 = i \sum_{j=0}^{N} m_j(i\xi)^j$, $m_j \in \mathbb{R}[\delta, \delta^{-1}]$, where $(1+K)$ is the symbol constructed above. The coefficients of $W^\pm$ and consequently of $M^\pm$ are expressed in terms of l_j. This allows us to write a system of equations for the functions l_j: $L_t = LM^+ - M^-L$. Since $(M^\pm)^*$, $(M^*)^\mp$, and ${}^t(M^\pm) = ({}^tM)^\mp$, we have $LM^+ - M^-L \in \mathfrak{ug}_+$ and consequently the equations for $\mathrm{Im}\, l_j$ are linear differential combinations of the equations for $\mathrm{Re}\, l_j$. Moreover, as

$$ LW^+ - W^-L = (1+K^-)(L_0M_0 - M_0L_0)(1+K^+)^{-1} = 0 $$

our equation can be written in the form $L_t = -LW_-^+ + W_-^-L$. This shows that the order with respect to ξ of the right hand side of (2) is $n-1$, i.e. the number of equations is equal to the number of unknown functions.

    Thus the system of equations for $\mathrm{Re}\, l_j$ is well defined. In the case $L_0 = -\xi + 1/(2i\delta)$, $M_0 = i\xi^2$ we obtain the ILW equation.

4. Gelfand-Dikii Symplectic Structure

Here we will briefly review the results of [9] in the form we need. Let now $\mathscr{B}$ be the ring $\mathscr{S} + \mathbb{C} = \{\varphi = \psi + c,\ \psi \in \mathscr{S},\ c \in \mathbb{C}\}$. Let $L_0 = i \sum_{k=0}^{r} c_k(i\xi)^k \in \mathfrak{ug}_+$, $c_k \in \mathbb{R}[\delta, \delta^{-1}]$, $c_r = 1$. Consider the subspace N of $\mathfrak{ug}_+$ defined by

$$ N = \left\{ L = L_0 + i \sum_{k=0}^{r-1} l_k(i\xi)^k \right\}. $$

For any $X \in \mathfrak{ug}_-$ we can construct a vector field V_X on N. It is given by the formula

$$ V_X(L) = -i(L(XL)_+ - (LX)_+ L). $$

The field V_X is uniquely defined by the initial part $X^r \in \mathfrak{ug}_- / \mathfrak{ug}^r_-$ of X, where $\mathfrak{ug}^r_-$ is the ideal in $\mathfrak{ug}_-$ consisting of the elements $Y = \sum\limits_{j=-\infty}^{-(r+1)} Y_j(i\xi)^j$. The fields V_X for a Lie algebra with the bracket

$$[V_X(L), V_Y(L)] = V_{[X,Y]_L},$$

where

$$[X, Y]_L = i\left(X(LY)_+ - (XL)_- Y - i\frac{\partial Y}{\partial L} V_X(L)\right)_- - (X \leftrightarrow Y)$$

and $[X, Y]_L \in \mathfrak{ug}_-$. There is 2-form $\omega(V_X, V_Y)(L) = \langle V_X(L), Y \rangle$ on the vector fields V_X. It is skew symmetric and closed.

5. The Ring of Functionals

Before we can define the Poisson brackets we need to define an appropriate space of functionals. The usual KdV theory deals with the integrals of densities which are differential polynomials in u (for brevity we will write $\int f$ instead of $\int\limits_{-\infty}^{\infty} f(x)dx$):

$$H = \int P[u, u_x, \ldots, u_x^{(k)}, \ldots].$$

Let us see what happens in the ILW case. The ILW may be written in the form:

$$u_t = \frac{d}{dx}\frac{\delta H}{\delta u},$$

where $H = -\int(u^3/3 + uTu_x/2 + u^2/(4\delta))$. The second term contains the nonlocal operator T, so the construction of densities is not obvious.

The characteristic difficulties of the nonlocal theory are as follows.

1. The operator T does not preserve the Schwarz space because

$$Tu(x) \xrightarrow[|x|\to\infty]{} -1/(2\delta)\,\mathrm{sign}(x)\int\limits_{-\infty}^{+\infty} u(y)dy.$$

So

a) there are densities to which we cannot apply the operator T. For instance $T(Tu)$ is not defined;

b) some densities cannot be integrated. For instance

$$(Tu)^2 \to 1/(2\delta)^2(\int u)^2,$$

so the integral of $(Tu)^2$ diverges.

