Commun. Math. Phys. 90, 1–26 (1983)

Communications in Mathematical Physics

Reconstruction of Singularities for Solutions of Schrödinger's Equation

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Abstract. We determine the behavior in time of singularities of solutions to some Schrödinger equations on R^n. We assume the Hamiltonians are of the form $H_0 + V$, where $H_0 = 1/2\Delta + 1/2 \sum_{k=1}^{n} \omega_k^2 x_k^2$, and where V is bounded and smooth with decaying derivatives. When all $\omega_k = 0$, the kernel $k(t,x,y)$ of $\exp(-itH)$ is smooth in x for every fixed (t,y). When all ω_1 are equal but non-zero, the initial singularity "reconstructs" at times $t = \frac{m\pi}{\omega_1}$ and positions $x = (-1)^m y$, just as if $V = 0$; k is otherwise regular. In the general case, the singular support is shown to be contained in the union of the hyperplanes $\{x | x_{js} = (-1)^l js_{y_{js}}\}$, when $\omega_j t/\pi = l_j$ for $j = j_1, \ldots, j_r$.

0. Introduction

Let $H = H_0 + V$ be a Schrödinger operator on $L^2(R^n)$, where H_0 is one of the model Hamiltonians:

(1) $-1/2\ \Delta$ Free Particle,
(2) $-1/2\ \Delta + 1/2\ |x|^2$ Isotropic Oscillator,
(3) $-1/2\ \Delta + 1/2 \sum_{k=1}^{n} \omega_k^2 x_k^2$ Anisotropic Oscillator,

and where the perturbing potential V is a 0-symbol on R^n, i.e. $|\partial_x^\alpha v| \leqq C_\alpha(1 + |x|)^{-|\alpha|}$. Then H generates a one parameter group of unitary operators $U(t) = \exp - itH$, whose Schwarz kernels we denote by $k_V(t,x,y)$ (called "propagators"). Our goal is to determine the wave front sets of these $k_V(t,x,y)$ when (t,y) are held fixed. This is the essential step in finding out how $U(t)$ propagates singularities—or, more correctly, how $U(t)$ smooths out and later reconstructs singularities.

The main problem is that although these distributions are oscillatory integral ones, i.e. of the form

$$k(t,x,y) = \int a(t,x,y,\theta) e^{iS(t,x,y,\theta)} d\theta,$$

they are not Lagrangian distributions (cf. 4, 7). Consequently, $WF(k(t,\cdot,y)) \nsubseteq$

$\Lambda_{S_{t,y}} = \{(x,\xi)|\xi = (\partial S/\partial x)(t,y,x,\theta), (\partial S/\partial\theta)(t,y,x,\theta) = 0\}$ and $WF(U(t)\varphi) \nsubseteq \Phi^t WF(\varphi)$, where Φ is the Hamiltonian flow for $H(x,\xi) = 1/2|\xi|^2 + 1/2 \sum_{k=1}^{n} \omega_k^2 x_k^2 + V(x)$.

Indeed, these relations fail for simple reasons. First, the Lagrangian manifolds $\Lambda_{S_{t,y}}$ and phase flow Φ^t are not even conic. Secondly, the amplitude a is not a symbol. Finally, $k(t,x,y)$ is known to be regular for small $|t|$ for a wider class of potentials (cf. [5, 6]). Hence singularities are instantly smoothed out, and the above relations would appear to be vacuous; however, singularities can appear at later times, and so the problem is really to locate them by a suitable replacement of these relations.

Our central point in this paper is that despite such problems the smoothness, decay and reconstruction of singularities for solutions of these Schrödinger equations can in fact be determined from the geometry of the phase flows Φ^t. The idea is this. An oscillatory integral wave function ψ should have a local singularity at x if and only if an "infinite amount" of its lagrangian projects over every neighborhood of x (under the projection $\pi(x,\xi) = x$). Indeed, the lagrangian represents the positions and momentum of the family of classical particles corresponding (in the semi-classical interpretation) to ψ. A singular point x of ψ should therefore correspond to an infinite density of these particles coinciding at x with various different momenta. Further, a co-direction ξ should be singular at such an x if an infinite density of these coinciding particles pass through x with momenta in the ξ-direction (i.e. in every conic neighborhood of ξ).

