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Classical Equations $dN_i/dr = \frac{1}{2} i\varepsilon_{ijk}[N_j, N_k]$

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Abstract. We study the first order system of equations $dN_i/dr = \frac{1}{2} i\varepsilon_{ijk}[N_j, N_k]$, where the N_i are classical, "non-abelian" gauge-Higgs fields with spherical symmetry. Exact solutions are constructed.

1.

Our starting point are the Bogomolny equations (vanishing self-coupling for the adjoint Higgs field), when there is spherical symmetry

$$\frac{d}{dr}\psi = \tfrac{1}{2}[N^+, N^-], \quad \frac{d}{dr}N^\pm = \pm[\psi, N^\pm], \tag{1}$$

$$[T_3, \psi] = 0, \qquad [T_3, N^\pm] = \pm N^\pm. \tag{2}$$

Here T_3 is a generator of the SO(3) subgroup of our gauge group G, and $\psi, N^\pm$ (related to the original Higgs and gauge fields) are elements of the Lie algebra $L(G)$ and satisfy $\psi = \psi^+$, $(N^+)^+ = N^-$ (Hermitean conjugates). The connection between these variables and the title variables N_i is $N_3 = -\psi$ and $N^\pm = N_i \pm iN_2$. For a derivation of the above equations see [1]. We also use the notion of the "grade" n of a generator X, if

$$[T_3, X] = nX.$$

The grade n is an eigenvalue of T_3, and therefore an integer or half-integer. We also need the star (*) operation

$$X^* = (-1)^{T_3} X^+ (-1)^{T_3},$$

which defines an involution of the subalgebra of $L(G)$ with integer grades.

Define $R = -\psi + N^-$ and $R^* = -\psi - N^+$. Using (1) one finds that

$$\frac{d}{dr}(R + R^*) = [R^*, R]. \tag{3}$$

The reverse is also true: given a Lie algebra element R which consists of a grade

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-1 part and a hermitean grade 0 part and satisfies (3), system (1) follows, by equating the different grades. We therefore look for such an R.

2.

We have adapted the method used by Leznov and Saveliev [2]. We shall work in the complexification G^c of our original gauge group. The corresponding Lie algebra is spanned by diagonal generators (we fix a Cartan subalgebra h and an ordering of the roots), and raising and lowering generators $E_{\pm\alpha}$ corresponding to the roots. (For definitions, see [3].) We assume that T_3 is a (hermitean) diagonal generator in the positive Weyl chamber. This means that raising generators ($E_\alpha, \alpha > 0$) have positive or zero grade.

Define the following subgroup: G^0 generated by the grade 0 elements of the Lie algebra $L(G^c)$. For an arbitrary embedding, G^0 may be non-abelian and non-semi-simple. The $G_\pm$ are generated by $\sum_\alpha \mathbb{C}E_\alpha (\alpha > 0$ or $\alpha < 0$ respectively). These are the maximal nilpotent subgroups of G^c. The corresponding nilpotent subgroups of G^0 are denoted $G^0_\pm$. Then M_+ are generated by the positive grade generators. It is isomorphic to G_+/G^0_+. Similarly for M_-. The above nilpotent subgroups are related through the action of the $*$ operation (we use $(E_\alpha)^+ = E_{-\alpha}$), e.g. $(G^0_+)^* = G^0_-$.

3.

We can now proceed with the construction. Pick a hermitean element Q of h (Cartan subalgebra), which in addition lies in the positive definite Weyl Chamber: $Q\cdot\alpha > 0$ for all roots $\alpha > 0$. Also pick grade 1 element M ($[T_3, M] = M$). One can now solve for $x \in M_+$ and

$$xQx^{-1} = Q + M. \tag{4}$$

This defines x uniquely [5]. We next consider the group element $xe^{Qr}x^*$, with r our single variable, and decompose it in a product of factors:

$$xe^{Qr}x^* = msa^2s^*m^* \tag{5}$$

with $m \in M_-$, $s \in G^0_-$, $a \in H$ and $a = a^*$. Here H is the Cartan subgroup. This decomposition is unique, provided that the left hand side is regular [4]. We will assume this to be the case since we have the freedom to vary M and Q. (Non-regular elements have measure zero in the group manifold.) In any specific representation m, s are lower triangular with 1 along the diagonal, a is diagonal and hermitean, m^*, s^* are upper triangular. They depend on the parameters of x and e^{Qr} through a system of linear equations. In particular we note that m, s, a, depend on r.