2. An integrable density may have a variational derivative which depends on x explicitly. For instance: let $I = \int T(uTu)$; then

$$\frac{\delta I}{\delta u} = \frac{1}{\delta}(xTu - T(xu)).$$

Indeed $\int uTu = 0$, so $uTu = d/dx\psi$, where $\psi \in \mathcal{S}$. Obviously ψ is equal to $\int\limits_{-\infty}^{x} (uTu)(y)dy$. Then

$$\int T d/dx\psi = T\psi|_{-\infty}^{\infty} = -1/(2\delta)[\text{sign}(x)\int\psi]|_{-\infty}^{\infty} = -1/\delta\int\psi.$$

Integrating by parts we obtain

$$I = -1/\delta \int\limits_{-\infty}^{\infty} \left(\int\limits_{-\infty}^{x} uTu(y)dy \right) dx = -1/\delta \left\{ \left[x \int\limits_{-\infty}^{x} uTudy \right]\Big|_{-\infty}^{\infty} - \int\limits_{-\infty}^{\infty} x(uTu)dx \right\}$$

$$= 1/\delta \int\limits_{-\infty}^{\infty} xuTudx.$$

Now it is easy to see that the variational derivative is given by the formula above.

3. The variational derivative may contain integrals, and even products of integrals. For instance: let $I = \int d/dx(Tu)^3$, then

$$\frac{\delta I}{\delta u} = -\frac{3}{4\delta^3}(\int u)^2.$$

Indeed

$$I = \int d/dx(Tu)^3 = (-1/2\delta)^3 \text{sign}(x)(\int u)^3|_{-\infty}^{\infty} = -1/(4\delta^3)(\int u)^3.$$

So $\delta I = -3/4\delta^3 \int(\int u)^2 \delta u$.

For this reason we choose our functionals to be of the form:

$$F = \sum_{\alpha} (\int P_{1,\alpha}) \cdots (\int P_{k,\alpha}).$$

We should like to stress that our space of functionals has a natural ring structure which would look artificial in the local theory.

4. As we saw earlier, the variational derivative may not lie in the space

$$\mathcal{S} + \mathbb{C} = \{\psi = \varphi + c, \ \varphi \in \mathcal{S}, \ c \in \mathbb{C}\}.$$

This fact destroys the usual technique (cf. [13]) because generally $\int d/dx P \neq 0$, so

$$\{H_1, H_2\} = \int \frac{\delta H_1}{\delta u} \frac{d}{dx} \frac{\delta H_2}{\delta u}$$

is not even a skew symmetric operation.

These difficulties indicate that the construction of the functionals in the nonlocal theory is a rather delicate problem; nevertheless, it can be solved.

We first construct a ring of functions $\tilde{\mathcal{R}}_{\ell}(u_0, ..., u_{n-1})$, $u_i \in \mathcal{S}_{\ell}$. Here ℓ denotes the field of real or complex numbers. (We will omit the index ℓ if it is not important.) By definition, an element of the ring $\tilde{\mathcal{R}}_{\ell}(u_0, ..., u_{n-1})$ is a linear combination of monomials with coefficients in ℓ.

The definition of monomials is given by induction. The main problem is to determine whether we are able to apply the operator T to a monomial. To control this we introduce the concept of method of construction (for brevity, simply path) for a monomial. Each path has a degree, which is also defined inductively. The definitions are as follows.

1) If $\lambda \in \mathscr{k}$, then λu_i is a monomial; $\gamma \lambda u_i = (\lambda u_i)$ is the path, and the degree of the path is $\deg \gamma \lambda u_i = 0$;

2) if Q is a monomial and γQ is a path to Q and $\deg \gamma Q \geq 0$, then TQ is a monomial, $\gamma TQ = (T\gamma Q)$ is a path to TQ and $\deg \gamma TQ = \deg \gamma Q - 1$;

3) if Q is a monomial, then $\partial_x Q$ is the monomial, $\gamma \partial_x Q = (\partial_x \gamma Q)$ is a path, and $\deg \gamma \partial_x Q = \deg \gamma Q + 1$;

