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Graded Gauge Theory

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Abstract. The mathematical background for a graded extension of gauge theories is investigated. After discussing the general properties of graded Lie algebras and what may serve as a model for a graded Lie group, the graded fiber bundle is constructed. Its basis manifold is supposed to be the so-called superspace, i.e. the product of the Minkowskian space-time with the Grassmann algebra spanned by the anticommuting Lorentz spinors; the vertical subspaces tangent to the fibers are isomorphic with the graded extension of the SU(N) Lie algebra. The connection and curvature are defined then on this bundle; the two different gradings are either independent of each other, or may be unified in one common grading, which is equivalent to the choice of the spin-statistics dependence. The Yang-Mills lagrangian is investigated in the simplified case. The conformal symmetry breaking is discussed, as well as some other physical consequences of the model.

1. Construction of a Graded Lie Algebra Associated with a Lie Group G

Let G be a Lie group of dimension N; in what follows, it will be supposed compact and semi-simple, unless explicitly stated otherwise. Let $\mathscr{A}_G$ denote its Lie algebra; for $X, Y \in \mathscr{A}_G$ their skew product is $[X, Y]$ and satisfies

$$[X, Y] = -[Y, X], \tag{1.1}$$

and the Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0. \tag{1.2}$$

The adjoint representation of $\mathscr{A}_G$ is defined as the mapping

$$\mathrm{ad} : \mathscr{A}_G \to L(\mathscr{A}_G, \mathscr{A}_G), \tag{1.3}$$

such that

$$\mathrm{ad}(X)\, Y = [X, Y]; \tag{1.4}$$

therefore

$$\mathrm{ad}(X)\,\mathrm{ad}(Y) - \mathrm{ad}(Y)\,\mathrm{ad}(X) = \mathrm{ad}([X, Y]). \tag{1.5}$$

In local coordinates $\mathrm{ad}(X)$ takes on the form of a $(N \times N)$-matrix $X^a C^b_{ac}$, C^b_{ac} being the structure constants of G.

The Cartan-Killing metric form in $\mathscr{A}_G$ is defined by

$$g_G(X, Y) = g_G(Y, X) = \mathrm{Tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y)). \tag{1.6}$$

For G compact and semi-simple, g_G is known to be negative-definite, non-degenerate, and satisfies the invariance condition

$$g_G([Z, X], Y) + g_G(X, [Z, Y]) = 0. \tag{1.7}$$

Consider now a faithful representation of $\mathscr{A}_G$ in a linear vector space E of dimension s (the lower bound on s will be discussed later). This representation, denoted by

$$\tau : \mathscr{A}_G \to L(E, E) \tag{1.8}$$

satisfies

$$\tau(X)\,\tau(Y) - \tau(Y)\,\tau(X) = \tau([X, Y]). \tag{1.9}$$

Let us introduce now the following mapping ϱ:

$$\varrho : E \times E \to \mathscr{A}_G, \tag{1.10}$$

satisfying

$$\forall_{u, v \in E}\ \varrho(u, v) = \varrho(v, u)\ ; \tag{1.11}$$

also

$$\mathrm{ad}(X)\,\varrho(u, v) = [X, \varrho(u, v)] = \varrho(\tau(X)u, v) + \varrho(u, \tau(X)v), \tag{1.12}$$

which can be interpreted as the formula for the derivation of $\varrho(u, v)$ through the derivation of its arguments; and finally

$$\forall_{u, v, w \in E}\ \tau(\varrho(u, v))w + \tau(\varrho(v, w))u + \tau(\varrho(w, u))v = 0. \tag{1.13}$$

The four identities (1.2), (1.9), (1.12), and (1.13) can be considered as a Z_2-graded Jacobi identity in $\mathscr{A}_G \oplus E$, which therefore acquires the properties of a Z_2-graded Lie algebra. It is enough to define the generalized product in $\mathscr{A}_G \oplus E$ as follows:

$$\{X, Y\} = [X, Y] \in \mathscr{A}_G, \tag{1.14a}$$

$$\{X, u\} = -\{u, X\} = \tau(X)u \in E, \tag{1.14b}$$

$$\{u, v\} = \varrho(u, v) \in \mathscr{A}_G, \tag{1.14c}$$

for any $X, Y \in \mathscr{A}_G$, $u, v \in E$.

Any two elements from $\mathscr{A}_G$ combine now to give an element of $\mathscr{A}_G$, any two elements of E combine to give an element of $\mathscr{A}_G$, whereas two elements from $\mathscr{A}_G$ and E combine to give an element of E. We may call $\mathscr{A}_G$ and E, respectively, the "even" and "odd" parts of $\mathscr{A}_G \oplus E$. If by $\mathscr{P}, \mathscr{R}, \mathscr{S}$ we denote elements of $\mathscr{A}_G \oplus E$, then the generalized Jacobi identity of our Z_2-graded Lie algebra can be written down simply as

$$\{\mathscr{P},\{\mathscr{R},\mathscr{S}\}\}+\{\mathscr{R},\{\mathscr{S},\mathscr{P}\}\}+\{\mathscr{S},\{\mathscr{P},\mathscr{R}\}\}=0. \tag{1.15}$$

The notation generalizing the Lie derivation of (yet hypothetical) corresponding vector fields may be sometimes useful, too; we shall write

$$\underset{X}{\pounds} Y=-\underset{Y}{\pounds} X=[X,Y], \tag{1.16a}$$

$$\underset{X}{\pounds} u=-\underset{u}{\pounds} X=\tau(X)u, \tag{1.16b}$$

$$\underset{u}{\pounds} v=\underset{v}{\pounds} u=\varrho(u,v), \qquad \forall_{X,Y\in\mathscr{A}_G,u,v\in E}. \tag{1.16c}$$

Let $\pi(\mathscr{P})$ denote the Grassmann parity of $\mathscr{P}$, i.e. $\pi(\mathscr{P})=0$ if $\mathscr{P}\in\mathscr{A}_G$, and $\pi(\mathscr{P})=1$ if $\mathscr{P}\in E$.

In the next paragraph we shall construct generalized differential operators which realize abstract relations (1.16) and satisfy

$$\underset{\mathscr{P}}{\pounds}\underset{\mathscr{R}}{\pounds}-(-1)^{\pi(\mathscr{P})\pi(\mathscr{R})}\underset{\mathscr{R}}{\pounds}\underset{\mathscr{P}}{\pounds}=\underset{\{\mathscr{P},\mathscr{R}\}}{\pounds}. \tag{1.17}$$

After generalizing the Jacobi identity for $\mathscr{A}_G\oplus E$, we proceed to generalize the definition (1.6) of the invariant Cartan-Killing metric. The mapping ϱ together with g_G enables us to define the following mapping ε from $E\times E$ onto $\mathbb{R}^1$: let

$$\forall_{u,v\in E,X\in\mathscr{A}_G},\ \varepsilon(\tau(X)u,v)=g_G(X,\varrho(u,v)). \tag{1.18}$$

Thus defined, ε will be non-degenerate because g_G was non-degenerate and τ faithful.

Lemma. *If we define*

$$\underset{X}{\pounds}(\varepsilon(u,v))=\varepsilon(\tau(X)u,v)+\varepsilon(u,\tau(X)v), \tag{1.19}$$

then

$$\underset{X}{\pounds}(\varepsilon(u,v))=0 \quad \textit{for any} \quad X,u,v \quad \textit{implies} \quad \varepsilon(u,v)=-\varepsilon(v,u), \tag{1.20}$$

and vice versa.

Proof. $\underset{X}{\pounds}\varepsilon(u,v)=0$ means $\varepsilon(\tau(X)u,v)=-\varepsilon(u,\tau(X)v)$; but $\varepsilon(\tau(X)u,v)=g_G(X,\varrho(u,v))$ $=g_G(X,\varrho(v,u))$ because ϱ is symmetric; therefore $\varepsilon(\tau(X)u,v)=\varepsilon(\tau(X)v,u)$, whence $\varepsilon(\tau(X)v,u)=-\varepsilon(u,\tau(X)v)$; the antisymmetry of ε follows because X was arbitrary. The inverse is obvious, too. The formula (1.19) generalizes the invariance property of g_G given by (1.7). Let us also notice that instead of defining ϱ first and the antisymmetric form ε by means of g_G, ϱ, and τ, we can start by defining an invariant antisymmetric form ε and then define the mapping ϱ with the aforementioned properties by means of ε, g_G, and τ.