Now, the unperturbed phase flows Φ_0^t for the Hamiltonians (1)–(3) are not conic, but they are of course linear. Consequently the lagrangian $\Lambda_y^0 = \{(x,\xi)|x = y\}$ for the initial data $x = y$ is carried by Φ_0^t into an affine lagrangian Λ_y^t, the lagrangian for $k_0(t,\cdot,y)$. One can check from the explicit formulas for $k_0(t,x,y)$ (Mehler formulas) that $WF(k_0(t,\cdot,y))$ consists exactly of the (vertical) rays in Λ_y^t, if such exist at time t, as would be predicted from the preceding remarks.

When the Hamiltonians (1)–(3) are perturbed by O-symbols V, the phase flows Φ^t remain asymptotic, as $|x| + |\xi| \to \infty$, to the Φ_0^t. Hence the $\Phi^t\Lambda_y^0$ are asymptotic to the Λ_y^t, and so one would predict that local singularities build up at the same places and in the same directions as for the unperturbed ones. Our main result is that the wave front sets are indeed stable under these perturbations.

This paper contains four sections. In Sect. 1 we treat perturbed free particle Hamiltonians, and show that $k_V(t,x,y)$ is smooth on $R_x^n x R_y^n$ for all t if V is bounded with bounded derivatives. In Sect. 2 we treat perturbed isotropic oscillators. Here we show that the amplitude of k_V inherits enough "symbol properties" from V to allow an analysis of singularities. The main point is to show that when $t = m\pi$, $k_V(t,\cdot,y)$ becomes both rapidly decreasing in x, and regular away from $x = (-1)^n y$, so that this latter point is forced to be singular. In Sect. 3, we derive containment relations for the wave front sets of perturbed anisotropic oscillators. Finally, in Sect. 4 we deal with some routine technical problems which come up in Sects. 1–3 and which are best confined to an appendix.

Section 1. Regularity of Perturbed Free Particle Propagators

In this section we wish to prove:

Theorem I. *Let* $V\in\mathscr{B}_{k+6([n/2])+1}(\mathbb{R}^n)$, *then*

$$k_V(t,x,y)=a(t,x,y)\frac{\exp((i|x-y|^2)/2t)}{(2\pi it)^{n/2}},$$

where $a\in\mathscr{B}_k(\mathbb{R}^n_x\times\mathbb{R}^n_y)$ *for each fixed* t.

Proof. From $(i\partial_t-H_0)U_V=V\cdot U_V$ we get the "Duhamel formula"

$$U_V(t)=U(t)+\frac{1}{i}\int_0^t U(t-s)VU_V(s)ds, \tag{1.1}$$

where U is the free propagator $e^{-it\Delta}$.

Iterating and replacing $U(s_j-s_{j+1})$ by $U(s_j)U(s_{j+1})^{-1}$, we get the norm convergent "Dyson Expansion:"

$$\begin{aligned}U_V(t)=U(t)+\sum_{l=1}^{\infty}(-i)^l\int_0^t\dots\int_0^{s_{l-1}}ds_1\dots ds_lU(t)\\ \cdot[U(s_1)^{-1}VU(s_1)]\dots[U(s_l)^{-1}VU(s_l)].\end{aligned} \tag{1.2}$$

Our first remark is that $U(s_j)^{-1}VU(s_j)$ is a ψDO whose amplitude is bounded with bounded derivatives.

$$\begin{aligned}U(s_j)^{-1}VU(s_j)\phi(z_j)&=\int\int\frac{dw_jdz_{j+1}}{(2\pi is_j)^n}\\ &\cdot\exp\left[i\left(-\frac{|z_j-w_j|^2}{2s_j}+\frac{|z_{j+1}-w_j|^2}{2s_j}\right)\right]V(w_j)\phi(z_{j+1})).\end{aligned} \tag{1.3}$$