4) if Q is a monomial and P is a monomial, then $Q \cdot P$ is a monomial $\gamma Q \cdot P = (\gamma Q \cdot \gamma P)$ is a path and

$$\deg \gamma Q \cdot P = \begin{cases} -1, & \text{if} \quad \deg \gamma Q = \deg \gamma P = -1, \\ 0, & \text{in all other cases}. \end{cases}$$

For any monomial G, we define

$$\deg G = \max \deg \gamma G,$$

as γG runs over all possible paths to G. The maximum is finite, since it is clearly bounded by the number of x-differentiations occurring in G. The simplest example of two paths with different degrees leading to the same monomial G is as follows.

Let $G = uu_x$, $\gamma G = ((u)/(\partial_x(u)))$; then $\deg \gamma G = 0$. There is another path $\gamma' G = (\partial_x(u/2(u)))$, for which $\deg \gamma' G = 1$. So $\deg G = 1$.

We introduce the filtration

$$\mathscr{R}_i = \{Q | \deg Q \geq i\}, \quad i \geq -1$$

on the set $\mathscr{R}$ of monomials. The Schwarz space $\mathscr{S}$ also has a filtration

$$\mathscr{S}_i = \{\psi | \psi = (d/dx)^i \varphi, \quad \varphi \in \mathscr{S}\}.$$

Define

$$\mathscr{S}_{-1} = \left\{ \psi | \psi = \int_{-\infty}^{x} \varphi(y) dy + \text{const}, \ \varphi \in \mathscr{S} \right\}.$$

It is easy to check that (for any $u_i \in \mathscr{S}$) $\mathscr{R}_i \subset \mathscr{S}_i$, but $\mathscr{R}_i \neq \mathscr{R} \cap \mathscr{S}_i$. For example $T(uTu) \in \mathscr{S}_0$, but $T(uTu) \in \mathscr{R}_{-1} \setminus \mathscr{R}_0$.

The degree of a linear combination of monomials is defined by

$$\deg \left(\sum_\alpha c_\alpha f_\alpha \right) = \min_\alpha \deg f_\alpha.$$

But the monomials are not linearly independent as functions from $\mathscr{S}_{-1}$. For instance

$$d/dx(uu_x) - u_x^2 - uu_{xx} = 0.$$

So different linear combinations of monomials may be equal to the same element from the ring $\tilde{\mathscr{R}}(u_0, \ldots, u_{n-1})$. We define the degree on $\tilde{\mathscr{R}}(u_0, \ldots, u_{n-1})$ by $\deg F = \max \deg \phi$, where ϕ runs over all possible representations of F as a linear combination of monomials. There is a filtration on $\tilde{\mathscr{R}}(u_0, \ldots, u_{n-1})$:

$$\tilde{\mathscr{R}}_j = \{F \in \tilde{\mathscr{R}} | \deg F \geq j\}, \quad j \geq -1.$$

Let $L = L_0 + L_1$ be a skew Hermitian symbol. Here $L_0 = i \sum_{k=0}^{n} c_k(i\xi)^k$, $c_k \in \mathbb{R}$, $c_n = 1$; and $L_1 = i \sum_{j=0}^{n-1} l_j(i\xi)^j$, $l_j \in \mathscr{S}_{\mathbb{C}}$. Then $\operatorname{Re} l_j = v_j \in \mathscr{S}_{\mathbb{R}}$ and $l_k = v_k + i \Lambda_{s>k}(v_s)$, where $\Lambda_{s>k}(v_s)$ is a certain linear combination of v_s and their derivatives. We define the ring of functionals $\tilde{\mathscr{F}}_{\mathbb{R}}(v_0, \dots, v_{n-1})$ to consist of all linear combinations with real coefficients of elementary functionals. The elementary functionals are:
$F_\alpha = \prod_{i=1}^{m} \int P_i$, where $P_i \in \tilde{\mathscr{R}}_{0,\mathbb{R}}(v_0, \dots, v_{n-1})$. The space $\mathscr{R}_{0,\mathbb{R}}(v_0, \dots, v_{n-1})$ can be embedded in $\mathscr{R}_{0,\mathbb{C}}(l_0, \dots, l_{n-1})$ as the subspace $\mathscr{R}_{0,\mathbb{C}}^{\sigma}(l_0, \dots, l_{n-1})$ of functions invariant under the operation of complex conjugation σ.