The next obvious step consists in generalizing the notion of the adjoint representation. Just as ad mapped $\mathscr{A}_G$ into $L(\mathscr{A}_G, \mathscr{A}_G)$, its graded extension, which we denote by $\mathcal{ad}$, will map

$$\mathcal{ad}: \mathscr{A}_G \oplus E \to L(\mathscr{A}_G \oplus E, \mathscr{A}_G \oplus E). \tag{1.21}$$

It is enough to define the action of $\mathcal{ad}(\mathscr{P})$ on any element $\mathscr{R} \in \mathscr{A}_G \oplus E$; more explicitly, as $\mathscr{P}$ and $\mathscr{R}$ may both denote either $X, Y \in \mathscr{A}_G$ or $u, v \in E$, we ask the following relations to be satisfied:

$$\mathcal{ad}(\mathscr{P})\mathscr{R} = \{\mathscr{P}, \mathscr{R}\}, \tag{1.22}$$

and

$$\mathcal{ad}(\mathscr{P})\,\mathcal{ad}(\mathscr{R}) - (-1)^{\pi(\mathscr{P})\pi(\mathscr{R})}\,\mathcal{ad}(\mathscr{R})\,\mathcal{ad}(\mathscr{P}) = \mathcal{ad}(\{\mathscr{P}, \mathscr{R}\}). \tag{1.23}$$

The formula (1.22) can be written in a more explicit form,

$$\mathcal{ad}(X)\,Y = [X, Y] = -\mathcal{ad}(Y)X, \tag{1.24a}$$

$$\mathcal{ad}(X)u = -\mathcal{ad}(u)X = \tau(X)u, \tag{1.24b}$$

$$\mathcal{ad}(u)v = \mathcal{ad}(v)u = \varrho(u, v). \tag{1.24c}$$

We see that $\mathcal{ad}(X)$ are even operators which map the even and odd components of $\mathscr{A}_G \oplus E$ onto themselves:

$$\mathcal{ad}(X): \mathscr{A}_G \to \mathscr{A}_G, \quad E \to E, \tag{1.25}$$

whereas $\mathcal{ad}(u)$ are odd operators, mapping $\mathscr{A}_G$ into E and E into $\mathscr{A}_G$:

$$\mathcal{ad}(u): \mathscr{A}_G \to E, \quad E \to \mathscr{A}_G. \tag{1.26}$$

We define now the "super-trace" for the $(N \times s) \times (N \times s)$ matrices $\mathcal{ad}(\mathscr{P})$ as the mapping Str of these matrices onto $\mathbb{R}^1$ which satisfies the following properties:

$$\mathrm{Str}(\mathcal{ad}(\mathscr{P}) + \mathcal{ad}(\mathscr{R})) = \mathrm{Str}(\mathcal{ad}(\mathscr{P})) + \mathrm{Str}(\mathcal{ad}(\mathscr{R}), \tag{1.27a}$$

$$\mathrm{Str}(\mathcal{ad}(\mathscr{P})\,\mathcal{ad}(\mathscr{R})) = (-1)^{\pi(\mathscr{P})\pi(\mathscr{R})}\,\mathrm{Str}(\mathcal{ad}(\mathscr{R})\,\mathcal{ad}(\mathscr{P})). \tag{1.27b}$$

The generalization of the definition of the invariant Cartan-Killing metric in $\mathscr{A}_G$ (1.6) extending it onto $\mathscr{A}_G \oplus E$ is now obvious. First, the even and odd subspaces of $\mathscr{A}_G \oplus E$ should be orthogonal to each other: if $\mathscr{P} = X \in \mathscr{A}_G$, $\mathscr{R} = u \in E$, then

$$\mathrm{Str}(\mathcal{ad}(\mathscr{P})\,\mathcal{ad}(\mathscr{R})) = \mathrm{Str}(\mathcal{ad}(X)\,\mathcal{ad}(u)) = 0. \tag{1.28}$$

This is obvious because of the "even" and "odd" properties of the corresponding matrices $\mathcal{ad}(X)$ and $\mathcal{ad}(u)$; as a matter of fact, the matrix $\mathcal{ad}(X)\,\mathcal{ad}(u)$ is off-diagonal, whence the result.

Next, we postulate

$$\mathrm{Str}(\mathcal{ad}(X)\,\mathcal{ad}(Y)) = \lambda g_G(X, Y), \tag{1.29a}$$

and

$$\mathrm{Str}(\mathcal{ad}(u)\,\mathcal{ad}(v)) = \mu\varepsilon(u, v). \tag{1.29b}$$

The normalizing constants λ and μ depend, of course, on the representation τ chosen (i.e. the dimension s of E).

Let us fix our representation by asking that the following independent identity is satisfied:

$$g_G(\varrho(u,v),\varrho(w,z))=\varepsilon(u,w)\,\varepsilon(v,z)+\varepsilon(u,z)\,\varepsilon(v,w)\,. \tag{1.30}$$

A simple calculus shows then that (up to a common multiplicative constant) we must have

$$\lambda=\frac{2s(s+1)}{N}, \qquad \mu=2(s+1)\,. \tag{1.31}$$

This condition fixes the representation τ up to an equivalence (automorphisms of E and $\mathscr{A}_G$); it is known that such a representation has the dimension given by

$$s=2^{\left[\frac{N}{2}\right]}, \tag{1.32}$$

$\left[\frac{N}{2}\right]$ being the integer part of $\frac{N}{2}$, and is called the spinor representation.

Summarizing, we may observe that the symplectic structure ε on E, invariant with respect to the action of the representation τ of $\mathscr{A}_G$, together with the definition of τ are enough to define canonically the Z_2-graded extension $\mathscr{A}_G\oplus E$ of a compact, semi-simple Lie algebra $\mathscr{A}_G$ we started with.

2. Representation of $\mathscr{A}_G\oplus E$ in Graded Differential Operators

The Lie algebra $\mathscr{A}_G$ could be identified with the set of left-invariant vector fields defined globally on G, and generated by the right action of G on itself; the Lie brackets of these N independent fields satisfied the commutation relations of $\mathscr{A}_G$. These vectors fields were also interpreted as invariant differential operators acting on the module of smooth real functions on G.

We would like to extend the analogy of the first paragraph and define some graded manifold including G as its even component, and then define some analogs of the invariant differential operators acting on functions over this graded manifold in such a way that their generalized Lie brackets, formally defined in (1.16), satisfy these commutation-anticommutation relations.

In order to do it, let us first introduce the exterior antisymmetric product in E,

$$\forall_{u,v\in E},\ uv=-vu\,. \tag{2.1}$$

Now E acquires the properties of a Grassmann algebra; let us denote by ΛE the exterior algebra of E, i.e. the linear space spanned by all independent formal powers of elements from E. If $\dim E=s$, then $\dim \Lambda E=2^s$.

We can exponentiate the action of $\mathscr{A}_G$ on E, thus obtaining a corresponding representation of G in $L(E,E)$, namely, for any $X\in\mathscr{A}_G$ if $\exp tX=g_t\in G$, then we put

$$\operatorname{Exp}\tau(tX)=\Delta(g_t)\in L(E,E)\,, \tag{2.2}$$

where $\operatorname{Exp}\tau(tX)$ means the usual exponential of an $s\times s$ matrix $\tau(tX)$. Obviously,

$$\Delta(g)\,\Delta(h)=\Delta(gh)\,, \tag{2.3a}$$

and

$$\Delta(g^{-1})=[\Delta(g)]^{-1}. \tag{2.3b}$$

Consider now the semi-direct product of G and E, denoted by $G\square E$, with the following composition law:

$$(h_1,u_1)(h_2,u_2)=(h_1h_2,u_1+\Delta(h_1)u_2), \qquad \forall h_1,h_2\in G, u_1,u_2\in E. \tag{2.4}$$

This law is obviously associative, the neutral element is $(e,0)$, e being the unit element of G, $0\in E$; the inverse of (h,u) is given by

$$(h,u)^{-1}=(h^{-1}, -\Delta(h^{-1})u) ; \tag{2.5}$$

$G\square E$ acquires the structure of a Lie group, E being its abelian subgroup, $\dim(G\square E)=N+s$.

We face the following problem now: in order to define differential operators (vector fields) we have to define first what we mean by module of functions upon which these operators shall act; once we imagine functions on $G\square E$ replacing functions on G, we are led to the whole ΛE, i.e. all possible exterior products of elements from E and finite polynomials of order $\leqq s$. So we have to extend $G\square E$ to a structure containing G and ΛE. The group G acts on ΛE by extension of the action on E defined by (2.2): for $g\in G$, $u,v\in E$, we put

$$\Delta(g)(uv)=(\Delta(g)u)(\Delta(g)v), \tag{2.6}$$

and then, by recurrence, for any two elements U, $W\in\Lambda E$, we define

$$\Delta(g)(UW)=(\Delta(g)U)(\Delta(g)W). \tag{2.7}$$

Consider now the semi-direct product $G\square\Lambda E$ with the following composition law:

$$\forall g_1,g_2\in G, U_1,U_2\in\Lambda E, (h_1,U_1)(h_2,U_2)=(h_1h_2,U_1(\Delta(h_1)U_2)). \tag{2.8}$$

This composition law is obviously associative, but there exists neither inverse, nor the group structure in $G\square\Lambda E$.

Consider the set of ΛE-valued functions on G; it is obviously a module (we can add them up together, and multiply them one by another). Our first set of vector fields can be induced by the action of the group $G\square E$ on the module of these "functions."