Rewrite the phase as $(z_{j+1}-z_j)\cdot\xi_j(s_j,z_j,z_{j+1},w_j)$, where

$$\xi_j(s_j,z_j,z_{j+1},w_j)=\frac{1}{s_j}\left(\frac{z_{j+1}+z_j}{2}-w_j\right). \tag{1.4}$$

Changing variables to ξ_j and noticing that the Jacobian $\left|\frac{\partial w_j}{\partial\xi_j}\right|=s_j^n$ cancels the denominators in (1.3), we get

$$\begin{aligned}&U(s_j)^{-1}VU(s_j)\phi(z_j)\\ &=\int\int\frac{\exp[i(z_{j+1}-z_j)\cdot\xi_j]}{(2\pi i)^n}V\left(\frac{z_{j+1}+z_j}{2}-s_j\xi_j\right)\phi(z_{j+1})dz_{j+1}d\xi_j\\ &=\int p(s_j,z_j,z_{j+1})\phi(z_{j+1})dz_{j+1},\end{aligned} \tag{1.5}$$

with

$$p(s_j,z_j,z_{j+1})=\int\exp[i(z_{j+1}-z_j)\cdot\xi_j]V\left(\frac{z_{j+1}+z_j}{2}-s_j\xi_j\right)\frac{d\xi_j}{(2\pi i)^n}.$$

By hypothesis, $V((z_{j+1}+z_j/2)-s_j\xi_j)\in\mathscr{B}_{k+6([n/2]+1)}\times(\mathbb{R}^n_{z_j}\times\mathbb{R}^n_{\xi_j}\times\mathbb{R}^n_{z_j+1})$, which concludes our first remark..

Next, taking kernels in (1.2) we get

$$k_V(t,x,y) = k_0(t,x,y) + \sum_{l=1}^{\infty} (-i)^l \int_0^t \ldots \int_0^{s_{l-1}} \overbrace{\int \ldots \int}^{l} k_0(t,x,z_1)p(s_1,z_1,z_2)\ldots p(s_l,z_l,y)d^lz. \tag{1.6}$$

We now concentrate on the l^{th} term

$$\frac{(-l)^l}{(2\pi it)^{n/2}} \int_0^t \ldots \int_0^{s_{l-1}} ds_1 \ldots ds_l \overbrace{\int \ldots \int}^{2l} d^lzd^l\xi \exp[i\Phi_l(t,x,\vec{z},\vec{\xi},y)]b_l(t,x,\vec{z},\vec{\xi},y), \tag{1.6l}$$

and

$$\Phi_l(t,x,\vec{z},\vec{\xi},y) = \frac{|x-z_1|^2}{2t} + (z_2 - z_1)\cdot\xi_1 + \cdots + (y - z_k)\cdot\xi_k$$

and

$$b_l(t,x,\vec{z},\vec{\xi},y) = \frac{1}{(2\pi i)^{n\cdot l}}\left(\frac{z_2+z_1}{2} - s_1\xi_1\right)\ldots V\left(\frac{z_k+y}{2} - s_k\xi_k\right),$$

(b_l is independent of x since the amplitude of $k_0(t,x,y)$ is).

To put this term into the desired form $a_l(t,x,y)\cdot\exp((i|x-y|^2)/2t)/(2\pi i)^{n/2}$, we first take the Taylor expansion of Φ_l about its critical point. Evidently,

$$C_{\Phi_l} = \left\{(t,x,\vec{z},\vec{\xi},y)|z_1 = \cdots = z_l = y, \right.$$

$$\left.\xi_1 = \cdots = \xi_l = \frac{x-z_1}{t} = \frac{x-y}{t}\right\}^2. \tag{1.7}$$[1]