We say that a functional $F \in \tilde{\mathscr{F}}_{\mathbb{R}}(v_0, \dots, v_{n-1})$ has a variational derivative if its first variation δF has the form $\int \sum_i A_i \delta v_i$. In this case the coefficients A_i are called the partial variational derivatives, denoted by $\delta F / \delta v_i$; they are given by

$$A_i = \delta F / \delta v_i = \sum_p A_{i,p}^1 \cdot A_{i,p}^2,$$

where $A_{i,p}^1 \in \tilde{\mathscr{F}}_{\mathbb{R}}(v_0, \dots, v_{n-1})$ and $A_{i,p}^2 \in \tilde{\mathscr{R}}_{\mathbb{R}}(v_0, \dots, v_{n-1})$.
For example if $I = (\int v) \cdot (\int v T v_x)$, then

$$\delta I = \int ((\int v T v_x + (\int v) \cdot (2 T v_x)) \delta v$$

and

$$\delta I / \delta v = \int v T v_x + 2(\int v)(T v_x).$$

Rewrite $F \in \tilde{\mathscr{F}}_{\mathbb{R}}(v_0, \dots, v_{n-1})$ in terms of the variables l_j. Then $\delta F = \int \sum_j B_j \delta l_j$ and

$$B_j = \delta F / \delta l_j = \sum_p B_{j,p}^1 \cdot B_{j,p}^2,$$

where $B_{j,p}^1 \in \tilde{\mathscr{F}}_{\mathbb{C}}(l_0, \dots, l_{n-1})$ and $B_{j,p}^2 \in \tilde{\mathscr{R}}_{\mathbb{C}}(l_0, \dots, l_{n-1})$.
Define $\nabla F \in \mathfrak{u}\mathfrak{g}_- / \mathfrak{u}\mathfrak{g}_-^n$ from the relation $\delta F = \int \operatorname{res} \nabla F \delta L$. We can take $-i \sum_{j=0}^{n-1} (i\xi)^{-(j+1)} \delta F / \delta l_j$ as a representative of ∇F.

Proposition 3. a) *Any* $F \in \tilde{\mathscr{F}}_{\mathbb{R}}$ *has a variational derivative;*
b) $\tilde{\mathscr{F}}_{\mathbb{R}}$ *is closed under the operation* $\{ , \}$ *defined by*

$$\{F_1, F_2\} = \omega(V_{\nabla F_1}, V_{\nabla F_2}) = \int \operatorname{res}(-i(L(\nabla F_1 L)_+ - (L \nabla F_1)_+ L)\nabla F_2).$$

Remark. The ring of symbols $\tilde{\mathscr{R}}_{\mathbb{R}}(v_0, \dots, v_{n-1})((\xi^{-1}))$ has an unusual property: the formula $\operatorname{tr}[\sigma_1, \sigma_2] = 0$ is false. The true formula is

$$\operatorname{tr}[\sigma_1, \sigma_2] = \int d/dx \operatorname{res}(\sigma_{1,\xi} \sigma_2).$$

For example $\operatorname{tr}[T v_0 \xi, (T v_0)^2 \xi^{-1}] = -(1/(4\delta^3))(\int v_0)^3 \neq 0$. So the operation $\{ , \}$ is not even skew symmetric on $\tilde{\mathscr{F}}_{\mathbb{R}}(v_0, \dots, v_{n-1})$.

Consider the subspace $\mathscr{F}_{\mathbb{R}} \subset \tilde{\mathscr{F}}_{\mathbb{R}}(v_0, \ldots, v_{n-1})$ consisting of functionals satisfying the additional condition

$$\nabla F = X_1 + X_2 + Y;$$

$$X_1 = \sum_k X_{1k}(i\xi)^k, \qquad X_{1k} = \sum_j X^1_{1kj} \cdot X^2_{1kj},$$

where $X^1_{1kj} \in \tilde{\mathscr{F}}_{\mathbb{C}}(l_0, \ldots, l_{n-1})$, $X^2_{1kj} \in \tilde{\mathscr{R}}_{0,\mathbb{C}}(l_0, \ldots, l_{n-1})$, $X_2 \in \mathbb{C}((\xi^{-1}))$, $Y \in \mathfrak{ug}^n_-$.