In order to define the action of $G\square E$ on $G\square\Lambda E$ it is enough to define it on simple elements of $G\square\Lambda E$, i.e. on the elements of the form:

$$(g,u_1u_2\ldots u_p), g\in G, u_1,u_2,\ldots,u_p\in E. \tag{2.9}$$

The group $G\square E$ acts on these elements from the right as follows:

$$\begin{aligned}R_{(h,v)}(g,u_1u_2\ldots u_p)&=(g,u_1u_2\ldots u_p)(h,v)\\&=(gh,(u_1+\Delta(g)v)(u_2+\Delta(g)v)\ldots(u_p+\Delta(g)v)).\end{aligned} \tag{2.10}$$

Obviously

$$((g,u_1u_2\ldots u_p)(h_1,v_1))(h_2,v_2)=(g,u_1u_2\ldots u_p)(h_1h_2,v_1+\Delta(h_1v_2). \tag{2.11}$$

This action is naturally extended onto any elements of $G\square\Lambda E$; the representation of $G\square E$ is obtained if we define

$$D(h,v)(g,W)=R_{(h,v)^{-1}}(g,W), \tag{2.12}$$

$(h,v)\in G\square E$, $(g,W)\in G\square\Lambda E$. Moreover, it has the covariance property with respect to the associative multiplication in $G\square\Lambda E$:

$$R_{(h,v)}((g_1,U_1)(g_2,U_2))=(R_{(h,v)}(g_1,U_1))(R_{(h,v)}(g_2,U_2)). \tag{2.13}$$

Now, if f is a function on $G\square E$, it can be represented as a linear combination

$$f=f_0(g)+f_1(g)\overset{(1)}{u}+f_2(g)\overset{(2)}{U}+\ldots+f_{2^s}(g)\overset{(2^s)}{U}=f(g,U), \tag{2.14}$$

where f_α are real smooth functions on G, $\alpha=0,1,\ldots,2^s$ and U symbolizes the element of ΛE given by $\overset{(1)}{u}\oplus\overset{(2)}{U}\oplus\ldots\oplus\overset{(2^s)}{U}$; $\overset{(\alpha)}{U}$ are the elements of $\Lambda^\alpha E$. These "functions," defined as in (2.14), form a module; we can add them up (by adding up the corresponding terms in the expansion), and multiply them, obtaining an entity of the same type. The group $G\square E$ acts in a natural way on this module: via the operator

$$(\mathcal{O}_{(h,u)}f)(g,U)=f(R_{(h,u)^{-1}}(g,U)). \tag{2.15}$$

We have now everything that is needed in order to define generalized left-invariant fields generated by $G\square E$ acting on our module. It is enough to consider $N+s$ different independent one-parameter subgroups of $G\square E$, N "even" ones belonging to G, and s "odd" ones generated by the elements of E. If we calculate

$$\Big(\underset{X}{\pounds}f\Big)(g,U)=\lim_{t\to 0}\frac{(\mathcal{O}_{(h_t,0)}f-f)(g,U)}{t}\left(X=\left.\frac{dh_t}{dt}\right|_{t=0}\right), \tag{2.16a}$$

and

$$\Big(\underset{u}{\pounds}f\Big)(g,U)=\lim_{t\to 0}\frac{(\mathcal{O}_{(e,tu)}f-f)(g,U)}{t}, \tag{2.16b}$$

we don't get a representation of (1.14) like that postulated in (1.16), but

$$\left.\begin{aligned}
&\underset{X}{\pounds}\,\underset{Y}{\pounds}-\underset{Y}{\pounds}\,\underset{X}{\pounds}=\underset{[X,Y]}{\pounds}\\
&\underset{X}{\pounds}\,\underset{u}{\pounds}-\underset{u}{\pounds}\,\underset{X}{\pounds}=\underset{\tau(X)u}{\pounds}\\
&\underset{u}{\pounds}\,\underset{v}{\pounds}+\underset{v}{\pounds}\,\underset{u}{\pounds}=2\underset{u+v}{\pounds}\,.
\end{aligned}\right\} \tag{2.17}$$

It is no wonder that we cannot obtain in this way the correct result (1.16); we have shown in the first paragraph that the graded structure was induced by the invariant symplectic structure ε on E, whereas in all our construction of "functions" and derivation no such structure has been used. On the other hand, such a symplectic structure is implicitly involved in the definition of *real* scalar functions on $G\square\Lambda E$.

Let us implement E (and ΛE) with the invariant symplectic structure given by $\varepsilon: E \times E \to \mathbb{R}^1$, satisfying the same axioms as in the first paragraph, i.e.

$$\left.\begin{aligned} &\varepsilon(u,v) = -\varepsilon(v,u)\,, \\ &\varepsilon \text{ nondegenerate}\,, \\ &\varepsilon(\tau(X)u,v) + \varepsilon(u,\tau(X)v) = 0\,, \end{aligned}\right\} \tag{2.18}$$

for any u, v, X. Also ε defines the canonical isomorphism between E and its dual E^*, and by obvious extension, between ΛE and ΛE^*; just as g_G could be interpreted as the canonical isomorphism between $\mathscr{A}_G$ and its dual $\mathscr{A}_G^*$; g_G^{-1} and ε^{-1} denote the corresponding inverse mappings. Now the form ϱ can be written symbolically as

$$\varrho = g_G^{-1} \circ \tau \circ \varepsilon\,, \tag{2.19}$$

which is equivalent with the definition by (1.18).

Let

$$\overset{(1)}{u}{}^* \in E^*,\ \overset{(2)}{U}{}^* \in \Lambda^2 E^*, \ldots,\ \overset{(2^s)}{U}{}^* \in \Lambda^{2^s} E^*\,. \tag{2.20}$$

We define a *real function* on $G \square \Lambda E$ as the linear combination

$$f = f_0 + f_1 \overset{(1)}{u}{}^* + f_2 \overset{(2)}{U}{}^* + \ldots + f_{2^s} \overset{(2^s)}{U}{}^*\,, \tag{2.21}$$

$f_0, f_1, \ldots, f_{2^s}$ being smooth functions on G. Now, for any $(g, U) \in G \square \Lambda E$, $g \in G$, $U = \overset{(1)}{u} \oplus \overset{(2)}{U} \oplus \ldots \oplus \overset{(2^s)}{U}$, the value of f at the point (g, U) is defined as

$$\begin{aligned} f(g,U) = f_0(g) + f_1(g) \overset{(1)}{u}{}^* \lrcorner \overset{(1)}{u} + f_2(g) \overset{(2)}{U}{}^* \lrcorner \overset{(2)}{U} + \ldots \\ \ldots + f_{2^s}(g) \overset{(2^s)}{U}{}^* \lrcorner \overset{(2^s)}{U}\,. \end{aligned} \tag{2.22}$$

These are the *real functions* on $G \square \Lambda E$; unfortunately, if we multiply them taking the product of their numerical values at the same point (g, U), the result is no longer a function of this type; i.e. the set of the functions is not a module. If we multiply just the expressions (2.21), there is no essential difference between these functions and the "functions" defined by (2.14); because of the duality between E and E^* they carry the same information. If we define the finite action of $G \square E$ on real functions of the type (2.22) as

$$(\mathcal{O}_{(h,v)} f)(g,U) = f(R_{(h,v)^{-1}}(g,U))\,, \tag{2.23}$$

the infinitesimal limit does not have properties of derivation with respect to the point-by-point multiplication (no analog of the Leibniz formula is possible). These difficulties are typical and amount to the impossibility of a correct definition of graded exponentiation. Let us therefore content ourselves with an explicit definition of the infinitesimal generators (differentiations) without being able to integrate them as in the classical case.

Let χ^A be the basis of E; $A, B = 1, 2, \ldots, s$;

$$\chi^A \chi^B + \chi^B \chi^A = 0\,. \tag{2.24}$$

Let L_b be a basis in $\mathscr{A}_G$, $a,b=1,2,\ldots,N=\dim G$,

$$[L_a, L_b]=C^d_{ab}L_d\,, \tag{2.25}$$

C^d_{ab} being the structure constants of $\mathscr{A}_G$. The left-invariant vector fields on G are given in local coordinates by the differential operators $S_a=S^b_a\partial_b$ such that:

$$S^b_a\partial_b S^f_d - S^b_d\partial_b S^f_a = C^b_{ad}S^f_b\,. \tag{2.26}$$

The representation τ shall be in these coordinates $\tau_a{}^A{}_B$, satisfying

$$\tau_a{}^A{}_B\tau_b{}^B{}_D - \tau_b{}^A{}_B\tau_a{}^B{}_D = C^d_{ab}\tau_d{}^A{}_D\,, \tag{2.27}$$

and the form ϱ has the components ϱ^a_{AB}.