Let $\bar{\xi} = (x-y)/t$; therefore $\Phi = (|x-y|^2/2t) + \frac{1}{2}(\vec{z}-y,\vec{\xi}-\bar{\xi})$. Hess (Φ_l) $\begin{bmatrix} \vec{z}-y \\ \vec{\xi}-\bar{\xi} \end{bmatrix}$, where $(\vec{z}-y,\vec{\xi}-\bar{\xi}) = (z_1-y,\ldots,z_l-y|\xi_1-\bar{\xi},\ldots,\xi_1-\bar{\xi})$, whence we get

$$\Phi = \frac{|x-y|^2}{2t} + \frac{1}{2}\frac{|z_1-y|^2}{t} + ((z_2-y)-(z_1-y))(\xi_1-\bar{\xi}) + \cdots +$$
$$+ -(z_l-y)(\xi_l-\bar{\xi}). \tag{1.8}$$

Factoring $\exp((i|x-y|^2)/2t)$ outside the integral, changing variables $\bar{z}_j = (z_j-y), \bar{\xi}_j = (\xi_j - \bar{\xi})$ and dropping the bars, we get for the l^{th} term

$$\frac{\exp((i|x-y|^2)/2t)}{(2\pi it)^{n/2}}a_l(t,x,y), \tag{1.9l}$$

with

$$a_l(t,x,y) = \int_0^t \ldots \int_0^{s_{l-1}} \overbrace{\int \ldots \int}^{2l} \exp\left[\left(\frac{1}{2}\frac{z_1^2}{t} + (z_2-z_1)\xi_1 + \cdots + (-z_l)\xi_l\right)\right]$$

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1 If $\Phi(x,\xi,y)$ is a phase function with (x,y) free variables and ξ the integration variables, $C_\Phi = \{(x,y)|\nabla_\xi\Phi = 0\}$

$$\cdot \prod_{j=1}^{l} V\left(\frac{z_{j+1}+z_j}{2} - s_j\xi_j + \frac{(t-s_j)}{t}y + \frac{s_j}{t}x\right)d\vec{z}d\vec{\xi},$$

and with $z_{l+1}=0$.

We now need to show that $a(t,x,y) = 1 + \sum_{l=1}^{\infty} a_l(t,x,y)$ converges in the space $\mathscr{B}_k(\mathbb{R}^n_x \times \mathbb{R}^n_y)$ for each fixed t,

The convergence proof is an integration-by-parts argument reminiscent of [11, Appendix]. We will break it up into a sequence of four claims; some of the proofs will be deferred to Sect. 4.

Claim I. $\partial_x^\alpha \partial_y^\beta a_l(t,x,y) =$

$$= \int_0^t \cdots \int_0^{s_{l-1}} \overbrace{\int \cdots \int}^{2l} e^{i\Phi_l'} \left[\langle Dz_1\rangle^2 \left(1 + \frac{n}{it} + \left(\frac{z_1}{t} - \xi_1\right)^2\right)^{-1}\right]^{n_0}$$

$$\cdot \prod_{j=2}^{l} \langle \xi_j - \xi_{j-1}\rangle^{-2n_0} \langle D_{z_j}\rangle^{2n_0} \times \prod_{j=1} \langle D_{\xi_j}\rangle^{2n_0} \langle z_{j+1} - z_j\rangle^{-2n_0}$$

$$\cdot \sum_{\substack{|\vec{\alpha}|=|\alpha| \\ |\beta|=|\vec{\beta}|}} \begin{bmatrix}\alpha \\ \vec{\alpha}\end{bmatrix}\begin{bmatrix}\beta \\ \vec{\beta}\end{bmatrix} \prod_{j=1}^{l} V^{(|\alpha_j|+|\beta_j|)}$$

$$\cdot \left(\frac{z_{j+1}+z_j}{2} - s_j\xi_j + \frac{t-s_j}{t}y + \frac{s_j}{t}x\right) \cdot \left(\frac{t-s_j}{t}\right)^{|\beta_j|} \left(\frac{s_j}{t}\right)^{|\alpha_j|} d\vec{z}d\vec{\xi},$$

where $\Phi_l' = \frac{1}{2}(|z_1|^2/t) + (z_2 - z_1)\xi_1 + \cdots + (-z_l)\xi_l$; n_0 is arbitrary, $z_{l+1} = 0$.[2]