Proposition 4. a) *The subspace $\mathscr{F}_{\mathbb{R}}$ is closed under the operation $\{,\}$.*
   b) *The operation $\{,\}$ defines a Lie algebra structure on $\mathscr{F}_{\mathbb{R}}$.*

6. The ILW-Type Equations are Hamiltonian

Rewrite (2) in the form:

$$L_t = -i(L(XL)_+ - (LX)_+ L). \tag{5}$$

If $M_0 = i \sum_{r=0}^m m_r(i\xi)^r$, $L_0 = i \sum_{k=0}^n c_k(i\xi)^k$, $c_n = 1$; $m_r, c_k \in \mathbb{R}$, then

$$X = [(1 + K^+)X_0(1 + K^-)^{-1}]_-,$$

where $X_0 = i \sum_{j=-n}^{m-n} \lambda_j(i\xi)^j$ and $\lambda_j \in \mathbb{R}$ are defined from the expansion: M_0/L_0

$= \sum_{j=-\infty}^{m-n} \lambda_j(i\xi)^j$.

Now, it is natural to expect that $[(1 + K^+)i(i\xi)^\alpha(1 + K^-)^{-1}]_-$ is the gradient of some functional $H_\alpha \in \mathscr{F}_{\mathbb{R}}$, i.e.

$$\delta H_\alpha = \int \mathrm{res}(1 + K^+)i(i\xi)^\alpha(1 + K^-)^{-1}\delta L. \tag{6}$$

Let us formulate some statements necessary for the proof of this fact.

Proposition 5. a) *If $\sigma_1, \sigma_2 \in \tilde{\mathscr{R}}(l_0, \ldots, l_{n-1})((\xi^{-1}))$, then*

$$\int d/dx\, \mathrm{res}(\sigma_1 \circ \sigma_2) = \int d/dx\, \mathrm{res}\,\sigma_1\sigma_2.$$

In the other words, the symbols are multiplied as series under the sign $\int d/dx\, \mathrm{res}$.
   b) *Let $\sigma_1, \sigma_2 \in \tilde{\mathscr{R}}(l_0, \ldots, l_{n-1})((\xi^{-1}))$; then*

$$\int \mathrm{res}[\sigma_1, \sigma_2] = \int d/dx\, \mathrm{res}(\sigma_{1,\xi}\sigma_2).$$

Set $V = -i(K^- - K^+)$.
   c) *Let K satisfy the condition of Proposition 2. Then*

$$T[(K^-)^r - (K^+)^r] = i[(K^-)^r + (K^+)^r] + c_r,$$

where

$$c_r = \begin{cases} 0, & \text{if} \quad r = 2k+1, \\ -i(\int V)^r/(2^{2r-1}\delta^r), & \text{if} \quad r = 2k. \end{cases}$$

d)

$$\int d/dx[(K^-)^r \delta K^-] = \begin{cases} 0, & \text{if } r=2k+1, \\ -1/(2^{2r+1}\delta^{r+1}) \cdot (\int V)^r (\int \delta V), & \text{if } r=2k. \end{cases}$$

A direct computation yields our main results.

Theorem 1. a) $H_\alpha = \text{res}\left[(iL_0 + L_{0,\xi}/(2\delta) i(i\xi)^\alpha) \right.$

$$\left. \sum_{k=0}^{\infty} (1/(4\delta))^{2k} (\int V)^{2k+1}/(2k+1) \right];$$

b) $\{H_\alpha, H_\beta\} = 0$.

(The proof can be found in [16].)

The formula for H_α is worth commenting on. The reality conditions for the symbol K show that $H_\alpha \in \tilde{\mathscr{F}}_\mathbb{R}(v_0, ..., v_{n-1})$. Since

$$VH_\alpha = [(1+K^+)i(i\xi)^\alpha (1+K^-)^{-1}]_-,$$

we have $H_\alpha \in \mathscr{F}_\mathbb{R}$ and in fact

$$H_\alpha = \int P_\alpha(v_0, ..., v_{n-1}), \quad \text{where} \quad P_\alpha \in \mathscr{R}_{0,\mathbb{R}}(v_0, ..., v_{n-1}).$$

The various terms in the formula for H_α play different roles. The term with $\int V$ contains the densities P_α, and the terms with $(\int V)^{2k+1}$, $k \geq 1$ cancel with the additional part of $\int V$ arising from the use of the formula (4).