Let us introduce the (graded) derivation with respect to the G-spinors χ^A as follows: it is linear, and

$$\partial_A\chi^B=\delta^B_A\,, \tag{2.28a}$$

$$\partial_A(\chi^B\chi^D)=\delta^B_A\chi^D-\delta^D_A\chi^B\,. \tag{2.28b}$$

Define the following operators acting on the ΛE-valued functions over G:

$$\left.\begin{aligned}\mathscr{D}_a &= S^b_a\partial_b+\tau_a{}^B{}_D\chi^D\partial_B\,, \qquad (2.29\text{a})\\ \mathscr{D}_B &= \partial_B+\varrho^a_{BD}\chi^D S^b_a\partial_b\,, \qquad (2.29\text{b})\end{aligned}\right\}$$

with

$$\varrho^a_{BD}=g^{ab}\varepsilon_{AB}\tau_b{}^A{}_D\,, \tag{2.30}$$

where g^{ab}, ε_{AB} are components of g_G and ε in our coordinates. It is easy to check that the operators defined by (2.29) satisfy

$$\left.\begin{aligned}[]
[\mathscr{D}_a, \mathscr{D}_b] &= C^d_{ab}\mathscr{D}_d\,,\\
[\mathscr{D}_a, \mathscr{D}_B] &= -\tau_a{}^D{}_B\mathscr{D}_D = C_a{}^D{}_B\mathscr{D}_D = -C_B{}^D{}_a\mathscr{D}_D\,,\\
\{\mathscr{D}_A, \mathscr{D}_B\}_+ &= 2\varrho^d_{AB}\mathscr{D}_d = C^d_{AB}\mathscr{D}_d = C^d_{BA}\mathscr{D}_d\,.
\end{aligned}\right\} \tag{2.31}$$

It is easy to check that these operators span the Z_2-graded Lie algebra defined by (1.14) and (1.16); moreover, if we define the $(N+s)\times(N+s)$ matrices

$$C_a=\left(\begin{array}{c|c} C_a{}^b{}_d & 0 \\ \hline 0 & C_a{}^B{}_D\end{array}\right), \qquad C_A=\left(\begin{array}{c|c} 0 & C_A{}^a{}_B \\ \hline C_A{}^B{}_d & 0\end{array}\right) \tag{2.32}$$

with the structure constants defined as in (2.31), then we obtain the adjoint representation of our Z_2-graded algebra:

$$\begin{aligned} C_aC_b-C_bC_a &= C^d_{ad}C_d\,,\\ C_AC_B-C_BC_a &= C^D_{aB}C_D\,,\\ C_AC_B+C_BC_A &= C^d_{AB}C_D\,.\end{aligned} \tag{2.33}$$

We shall also use the generalized indices ψ, ϕ standing for a or for B; introducing Grassmann parity $\pi(\phi)$ as $\pi(a)=0$, $\pi(B)=1$, we can rewrite (2.33) as the graded

Jacobi identity

$$C^{\chi}_{\phi\Omega}C^{\Omega}_{\psi\Delta}-(-1)^{\pi(\phi)\pi(\psi)}C^{\chi}_{\psi\Omega}C^{\Omega}_{\phi\Delta}=C^{\Omega}_{\phi\psi}C^{\chi}_{\Omega\Delta}\,. \tag{2.34}$$

The fact that C_a's are even operators and C_B's are odd is visualized in their matrix form (2.32). Finally, the normalization relations in coordinates are

$$g_{ab}\varrho^{a}_{BD}\varrho^{b}_{EF}=\varepsilon_{BE}\varepsilon_{DF}+\varepsilon_{BF}\varepsilon_{DE}\,.$$

Summarizing we may say that we have come as close as we could to the notion of a graded Lie group. The essential difference with the ordinary Lie group G is the fact that whereas for G the "group" and the "group manifold" on which it acted as a group of transformations were identical; in the case with grading, $G\Box E$ has the structure of a graded Lie group, whereas the "manifold" $G\Box\Lambda E$ has not, and the exponentiation is not well defined. Nevertheless, the graded vector fields (2.29), the graded adjoint representation (2.32) and the module of ΛE-valued functions on G are sufficient to define the graded analog of gauge theory.

3. Graded Fiber Bundles, Graded Connections

The impossibility of definition of the graded Lie group makes somewhat difficult a definition of a principal fiber bundle; for our purposes, however, it will be enough to define a product space of $G\Box\Lambda E$ with some basis manifold; the graded vector fields defined by (2.31) act on ΛE-valued functions of G and of the basis manifold (leaving the parameters of the basis manifold unchanged) and are the analogs of the vertical vector fields in a principal fiber bundle. We shall assume that our bundles are globally trivial; now we draw our attention to the generalized graded bundles in which the basis space is also a Z_2-graded manifold. This basis space will be assumed in its simplest well-known version, [1, 2], i.e. the product of the Minkowskian space-time M_4 with the linear space of anticommuting Majorana spinors

$$\theta^{\alpha}\theta^{\beta}+\theta^{\beta}\theta^{\alpha}=0\,,\qquad \theta^{\alpha}\bar{\theta}^{\dot\beta}+\bar{\theta}^{\dot\beta}\theta^{\alpha}=0\,,\qquad \bar{\theta}^{\dot\alpha}\bar{\theta}^{\dot\beta}+\bar{\theta}^{\dot\beta}\bar{\theta}^{\dot\alpha}=0\,, \tag{3.1}$$

$\alpha,\beta=1,2$; $\dot\alpha,\dot\beta=\dot1,\dot2$; we denote symbolically this superspace by $M_4\times\{\theta\}$. The corresponding graded manifold is $M_4\times\Lambda\{\theta\}$, where $\Lambda\{\theta\}$ denotes the Grassmann algebra of $\{\theta\}$. We shall call "functions" on $M_4\times\{\theta\}$ the $\Lambda\{\theta\}$-valued functions on M_4; any such "function" is decomposed as

$$\begin{aligned}\phi(x,\theta)=\phi_0(x)+\phi_\alpha(x)\theta^\alpha+\phi_\beta(x)\bar\theta^{\dot\beta}+\phi_{\alpha\beta}(x)\theta^\alpha\theta^\beta+\dots\\ \dots+\phi_{\alpha\beta\dot\gamma\dot\delta}(x)\theta^\alpha\theta^\beta\bar\theta^{\dot\gamma}\bar\theta^{\dot\delta}\,,\end{aligned} \tag{3.2}$$

all coefficients ϕ being smooth functions of $x\in M_4$.

Defining the (graded) derivation with respect to the spinor variables θ as follows:

$$\partial_\alpha\theta^\beta=\delta^\beta_\alpha,\ \partial_\alpha\bar\theta^{\dot\beta}=0,\ \partial_{\dot\alpha}\bar\theta^{\dot\beta}=\delta^{\dot\beta}_{\dot\alpha},\ \partial_{\dot\alpha}\theta^\beta=0\,, \tag{3.3a}$$

with the (anti)-Leibniz rule

$$\partial_\alpha(\theta^\beta\theta^\delta)=\delta^\beta_\alpha\theta^\delta-\delta^\delta_\alpha\theta^\beta\,, \tag{3.3b}$$

so that $\partial_\alpha\partial_\beta+\partial_\beta\partial_\alpha=0$, etc.

We can define the graded extension of the Poincaré algebra as follows:

$$\begin{aligned} &P_j=\partial_j,\ J_{kl}=x_kP_l-x_lP_k+\sigma_{kl}{}^{\alpha}{}_{\dot\beta}\,\bar\theta^{\dot\beta}\partial_\alpha+\sigma_{kl\alpha}{}^{\dot\beta}\,\theta^\alpha\partial_{\dot\beta}\,,\\ &\mathscr{D}_\alpha=\partial_\alpha+\sigma^j{}_{\alpha\dot\beta}\bar\theta^{\dot\beta}P_j,\ \bar{\mathscr{D}}_{\dot\beta}=\partial_{\dot\beta}+\sigma^j{}_{\alpha\dot\beta}\theta^\alpha P_j\,. \end{aligned} \tag{3.4}$$

These generators satisfy the following commutation-anticommutation relations:

$$\begin{aligned} &[P_j, P_m]=0\,,\\ &[J^{kl}, P_m]=\delta^k_m P^l-\delta^l_m P^k\,,\\ &[J^{kl}, J^{mn}]=g^{km}J^{ln}+g^{ln}J^{km}-g^{lm}J^{kn}-g^{kn}J^{lm}\,,\\ &[J^{mn}, \mathscr{D}_\alpha]=\sigma^{mn}{}_\alpha{}^{\dot\beta}\,\bar{\mathscr{D}}_{\dot\beta}\,,\\ &[J^{mn}, \bar{\mathscr{D}}_{\dot\beta}]=\sigma^{mn\alpha}{}_{\dot\beta}\,\mathscr{D}_\alpha\,,\\ &[P_k, \mathscr{D}_\alpha]=0,\ [P_k, \bar{\mathscr{D}}_{\dot\beta}]=0\,,\\ &\{\mathscr{D}_\alpha, \mathscr{D}_\beta\}_+=0,\ \{\bar{\mathscr{D}}_{\dot\alpha}, \bar{\mathscr{D}}_{\dot\beta}\}_+=0\,,\\ &\{\mathscr{D}_\alpha, \bar{\mathscr{D}}_{\dot\beta}\}_+=2\sigma^j{}_{\alpha\dot\beta}\,P_j\,. \end{aligned} \tag{3.5}$$

Here

$$\begin{aligned} &\sigma^{j\alpha}{}_{\dot\beta}=-(\gamma^j)^\alpha{}_{\dot\beta},\ (\gamma^j)^\alpha{}_\beta=0\,,\\ &\sigma^j{}_\beta{}^{\dot\alpha}=(\gamma^j)^{\dot\alpha}{}_\beta,\ (\gamma^j)^{\dot\alpha}{}_{\dot\beta}=0\,; \end{aligned} \tag{3.6}$$