Proof. Pass $\partial_x^\alpha \partial_y^\beta$ under the sign of integration in (1.9l); since Φ_l' is independent of (x,y) one may immediately expand

$$\partial_x^\alpha \partial_y^\beta \prod_{j=1}^{l} V\left(\frac{z_{j+1}+z_j}{2} - s_j\xi_j + \frac{(t-s_j)}{t}y + \frac{s_j}{t}x\right)$$

by Leibniz' rule and the chain rule. Next integrate by parts using:

$$\langle z_{j+1} - z_j)\rangle^{-2} \langle D_{\xi_j}\rangle^2 e^{i\Phi'} = e^{i\Phi'},\ j = 1,..l,(z_{l+1}=0), \tag{1.10a}$$

$$\langle \xi_j - \xi_{j-1}\rangle^{-2} \langle D_{z_j}\rangle^2 e^{i\Phi'} = e^{i\Phi'},\ j = 2,\ldots,l, \tag{1.10b}$$

$$\left[1 + \frac{n}{it} + \left(\frac{z_1}{t} - \xi_1\right)^2\right]^{-1} \langle D_{z_1}\rangle^2 e^{i\Phi'} = e^{i\Phi'}, \tag{1.10c}$$

where we recall that $\langle u\rangle = (1+|u|^2)^{1/2}, D_{z_j} = (1/i)\nabla_{z_j}$, $\langle D_{z_j}\rangle^2 = (1 - \Delta_j)$, etc.

Using the product of the operator of (1.10b) to the n_0 power, followed by those of (1.10a) and (1.10c) to the n_0 power, and integrating by parts (taking transposes) we get the claimed expression for $\partial_x^\alpha \partial_y^\beta a_l$.

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2 $\langle u\rangle = (1+|u|^2)^{1/2}$

Claim II. $\partial_x^\alpha \partial_y^\beta a_l$ is a sum of terms of the form

$$\int_0^t \cdots \int_0^{s_{l-1}} \overbrace{\int \cdots \int}^{l} e^{i\Phi'} \cdot \prod_{j=1}^{l} \langle z_{j+1} - z_j \rangle^{-2n_0} \cdot \left[1 + \frac{n}{it} + \left(\frac{z_1}{t} - \xi_1\right)^2\right]^{-n_0}$$

$$\cdot \prod_{j=2}^{l} \langle \xi_j - \xi_{j-1} \rangle^{-2n_0} P_l(s_1 \ldots, s_l, t, \vec{z}, \vec{\xi}) \prod_{j=1}^{l} V_j^{|\alpha_j| + |\beta_j| + \leqq 6n_0},$$

where $\| P_l \|_\infty \leqq C_{n_0}(t)^l$, and $V_j = V((z_{j+1} + z_{z_j})/2 - s_j\xi_j + ((t-s)/t) + (s_j/t)x)$. Here and hereafter $C_{n_0}^l(t)$ is a constant depending only on t and n_0, raised to the l^{th} power.[3]

Proof. We have only applied Leibniz' law to the expression in Claim I. Differentiations of bracket factors such as $\langle z_{j+1} - z_j \rangle^{-2n_0}$ only produce bracket factors to a lower order, and we may absorb the extra decaying factors in P_l. (P_l does not decay altogether, since some terms involve no differentiations of bracket factors.) Differentiations of $V_j^{(|\alpha_j| + |\beta_j|)}$ can go no higher than $6n_0$, since V_j depends only on (z_j, z_{j+1}, ξ_j), and one can only perform $2n_0$ differentiations with respect to each. The factors of s_j may be absorbed in P_l. Proof that $\| P_l \|_\infty \leqq C_{n_0}(t)^l$ and further details will be given in Sect. 4.