So the ILW-type equations (5) are Hamiltonian with respect to the Gelfand-Dikii symplectic structure with Hamiltonia

$$H = \sum_{j=-n}^{m-n} \lambda_j H_j,$$

and have an infinite number of conservation laws in involution.

In the simplest case of the original ILW equation,

$$L_0 = -\xi + 1/(2i\delta), \quad L = L_0 - (K^-_{-1} - K^+_{-1}) = -\xi + 1/(2i\delta) - iu,$$

and

$$H_\alpha = \text{res}\left[-i(i\xi)^{\alpha+1} \sum_{k=0}^{\infty} (1/(4\delta))^{2k} (\int V)^{2k+1}/(2k+1) \right].$$

The first few Hamiltonians are:

$$H_{-1} = \int u, \quad H_0 = \int u^2/2, \quad H_1 = \int (u^3/3 + uTu_x/2 + u^2/(4\delta)),$$

$$H_2 = \int (u^4/4 + 3/4 \cdot u^2 Tu_x + 3/8 \cdot (Tu_x)^2 + (u_x)^2/8 + u^3/(3\delta)$$
$$+ uTu_x/(2\delta) + u^2/(8\delta^2)).$$

The Hamiltinian H for ILW is

$$-(H_1 + H_0/(2\delta)) = -\int (u^3/3 + uTu_x/2 + u^2/(2\delta)).$$

For the ILW equation, (5) has the form

$$u_t = \frac{d}{dx} \frac{\delta H}{\delta u}.$$

The involution statement means that

$$\int \frac{\delta H_\alpha}{\delta u} \frac{d}{dx} \frac{\delta H_\beta}{\delta u} = 0.$$

For BO and ILW equations this result is well known [17, 18].

The Hamiltonians H_α can be rewritten in a more convenient form.

Theorem 2.

$$H_\alpha = \int \mathrm{res}(iL_0 + L_{0,\,\xi}/(2\delta))(i\xi)^\alpha[\ln(1+K^-) - \ln(1+K^+)].$$

Here $(1+K^\pm)$ *is a series in a commutative parameter* ξ, *that is, in the above formula the terms are multiplied simply as formal series (not as symbols).*

Define

$$i\chi = \ln(1+K^-) - \ln(1+K^+).$$

Given the symbol L, we construct the differential operator $\hat{L} = \sum\limits_{\alpha \geqq 0} 1/\alpha! \, \partial_\xi^\alpha(L) \partial_x^\alpha$. (Here, too, ξ is a commutative parameter.)

Finally, for any differential operator $A = \sum\limits_{i=0}^N a_i \partial_x^i$, define the expression $A\{f\}$ by

$$A\{f\} = \sum_{i=0}^N a_i P_i(f),$$

where P_s is the differential polynomial defined by

$$P_s(f_x' f^{-1}) = f_x^{(s)} f^{-1}.$$

Now we can state our last observation.

Theorem 3. a) $\chi \in \mathscr{S}((\xi^{-1}))$;
     b) $\hat{L}\{(T\chi_x - i\chi_x)/2\} = L_0 e^{i\chi}$.

Corollary. *For* $L = -\xi + 1/(2i\delta) - iu$ *(the ILW case), we have*

$$\hat{L} = -\partial_x - \xi + 1/(2i\delta) - iu,$$

and the "Riccati" equation Theorem 3b) above takes the form:

$$-T\chi_x/2 + i\chi_x/2 - \xi + 1/(2i\delta) - iu = (-\xi + 1/(2i\delta))e^{i\chi}.$$

Remark 1. Expanding χ as a series in ξ^{-1}

$$\chi = \sum_{j=1}^\infty \chi_j \xi^{-j},$$

we can obtain recurrence formulas for χ_j from which we can calculate χ_j in terms of the coefficients of L.

Remark 2. Theorems 2 and 3 show that in the ILW case our formulas for the Hamiltonians agree with the results of [3].

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