γ^k are the standard Dirac matrices, and

$$\sigma^{kl}=\tfrac{1}{8}(\gamma^k\gamma^l-\gamma^l\gamma^k)\,. \tag{3.7}$$

The indices α, $\dot\beta$ are raised and lowered by means of the invariant spinorial "metric" $\varepsilon_{\alpha\beta}$, $\varepsilon_{\dot\alpha\dot\beta}$ and its inverse $\varepsilon^{\alpha\beta}$, $\varepsilon^{\dot\alpha\dot\beta}$; $\varepsilon_{12}=-\varepsilon_{21}=1$, $\varepsilon_{12}=-\varepsilon_{21}=1$, so that

$$\partial_\alpha\theta_\beta=\varepsilon_{\alpha\beta},\ \partial_{\dot\alpha}\bar\theta_{\dot\beta}=\varepsilon_{\dot\alpha\dot\beta}\,. \tag{3.8}$$

The exterior calculus is easily generalized on $M\times\Lambda\{\theta\}$ (cf. [3, 4]); we introduce the exterior 1-forms $d\theta^\alpha$ and $d\bar\theta^{\dot\beta}$ such that

$$d\theta^\alpha(\partial_\beta)=\delta^\alpha_\beta,\ d\bar\theta^{\dot\alpha}(\partial_{\dot\beta})=\delta^{\dot\alpha}_{\dot\beta}\,, \tag{3.9}$$

and dx^j, together with the generalized exterior product

$$\begin{aligned} &dx^i\wedge dx^j=-dx^j\wedge dx^i\\ &dx^i\wedge d\theta^\alpha=-d\theta^\alpha\wedge dx^i\\ &d\theta^\alpha\wedge d\theta^\beta=d\theta^\beta\wedge d\theta^\alpha\,, \end{aligned} \tag{3.10}$$

or, if we introduce the generalized induces K, L designing both j or α, $\dot\beta$, then symbolically

$$dz^K\wedge dz^L+(-1)^{\pi(K)\pi(L)}\,dz^L\wedge dz^K=0\,, \tag{3.11}$$

$\pi(\alpha)=\pi(\dot\beta)=1$, $\pi(j)=0$, z^K standing for θ^α, $\bar\theta^{\dot\beta}$ or x^j.

We shall often use the non-holonomic basis dual to the vector fields $\mathscr{D}_\alpha$, $\bar{\mathscr{D}}_{\dot\beta}$, ∂_k

$$e^\alpha(\mathscr{D}_\beta)=\delta^\alpha_\beta,\ \bar{e}^{\dot\alpha}(\bar{\mathscr{D}}_{\dot\beta})=\delta^{\dot\alpha}_{\dot\beta}, \qquad (3.12a)$$
$$e^j(\partial_k)=\delta^j_k,\ e^j(\mathscr{D}_\alpha)=0,\ e^\alpha(\partial_j)=0,\ \text{etc.},$$

so that

$$e^\alpha=d\theta^\alpha,\ \bar{e}^{\dot\beta}=d\bar\theta^{\dot\beta}, \qquad (3.12b)$$
$$e^j=dx^j-\sigma^j{}_{\alpha\dot\beta}\,\theta^\alpha\, d\bar\theta^{\dot\beta}-\sigma^j{}_{\alpha\dot\beta}\,\bar\theta^{\dot\beta}\, d\theta^\alpha\,.$$

The integration rules are the following:

$$\int d\theta^\alpha=0,\ \int d\bar\theta^{\dot\beta}=0,\ \int\theta^\alpha\, d\theta^\beta=\varepsilon^{\alpha\beta},\ \int\bar\theta^{\dot\alpha}\, d\bar\theta^{\dot\beta}=\varepsilon^{\dot\alpha\dot\beta}, \qquad (3.13)$$

and the "volume" of the θ-space may be normalized to 1:

$$\int\theta^1\theta^2\bar\theta^{\dot 1}\bar\theta^{\dot 2}\, d\theta^{\dot 1}\, d\theta^{\dot 2}\, d\bar\theta^{\dot 1}\, d\bar\theta^{\dot 2}=1\,. \qquad (3.14)$$

A connection in a classical principal fiber bundle $P(M_4, G)$ was given by a Lie-algebra $\mathscr{A}_G$-valued left-invariant 1-form ω over P. This implied

$$\underset{X}{\pounds}\,\omega=-\operatorname{ad}(X)\omega \qquad (3.15)$$

for any left-invariant vertical vector field X generated by the right action of G on $P(M_4, G)$. Let σ be the canonical isomorphism from $\mathscr{A}_G$ onto the tangent spaces to the fibers in $P(M_4, G)$. If X is a left-invariant vector field, then

$$\sigma\circ\omega(X)=X\,. \qquad (3.16)$$

A field X called horizontal if $\omega(X)=0$. Any field can be decomposed into its horizontal and vertical parts:

$$X=\mathrm{hor}X+\mathrm{ver}X=[X-\sigma\circ\omega(X)]+\sigma\circ\omega(X)\,. \qquad (3.17)$$

The curvature of ω is its covariant differential, i.e. a two-form defined by

$$\Omega(X,Y)=D\omega(X,Y)=d\omega(\mathrm{hor}X,\mathrm{hor}\,Y)\,. \qquad (3.18)$$

The covariance property (3.15) enables us to write

$$D\omega(X,Y)=d\omega(X,Y)+\tfrac{1}{2}[\omega(X),\omega(Y)]_{\mathscr{A}_G}\,. \qquad (3.19)$$

Finally, if g_M is a metric in M_4, g_G a metric in G (which, when not explicitly stated otherwise, is supposed to be the Killing-Cartan metric), then a connection ω enables us to define a canonical metric on $P(M_4, G)$:

$$g_P(X,Y)=g_{M_4}(d\pi(X), d\pi(Y))+g_G(\omega(X),\omega(Y))\,, \qquad (3.20)$$

where $d\pi$ is the differential of the canonical projection $\pi: P(M_4, G)\to M_4$. In local coordinates ω can be decomposed as

$$\omega=\omega^a L_a=\omega^a_j L_a\, dx^j+\omega^a_b L_a e^b\,, \qquad (3.21)$$

where $[L_a, L_b]=C^d_{ab}L_d$ is a basis in $\mathscr{A}_G$, e^b are the invariant 1-forms on G, x^j are some coordinates in M_4. We have also for the components of the curvature 2-form

Ω:

$$\Omega^a_{bd} = \partial_b \omega^a_d - \partial_d \omega^a_b + C^a_{gf} \omega^g_b \omega^f_d = 0\,,$$
$$\Omega^a_{jd} = \partial_j \omega^a_d - \partial_d \omega^a_j + C^a_{gf} \omega^g_j \omega^f_d = 0\,, \qquad (3.22)$$
$$\Omega^a_{jk} = \partial_j \omega^a_k - \partial_k \omega^a_j + C^a_{gf} \omega^g_j \omega^f_k \neq 0\,.$$

The gauge invariant quantity

$$-\tfrac{1}{4}\operatorname{Tr}(g^{ij} g^{kl} \Omega_{ik} \Omega_{jl}) = -\tfrac{1}{4} g_{ab} g^{ij} g^{kl} \Omega^a_{ik} \Omega^b_{jl} \qquad (3.23)$$

is called the lagrangian of the gauge field Ω^a_{ij} (g_{ab} are the components of g_G, g^{ij} the components of g_{M_4}).

In order to generalize this formalism to the graded fiber bundles, we shall carefully proceed by steps. First let us replace the base space M_4 by the "superspace" $M_4 \times \{\theta\}$, leaving the same structural group G. The rules for the exterior differentiation are maintained, only the symmetry properties of the p-forms are modified, i.e.

$$e^K \wedge e^L + (-1)^{\pi(K)\pi(L)} e^L \wedge e^K = 0\,, \qquad (3.24)$$

therefore if A is a 1-form $A_K dz^K$, then its differential is

$$\Theta = dA = \partial_L A_K \, dz^L \wedge dz^K\,, \qquad (3.25)$$

which gives the following expressions for the components:

$$\Theta_{ij} = \partial_i A_j - \partial_j A_i = -\Theta_{ji}$$
$$\Theta_{i\beta} = \partial_i A_\beta - \partial_\beta A_i = -\Theta_{\beta i} \qquad (3.26)$$
$$\Theta_{\alpha\beta} = \partial_\alpha A_\beta + \partial_\beta A_\alpha = \Theta_{\alpha\beta}\,.$$

If we want the *covariant* differential to have the same symmetry properties, it implies that the second term in the definition (3.19) has them too, i.e. in local coordinates

$$C^a_{bd} A^b_i A^d_j = -C^a_{bd} A^b_j A^d_i\,,$$
$$C^a_{bd} A^b_i A^d_\beta = -C^a_{bd} A^b_\beta A^d_i\,, \qquad (3.27)$$
$$C^a_{bd} A^b_\alpha A^d_\beta = C^a_{bd} A^b_\beta A^d_\alpha\,.$$

This in turn is possible only if

$$A^b_i A^d_j = A^d_j A^b_i\,,$$
$$A^b_i A^d_\beta = A^d_\beta A^b_i \quad \text{and} \quad A^b_\alpha A^d_\beta = -A^d_\beta A^b_\alpha\,, \qquad (3.28)$$

which symbolically can be written as

$$\pi(A^b_K) = \pi(K)\,, \qquad (3.29)$$

and

$$A^b_K A^d_L = (-1)^{\pi(K)\pi(L)} A^d_L A^b_K\,. \qquad (3.30)$$

The connection coefficients are $\Lambda\{\theta\}$-valued functions on M_4; the condition (3.29) limits their form to quite a limited development, i.e. A^d_j contain only even powers of θ's, whereas A^d_α contain only odd powers of θ's. If the negative-energy states are to be avoided, the connection has to be hermitian, i.e.

$$(A^d_\alpha)^\dagger = A^d_{\dot\alpha}. \tag{3.31}$$

This fixes the development of A^d_K in the anaholonomic basis e^j, $d\theta^\alpha$, $d\bar\theta^{\dot\beta}$ defined by (3.12a) and (3.12b), as follows:

$$\begin{aligned} A^b_j &= B^b_j(x) + \frac{1}{l}\sigma_{j\alpha\dot\beta}\theta^\alpha\bar\theta^{\dot\beta}\phi^b(x) \\ A^b_\alpha &= \phi^b(x)\theta_\alpha\,;\; A^b_{\dot\beta} = (A^b_\beta)^\dagger = \phi^b\bar\theta_{\dot\beta}. \end{aligned} \tag{3.32}$$

The higher order terms in θ's could be introduced, too, but it is quite easy to verify that they will not contribute to the fourth-power term in the final Lagrangian.