Claim III. Each term in the sum of Claim **II** is bounded by $C_{n_0}(t)^l \| V \|_{6n_0 + |\alpha| + |\beta|}^l (t^l/l!)$ for $n_0 \geqq [n/2] + 1$.

Proof. We have only to estimate

$$\int_0^t \cdots \int_0^{s_{l-1}} \overbrace{\int \cdots \int}^{2l} \prod_{j=1}^{l} \langle z_{j+1} - z_j \rangle^{-2n_0} \left| 1 + \frac{n}{it} + \left(\frac{z_1}{t} - \xi_1\right)^2 \right|^{-n_0}$$

$$\cdot \prod_{j-2}^{l} \langle \xi_j - \xi_{j-1} \rangle^{-2n_0} \times \prod_{j=1}^{l} |V_j^{(|\alpha_j| + |\beta_j|) + \leqq 6n_0}| d^l \vec{z} d^l \vec{\xi}.$$

First, change variables to $y_j = z_{j+1} - z_j, \eta_1 = -z_1/t + \xi_1, \eta_j = \xi_j - \xi_{j-1}$ for $j \geqq 2$.

The Jacobian determinant may be computed by adding the l^{th} column to the $(l-1)^{st}$ (note $y_l = - z_l$) and repeating; this puts the matrix in upper triangular form and shows $|\det J| = 1$. Then bound

$$\| V_j^{(\alpha_j + \beta_j) + \leqq 6n_0} \|_\infty \leqq \| V \|_{|\alpha| + |\beta| + 6n_0}.$$

We are then reduced to $\| V \|_{|\alpha| + |\beta| + 6n_0}^l \int_0^t \cdots \int_0^{s_{l-1}} \overbrace{\int\int}^{2l} \prod_{j=1}^{2l} \langle v_j \rangle^{-2n_0} dv_j$, aside from some harmless factors of n/it. For $2n_0 > n$, the integrals converge, so take $n_0 \geqq [n/2] + 1$. Absorbing the bound for $\int \langle v_j \rangle^{-2n_0} dv_j$ into the bound for P_l, and integrating over t we get $\| V \|_{(|\alpha| + |\beta| + [n/2] + 1)}^l C_{n_0}^1(t) (t^l/l!)$ as a bound for the expression above.

Claim IV. The number of terms in the sum of Claim **II** is bounded by $C_{n_0}^l$.

Proof. This is again a consequence of Leibniz's law, and is deferred to Sect. 4. The

3 $V_j^{(|\alpha_j| + |\beta_j| + \leqq 6n_0)}$ is the result of (a) differentiating V $|\alpha_j| + |\beta_j| +$ (no more than $6n_0$) times and then substituting $\cos s_j((z_{j+1} + z_j)/2) - \sin s_j \xi_j + (\sin(t - s_j)/\sin t)y + (\sin s_j/\sin t)x$ in for the argument

main point is that although there are l factors of $\langle z_{j+1} - z_j \rangle^{-2n_0}$ and of V_j, each depends on only two z_j variables; hence the number of terms for the product grows like a power of the number for each factor, which is independent of l.

Modulo the remaining details in part 4, we have proved Claims **I–IV**. Summing up, let us state the

Conclusion. Let $a = 1 + \sum_{l=1}^{\infty} a_l(t,x,y)$. Then if $V \in \mathscr{B}_{k+6([n/2]+1)}$, $a \in \mathscr{B}_k$ for each t.

Proof. According to Claims **I–IV**, $\|a_l\|_{|\alpha|+|\beta|} \leqq C_{n_0}(t)^l(t/l!)\|V\|^l_{(|\alpha|+|\beta|+6[n/2]+1)}$. Summing over l, we get $\|a\|_{|\alpha|+|\beta|} \leqq \exp(tC_{n_0}(t) \cdot \| \quad \|_{(|\alpha|+|\beta|+6[n/2]+1)}$. Taking the maximum over $\|\alpha\| + \|\beta\| \leqq k$ yields the conclusion, and thus the proof of Theorem **I**.