The above decomposition is the key to the solution of (3). Set $R = t^{-1}\dot{t}$, where $\cdot$ denotes d/dr and $t \equiv msa$. We will show that R satisfies (3). Using the definition of t and (4), (5) we see that

$$\frac{d}{dr}(tt^*)(tt^*)^{-1} = xQx^{-1} = Q + M.$$

But also

$$\frac{d}{dr}(tt^*)(tt^*)^{-1} = t(t^{-1}\dot{t} + \dot{t}^* t^{*-1})t^{-1} = t(R + R^*)t^{-1},$$

and we can conclude that

$$t(R + R^*)t^{-1} = Q + M. \tag{6}$$

At this point we observe that $R = t^{-1}\dot{t}$ contains by construction non-positive grade generators and R^* only non-negative. Further from (6) we conclude that the positive grade part of R^* must be entirely grade 1, since we have only grades 0, 1 on the right hand side. Therefore R has the required "grade" structure and by differentiating (6) once more, we see that R satisfies (3):

$$\begin{gathered} \dot{t}(R + R^*)t^{-1} + t(\dot{R} + \dot{R}^*)t^{-1} + t(R + R^*)\dot{t}^{-1} = 0 \;\Rightarrow \\ t^{-1}\dot{t}(R + R^*) + (\dot{R} + \dot{R}^*) - (R + R^*)t^{-1}\dot{t} = 0 \;\Rightarrow \\ \dot{R} + \dot{R}^* = [R + R^*, R]. \end{gathered}$$

From (6) we see that the positive grade part of R^* is $a^{-1}s^{-1}Msa$. So we can finally write

$$R = as^*M^*s^{*-1}a^{-1} + a^{-1}s^{-1}\dot{s}a + a^{-1}\dot{a}. \tag{7}$$

We further conjecture that every solution of (3) can be gotten this way (up to gauge transformation).

4.

The grade 0 part of R that we have just constructed will not be in general hermitean. However we still have a residual gauge symmetry at our disposal:

$$R \to u^+ Ru + u^+\dot{u} \text{ with } u \in G^0, u^+u = 1.$$

This "grade 0" gauge transformation leaves our system (3) invariant. We should therefore choose u such that

$$\begin{gathered} u^+a^{-1}s^{-1}\dot{s}au + u^+\dot{u} = u^+a\dot{s}^+(s^+)^{-1}a^{-1}u - u^+\dot{u} \;\Rightarrow \\ \dot{u}u^+ = \tfrac{1}{2}(a\dot{s}^+(s^+)^{-1}a^{-1} - a^{-1}s^{-1}\dot{s}a). \end{gathered} \tag{8}$$

This determines u up to a constant. To summarize let us write the expressions for our original field variables:

$$\psi = -\tfrac{1}{2}u^*(a^{-1}s^{-1}\dot{s}a + a\dot{s}^*(s^*)^{-1}a^{-1} + 2a^{-1}\dot{a})u, \tag{9}$$

$$N^- = u^*as^*M^*s^{*-1}a^{-1}u, \tag{10}$$

where a and s are determined by (5) and u determinded by (8). Here M contains the "integration constants" together with Q which enters in (5).

The case of maximal embedding [1] can be derived as a special case of the above. The G^0 is diagonal and therefore $s = 1$, and u are constant phase transformations and can be used to make M real. Then $\psi = -a^{-1}\dot{a}$, $N^- = aM^*a^{-1}$. In [1] the regularity conditions at the origin could be satisfied by an appropriate choice of the integration constants. These conditions still have to be worked out in detail for our general solutions (9), (10); we are presently studying this problem.

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