A parameter l with the dimension of length has to be introduced because $\dim|\theta^\alpha| = \mathrm{cm}^{1/2}$, $\dim\phi^b = \mathrm{cm}^{-1}$, and $\dim B^b_j = \mathrm{cm}^{-1}$. The term containing this parameter in the development (3.32) manifestly breaks the conformal symmetry; scale invariance is recovered when $l \to \infty$. The lagrangian of the theory is, by analogy, the same as given by (3.23)

$$\begin{aligned} L = -\tfrac{1}{32}\sqrt{|g|}\,g_{ab}\Big[& g^{ik}g^{jl}F^a_{ij}F^b_{kl} \\ & + \frac{2}{l}\left(g^{ij}\varepsilon^{\alpha\beta}F^a_{i\alpha}F^b_{j\beta} + g^{ij}\varepsilon^{\dot\alpha\dot\beta}F^a_{i\dot\alpha}F^b_{j\dot\beta}\right) \\ & + \frac{1}{l^2}\left(\varepsilon^{\alpha\gamma}\varepsilon^{\beta\delta}F^a_{\alpha\beta}F^b_{\gamma\delta} + 2\varepsilon^{\alpha\gamma}\varepsilon^{\dot\beta\dot\delta}F^a_{\alpha\dot\beta}F^b_{\gamma\dot\delta} + \varepsilon^{\dot\alpha\dot\gamma}\varepsilon^{\dot\beta\dot\delta}F^a_{\dot\alpha\dot\beta}F^b_{\dot\gamma\dot\delta}\right)\Big]. \end{aligned} \tag{3.33a}$$

The fourth-power term in θ is then equal to

$$L^{(4)} = -\frac{3}{8l^2}(\nabla_i\phi^b)(\nabla^i\phi_b) - \frac{1}{4l^2}G^a_{ij}G^{ij}_a - \frac{4}{l^4}\phi^b\phi_b, \tag{3.33b}$$

with

$$\begin{aligned} G^a_{ij} &= \partial_i B^a_j - \partial_j B^a_i + C^a_{bd}B^b_i B^d_j, \\ \nabla_j\phi^a &= \partial_j\phi^a + C^a_{bd}B^b_j\phi^d. \end{aligned}$$

4. Double Grading and the Spin-Statistics Dependence

Now we proceed to the definition of the real goal of our construction, i.e. the bundle in which both the base space and the typical fiber are graded manifolds. Unless stated explicitly otherwise, we shall use local non-holonomic systems in the basis graded manifold $M_4 \times \{\theta\}$, and in the fiber graded manifold $G\square\{\chi\}$ [(3.12b), (2.29a), (2.29b)]. The connection 1-form A in this bundle is decomposed as follows: (we take into account only non-trivial components)

$$\begin{aligned} A &= A_K e^K = A^\Phi_K e^K \mathscr{D}_\Phi \\ &= (A^b_j e^j + A^b_\alpha e^\alpha + A^b_{\dot\beta}\bar e^{\dot\beta})\mathscr{D}_b + (A^D_j e^j + A^D_\alpha e^\alpha + A^D_{\dot\beta}\bar e^{\dot\beta})\mathscr{D}_D, \end{aligned} \tag{4.1}$$

the coefficients A_K^ϕ being functions on $M_4 \times \{\theta\}$ and on $G\square\{\chi\}$. The dependence on $G\square\{\chi\}$ is fully determined by the condition of left-invariance (or horizontality of the curvature F, i.e. vanishing of the components $F_{\psi\Omega}^\phi$, $F_{\psi K}^\phi$). The only non-vanishing components are the horizontal ones, F_{KL}^ϕ.

However, if we want to generalize the definition of the curvature 2-form, i.e. to put:

$$F_{KL}^\phi = \mathscr{D}_K A_L^\phi e^K \wedge e^L + C_{\chi\psi}^\phi A_K^\chi A_L^\psi e^K \wedge e^L , \tag{4.2}$$

the problem of Grassmann parity counting is more complicated than in the previous example [formulae (3.26)–(3.30)], because now the commutation or anticommutation properties of A_K^ϕ depend not only on the parity of the power of θ's they contain, but also on the powers of G-spinors χ; the result will be different depending on the hypothesis we make about the commutation or anticommutation between θ's and χ's.

Two assumptions are possible: either

$$\theta^\alpha \chi^B = \chi^B \theta^\alpha , \tag{4.3a}$$

or

$$\theta^\alpha \chi^B = -\chi^B \theta^\alpha . \tag{4.3b}$$

Let us treat the two cases separately.

a) In the case of commuting θ's and χ's the two different Grassmann parities do not influence each other and add up separately; therefore the parity rule for the coefficients A_K^ϕ is

$$A_K^\phi A_L^\psi = (-1)^{\pi(\phi)\pi(\psi) + \pi(K)\pi(L)} A_L^\psi A_K^\phi , \tag{4.4}$$

yielding the following Table 1:

Table 1

	A_j^b	A_α^b	A_j^B	A_α^B
A_i^d	0	0	0	0
A_β^d	0	1	0	1
A_i^D	0	0	1	1
A_β^D	0	1	1	0

(4.5)

in which 0 means commutation, and 1 means anticommutation between the respective entities. It can be easily checked that these properties combined with the symmetry $C_{\chi\Omega}^\phi + (-1)^{\pi(\chi)\pi(\Omega)} C_{\Omega\chi}^\phi = 0$ assure the required symmetry of F_{KL}^ϕ, namely

$$F_{KL}^\phi = (-1)^{\pi(K)\pi(L)} F_{LK}^\phi . \tag{4.6}$$

In order to have the commutation – anticommutation properties (3.37), the development of A_K^ϕ in powers of χ^D and θ^α must be of a particular form, namely, if $\pi(\phi) = 0$, the corresponding A_K^ϕ contains only even powers of χ^D, and if $\pi(\phi) = 1$, the corresponding A_K^ϕ contains only odd powers of χ^D; the same is true for the powers of θ^α's and the Grassmann parity $\pi(K)$. Therefore, if we keep only zeroth and first powers of χ, and do not introduce any dimension parameters, the unique development becomes:

$$\begin{gathered} A_j^b = B_j^b(x), \\ A_\alpha^b = \phi^b(x)\theta_\alpha, \ A_{\dot\alpha}^b = \phi^b(x)\bar\theta_{\dot\alpha}, \\ A_j^B = \chi^B W_j(x), \\ A_\alpha^B = \chi^B\theta_\alpha D(x), \ A_{\dot\alpha}^B = \chi^B\bar\theta_{\dot\alpha} D(x). \end{gathered} \tag{4.7}$$

We see that due to the separate adding up of the space-time and group spinor parities the fermionic sector is completely eliminated, leaving only massless fields $D(x)$, $W_j(x)$, $\phi^a(x)$, and $B_j^a(x)$. The conformal symmetry can be broken and masses introduced as in the example given by (3.32), i.e. if we introduce the universal length scale l and enlarge the supergauge conditions to:

$$\begin{gathered} A_j^b = B_j^b(x) + \frac{1}{l}\phi^b(x)\sigma_{j\alpha\dot\beta}\theta^\alpha\bar\theta^{\dot\beta}, \\ A_\alpha^b = \phi^b(x)\theta_\alpha, \\ A_j^B = \chi^B W_j(x) + \frac{1}{l}\chi^B\sigma_{j\alpha\dot\beta}\theta^\alpha\bar\theta^{\dot\beta} D(x), \\ A_\alpha^B = \chi^B\theta_\alpha D(x) + \frac{1}{l}\chi^B\sigma^j{}_{\gamma\dot\delta}\theta^\gamma\bar\theta^{\dot\delta}\theta_\alpha W_j(x). \end{gathered} \tag{4.8}$$

Of course, even then such a theory has no interest because of the absence of fermions; therefore, we proceed directly to the alternative parity counting, corresponding to (4.3b):

b) Now the two different Grassmann parities add up together, and the parity rule for the coefficients A_K^ϕ becomes

$$\pi(A_K^\phi) = \pi(\phi) + \pi(K), \tag{4.9}$$

i.e.

$$A_K^\phi A_L^\psi = (-1)^{[\pi(\phi)+\pi(K)][\pi(\psi)+\pi(L)]} A_L^\psi A_K^\phi, \tag{4.10}$$

yielding the following rule

Table 2

	A_j^b	A_α^b	A_j^B	A_α^B
A_i^d	0	0	0	0
A_α^d	0	1	1	0
A_i^D	0	1	1	0
A_α^D	0	0	0	0

(4.11)

Now the symmetry properties of F_{KL}^ϕ are destroyed, e.g. the expressions like

$$C_{bD}^A A_j^b A_\alpha^D + C_{Db}^A A_j^D A_\alpha^b = C_{bD}^A (A_j^b A_\alpha^D - A_j^D A_\alpha^b), \tag{4.12a}$$

or

$$C^{a}_{BD}A^{B}_{j}A^{D}_{\alpha} \tag{4.12b}$$

have no determined symmetry, i.e. they are neither symmetric nor antisymmetric. In such a case neither a stationary Lagrangian nor the Hamiltonian can be positive definite. In order to restore the positive definiteness of the Hamiltonian and to eliminate the ghost fields, we have imposed more strict (super) gauge conditions, which however will not affect the usual gauge symmetry.

The parity counting defined by (3.43) can be found more attractive also for the following reasons. When we want to take into account the fact that the Lorentz fermions are anti-commuting quantities, whereas the bosons are commuting quantities, we imply that $\psi_\alpha(x)$ or $\psi_{j\alpha}(x)$ have their values in some anti-commutative ring. Supposing that this ring is of finite dimension, it is natural to decompose, e.g. $\psi_\alpha = \psi^A_\alpha \chi_A$, where χ_A is the basis of the anti-commuting generators of the corresponding Grassmannian.

In the usual supersymmetric theories the anti-commuting quantities χ_A were identified with the duals of the anti-commuting Lorentz spinors θ, so that ψ_α was supposed to be proportional to some anti-commuting ε_α. However, there is no a priori reason to do so; the anti-commuting basis χ_A may be chosen quite independently of the Lorentz spinors θ.

On the other hand, and in the spirit of the grand unification, one is led to believe that both the Lorentz spinors θ and the G-spinors χ^A have common origin and are the split and reduced parts of the higher-dimensional spinors which we denote by ξ, and which correspond to the Riemannian metric constructed on the unified space containing also the internal degrees of freedom, which manifest themselves in the gauge group G. This is visualized on the following scheme:

Manifolds	M Lorentz manifold	$M \times G$	$P(M, G)$ unified manifold
Corresponding spinors	Lorentz spinors θ	Cartesian product $\theta \times \chi$	Unified spinors ξ

In such a case it is natural to assume that even after the dimensional reduction and the corresponding splitting up of spinors has occured, the reduced parts of the unified spinors ξ still anti-commute between themselves.

Therefore we propose to realize the Grassmann parity counting by including the dependence on χ's, and in order to keep the required symmetry properties of F^{ϕ}_{KL} given by (3.39) we impose the following supergauge conditions (cf. [7])

$$A^{B}_{j}=0,\ A^{b}_{\alpha}=0,\ A^{d}_{\beta}=0, \tag{4.13}$$

which means that only commuting fields A^{B}_{α}, A^{a}_{j} do not vanish.

As we have already stated, only the term proportional to $\theta^1\theta^2\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}}$ will be considered as the relevant part of the final Lagrangian. If the χ's are present explicitly, one has to ask what powers of χ's are to be considered as relevant, too. The power of the "volume element" in the Grassmannian $\Lambda\{\chi\}$ depends on the dimension of G and is not a good candidate; the only other universal invariant is the bilinear combination $\varepsilon_{AB}\chi^A\chi^B$, which has the same form independent of the choice of G. Therefore we propose to develop A^ϕ_K only in zeroth and first powers of χ^B, and consider only the coefficient of $\varepsilon_{AB}\chi^A\chi^B\theta^1\theta^2\bar\theta^{\dot 1}\bar\theta^{\dot 2}$ as the relevant part of the final Lagrangian.

The most general development of A^ϕ_K under these conditions and without the conformal symmetry breaking is:

$$\begin{aligned}A^b_j &= B^b_j(x) + \chi^A\tau^b_{AB}(\psi^B_\alpha\sigma_j{}^\alpha{}_{\dot\beta}\bar\theta^{\dot\beta} + \theta_\alpha\sigma_j{}^\alpha{}_{\dot\beta}\bar\psi^{B\dot\beta})\\ &\quad + \chi^A\tau^b_{AB}(\psi^B_{j\alpha}\theta^\alpha + \bar\psi^B_{j\dot\beta}\bar\theta^{\dot\beta})\,;\\ A^B_\alpha &= \psi^B_\alpha(x) + \psi^B_{j\alpha}\sigma^j{}_{\gamma\dot\delta}\theta^\gamma\bar\theta^{\dot\delta} + D(x)\chi^B\theta_\alpha\\ &\quad + W_j(x)\sigma^j{}_{\alpha\dot\beta}\bar\theta^{\dot\beta}\chi^B + \phi^a(x)\tau_a{}^B{}_D\chi^D\theta_\alpha\\ &\quad + \tau^B_{aD}\chi^D B^a_j\sigma^j{}_{\alpha\dot\beta}\bar\theta^{\dot\beta}\,;\; A^B_{\dot\alpha} = (A^B_\alpha)^\dagger\,.\end{aligned} \tag{4.14}$$

The theory introduces in a natural way the following bosons: scalar $D(x)$, vector $W_j(x)$, Higgs scalar multiplet ϕ^a and the gauge field $B^a_j(x)$; and the fermions: spin 1/2 multiplet ψ^B_α, and spin 3/2 multiplet $\psi^B_{\alpha j}$, both transforming under the spinorial representation $\tau_a{}^A{}_B$ of G. All these fields are massless. As in the cases discussed above, one may generalize the θ-dependence of the potentials keeping still the required Grassmann parity, but introducing the elementary length l and breaking the conformal symmetry. This is achieved by replacing:

$$\begin{aligned}&B^a_j \quad\text{by}\quad B^a_j + \frac{f}{l}\phi^a\sigma_{j\alpha\dot\beta}\theta^\alpha\bar\theta^{\dot\beta}\,,\\ &\psi^B_\alpha \quad\text{by}\quad \psi^B_\alpha + \frac{\lambda}{l}(\psi^B_\gamma\theta^\gamma + \bar\psi^B_{\dot\gamma}\bar\theta^{\dot\gamma})\theta_\alpha + \frac{v}{l^2}\psi^B_\alpha\theta^1_1\theta^2_2\bar\theta^{\dot 1}\bar\theta^{\dot 2}\,,\\ &\psi^B_{j\alpha} \quad\text{by}\quad \psi^B_{j\alpha} + \frac{p}{l}(\psi^B_{j\gamma}\theta^\gamma + \bar\psi^B_{j\dot\gamma}\bar\theta^{\dot\gamma}) + \frac{q}{l^2}\psi^B_{j\alpha}\theta^1\theta^2\bar\theta^{\dot 1}\bar\theta^{\dot 2}\,,\\ &D \quad\text{by}\quad D + \frac{d}{l}D(\theta_\gamma\theta^\gamma + \bar\theta_{\dot\gamma}\bar\theta^{\dot\gamma}) + \frac{\delta}{l^2}D\theta^1\theta^2\bar\theta^{\dot 1}\bar\theta^{\dot 2}\,,\\ &W_j \quad\text{by}\quad W_j + \frac{w}{l}W_j(\theta_\gamma\theta^\gamma + \bar\theta_{\dot\gamma}\bar\theta^{\dot\gamma}) + \frac{u}{l^2}W_j\theta^1\theta^2\bar\theta^{\dot 1}\bar\theta^{\dot 2}\,.\end{aligned} \tag{4.15}$$

Here too, like in example (3.32), the highest order terms most probably will not contribute to the final Lagrangian, but we have not the formal proof for that.

It is worthwhile to note that the introduction of the dimensional parameter l into the development corresponds exactly to what is usually done in the supersymmetric theories, namely the introduction of the fields having anomalous dimensions. We do exactly the same, but the anomalous dimensions are taken care of through the introduction of the dimensional multiplier l, in order to visualise them better. Here f, λ, v, p, q, δ, w, u are some dimensionless constants which later may be absorbed in the renormalized fields.

The generalized lagrangian is defined by

$$L = -\frac{1}{V}g_{\phi\Omega}g^{KL}g^{MN}F^\phi_{KM}F^\Omega_{LN}\,, \tag{4.16}$$

which is written more explicitly as

$$L = -\frac{1}{V}\left[g_{ab} F^a_{KM} F^b_{LN} + \varepsilon_{AB} F^A_{KM} F^B_{LN}\right] g^{KL} g^{MN}, \tag{4.17}$$

where

$$g^{KL} g^{MN} F^a_{KM} F^b_{LN} = g^{ij} g^{km} F^a_{ik} F^b_{jm} + \frac{2}{l} g^{ij} \varepsilon^{\alpha\beta} F^a_{i\alpha} F^b_{j\beta} + \frac{1}{l^2} \varepsilon^{\alpha\beta} \varepsilon^{\gamma\delta} F^a_{\alpha\gamma} F^b_{\beta\delta}, \tag{4.18a}$$

and similarly,

$$g^{KL} g^{MN} F^A_{KM} F^B_{LN} = g^{ij} g^{km} F^A_{ik} F^B_{jm} + \frac{2}{l} g^{ij} \varepsilon^{\alpha\beta} F^A_{i\alpha} F^B_{j\beta} + \frac{1}{l^2} \varepsilon^{\alpha\beta} \varepsilon^{\gamma\delta} F^A_{\alpha\gamma} F^B_{\beta\delta} \tag{4.18b}$$

(where for simplicity we did not distinguish between the dotted and un-dotted fermion indices α, β and $\dot{\alpha}$, $\dot{\beta}$; in real calculations the full development must be taken into account).

In the final expression, only the terms containing $\varepsilon_{AB}\chi^A\chi^B\theta^1\theta^2\bar{\theta}^{\dot{1}}\bar{\theta}^{\dot{2}}$ are taken under consideration as the physical Lagrangian. Unless the conformal symmetry is broken in the definition of connection coefficients [with the substitution (3.51)], the "elementary length" l appears in a homogeneous manner in this highest power term, as $\frac{1}{l^2}$, and therefore may be disregarded in the variational principle, analogously to example (3.33).

5. Discussion. Prospects and Conclusions

It is easy to see that the full calculus of the Lagrangian (3.52) is a very tedious one, and it is no wonder that we are still unable to determine it at the present stage, especially with the conformal symmetry breaking as in (3.51), although the problem remains purely technical. However, some important observations can be made without the thorough computation of all the coefficients in the development. The main features of the model are visible already at this stage.

a) First of all, let us comment on the purely algebraic properties of the graded gauge theory. The principle of summing up the G-spinor and fermionic Grassmann parities, together with the graded symmetry requirements imposed upon the curvature 2-form F serve as the super-selection rules, which assign well determined group representations to the bosonic and fermionic sectors. This can be summarized up in Table 3:

Table 3

G-spin \ Space-time spin	0	1/2	1	3/2	2
0	$D(x)$	—	$W_j(x)$	—	$h_{ij}(x)$
1/2	—	$\psi^A(x)$	—	$\psi^A_{\alpha j}(x)$	—
1	$\phi^a(x)$	—	$B^a_j(x)$	—	$f^a_{ij}(x)$

Here we have laid down all the multiplets appearing in the model and put the bar in place of the forbidden multiplets. We have included in our table the spin-2 multiplets, which are in principle allowed by the model.

It is interesting to note that together with the classical spin-2 field (represented by the traceless symmetric tensor $h_{ij}(x)$, appearing in the last column) there should appear an G-adjoint multiplet of such tensors, $f^a_{ij}(x)$. The interpretation of this field seems yet physically unclear; it should appear in the development of the gauge potential A^b_j in the following combination:

$$A^b_j = B^b_j(x) + f^b_{ij}\sigma^i{}_{\alpha\dot\beta}\theta^\alpha\bar\theta^{\dot\beta} + \dots. \tag{5.1}$$

At least in principle, this table could be continued in both directions, including the G-spin 3/2, and so forth.

Another important point concerns the G-spinorial representation for the fermions, which is obviously reducible except for the case $G=\mathrm{SU}(2)$. In order to be able to identify our fields with any "elementary" particles, the decomposition of this representation into a sum of irreducible representations describing the quarks will not appear automatically in such a decomposition.

In the case of $G=\mathrm{SU}(2)\,\dim G=2$, and $s=2$; the G-spin representation is irreducible and lowest-dimensional. However, if we go to $G=\mathrm{SU}(3)$, then $\dim G=8$ and $s=16$. This G-spin representation decomposes as $16=8\oplus\bar{8}$ into two adjoint representations, and there is *no place for quarks* in the model.

The fundamental representation begins to appear in the case when $G=\mathrm{SU}(4)$. Then $\dim G=15$, $s=2^7=128$. This representation decomposes as follows:

$$128=15\oplus15\oplus\dots\oplus15\oplus4\oplus\bar{4}, \qquad \text{i.e.}$$
$$\leftarrow \text{eight times} \rightarrow$$

we have eight adjoint representations and *two fundamental ones.* As a by-product, our model gives the explanation why there are *four quarks and four anti-quarks*, and not *three quarks* as in the first quark models which finally had to be extended to a four-quark model.

b) Although the full Lagrangian is difficult to calculate, it is quite easy to obtain just the dynamical terms without interactions for each of the fields separately. When there is no conformal symmetry breaking, i.e. with the ansatz (3.50), these turn out to be; (up to a normalizing multiplicative constant, different for each field):

$$g^{ij}\partial_i D\partial_j D \qquad \text{for the scalar field}, \tag{5.2a}$$

$$g^{ij}g^{km}(\partial_i W_k - \partial_k W_i)(\partial_j W_m - \partial_m W_j) \qquad \text{for the vector field}, \tag{5.2b}$$

$$g_{ab}g^{ij}(\partial_i\phi^a)(\partial_j\phi^b) \qquad \text{for the Higgs multiplet}, \tag{5.2c}$$

$$g_{ab}g^{ij}g^{kl}G^a_{ik}G^b_{jl} \qquad \text{for the Yang-Mills field}, \tag{5.2d}$$

where $G^a_{ik}=\partial_i B^a_k-\partial_k B^a_i + C^a_{bd}B^b_i B^d_k$.

As in the simplified version of the theory without the explicit dependence of potentials on the G-spinors, the fermionic sector does not appear in the highest order θ-term in the Lagrangian unless we introduce the elementary length l into the development, replacing everywhere

$$\begin{aligned} &\psi_\alpha^B \quad \text{by} \quad \psi_\alpha^B + \frac{\lambda}{l}(\psi_\gamma^B \theta^\gamma + \bar{\psi}_{\dot{\gamma}}^B \bar{\theta}^{\dot{\gamma}})\theta_\alpha + \text{etc.}, \\ &\psi_{j\alpha}^B \quad \text{by} \quad \psi_{j\alpha}^B + \frac{p}{l}(\psi_{j\gamma}^B \theta^\gamma + \bar{\psi}_{j\dot{\gamma}}^B \bar{\theta}^{\dot{\gamma}})\theta_\alpha + \dots, \end{aligned} \tag{5.3}$$

etc. The Lagrangian then contains ψ_β^A via the following dynamical terms:

$$g^{ij}(\nabla_i \psi^\dagger)(\nabla_j \psi) \quad \text{and} \quad [\psi^\dagger \gamma^j \nabla_j \psi - (\gamma^j \nabla_j \psi^\dagger)\psi]. \tag{5.4}$$

The d'Alembertian-type term $g^{ij}(\nabla_i \psi^\dagger)(\nabla_j \psi)$ appears also for the vector-spinor field $\psi_{\beta j}^B$. Both may be removed leaving only the Dirac type lagrangian if we choose the renormalizing constant λ in an appropriate way. We lack the physical motivation which would fix λ in that manner. Most probably only introducing the non-linear terms of higher order would bring in some additional symmetry breaking and the corresponding Higgs' mechanism would fix some more constants by making some solutions stable as compared to all other ones.

c) The conformal symmetry breaking that has been introduced here comes from the anomalous dimensions of the fields in the expressions containing the elementary length l. We don't think that it is unnatural, because sooner or later it must be introduced if we carry the spirit of supersymmetry to its logical end. As a matter of fact, from the beginning we want to form the linear superpositions of the fields of different spin; on the other hand, their dimensions are not the same, so we have to introduce the coefficients which take care of these dimensional differences. For example, if we choose the Pauli matrices σ^j dimensionless, and if we want to add up x^j (with the dimension of cm) and the expressions $\bar{\theta}\sigma^j\theta$, this can be done only if the dimension of the spinor θ is equal to $\text{cm}^{1/2}$. The scale is not given a priori; that is why we visualize it by introducing the length factor l. However, in the final expressions it would come out homogeneously, giving some overall factor for the lagrangian density. The situation becomes less simple if we push the unification further, including the group dimensions and the G-spinors. Unifying the group dimensions with the space-time dimensions (including all in a principal fiber bundle) means also introducing a length scale in order to give the proper dimension to the group manifold variables; let us call this constant e (usually we see it in front of the structure constants). Finally, the G-spinors must also have the dimension $\text{cm}^{1/2}$; therefore the third constant g of the dimension of length is needed. Even if the final lagrangian is still homogeneous in dimension, it contains different products of these dimensional constants; the relevant information left in the theory will be contained in the three independent dimensionless ratios l/g, eg, and el. It seems therefore quite attractive that the graded gauge theory provides us naturally with three different scales of interactions. Even if we suppose that all these ratios are equal to 1, it provides us with quite big factors $s(s+1)/N$ in front of several terms in the lagrangian. As $s=2^{N/2}$, these factors grow very rapidly with the dimension of the gauge group, e.g. $s(s+1)/N$ is equal to 34 for SU(3), and to 6.99×10^5 for SU(5), etc.

d) Another feature of the model is worth noting, namely the fact that the Yukawa, Fermi and current-current couplings are related to the masses obtained via the conformal symmetry breaking. As both these quantities are measurable, this makes possible the confrontation with experiment; however, we must add that the particles described by the unified supersymmetric fields should correspond to quarks rather than to the less elementary observed particles.

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