Section 2. Reconstruction of Singularities for Perturbed Oscillator Propagators

In this section we will prove the following theorems:

Theorem II. *Let* $V \in S^0(\mathbb{R}^n), H = -1/2\Delta + \frac{1}{2}|x|^2 + V(x)$ *and* $k_V(t,x,y)$ *be the Schwartz kernel for* $\exp(-itH)$. *Then*

$$\operatorname{sing\,supp} k_V(t,\cdot,y) = \begin{cases} \phi & \text{if } t \neq m\pi \\ \{(-1)^m y\} & t = m\pi. \end{cases}$$

Moreover when $t = m\pi, k_V$ *is rapidly decreasing in* x *away from the singularity.*

Theorem III. *Let* $V \in \mathscr{B}(\mathbb{R}^n), H = -1/2\Delta + \frac{1}{2}|x|^2 + V(x)$, *and* $U(t) = \exp - itH$. *Then* $S(t) = \operatorname{tr} U(t)$ *is a temperate distribution on* $\mathbb{R}$, *and* sing supp $S \subseteq \{2\pi m\}$, *the period set of the unperturbed motion.*

Remark. Most likely, $WF(k_V(m\pi,\cdot,y)) = \{(-1)^m y,\xi) | \xi \in \mathbb{R}^n\}$[4], i.e. there are no regular directions at the singularity. This is certainly predicted by the phase space picture.

The key element in the proof of these theorems is the following description of the amplitude and phase functions of the perturbed propagators:

Definition 2.1. Let $a(x,\xi,y)$ be a complex-valued function on $\mathbb{R}^n_x \times \mathbb{R}^m_\xi \times \mathbb{R}^n_y$. Then a is an isotropic multi-symbol of order 0, written $a \in IS^0(\mathbb{R}^n_x \times \mathbb{R}^m_\xi \times \mathbb{R}^n_y)$ if

(i) $|\partial^\alpha_x \partial^\beta_y \partial^\gamma_\xi a| \leqq A^\rho_{(\alpha,\beta,\gamma)} \langle x \rangle^{-\rho} \langle y \rangle^\rho \langle \xi \rangle^\rho$, $0 \leqq \rho \leqq |\alpha|$,

(ii) $|\partial^\alpha_x \partial^\beta_y \partial^\gamma_\xi a| \leqq B^\rho_{(\alpha,\beta,\gamma)} \langle x \rangle^{\rho} \langle y \rangle^{-\rho} \langle \xi \rangle^\rho$, $0 \leqq \rho \leqq |\beta|$,

(iii) $|\partial^\alpha_x \partial^\beta_y \partial^\gamma_\xi a| \leqq C^\rho_{(\alpha,\beta,\gamma)} \langle x \rangle^{\rho} \langle y \rangle^{\rho} \langle \xi \rangle^{-\rho}$, $0 \leqq \rho \leqq |\gamma|$,

for some constants $A^\rho_{(\alpha,\beta,\gamma)}$ etc. Here $\langle u \rangle = (1 + |u|^2)^{1/2}$. If there are no ξ-variables, i.e. $m = 0$, we speak of an isotropic bi-symbol. The word isotropic is used because differentiations in any component of the x, y or ξ variables produces equal decay in all of them.

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4 This has been verified by Alan Weinstein, in "A symbol class for some Schrödinger Equations on $\mathbb{R}^n$," to appear in the Am. J. Math.

We can now state the basic lemmas.

Lemma 2.I. *Let $H = -1/2\Delta + 1/2|x|^2 + V(x)$, with $V \in S^0(\mathbb{R}^n)$, and let k_V be as above. Then for $t \neq m\pi$,*

$$k_V(t,x,y) = \frac{a(t,x,y)e^{iS(t,x,y)}}{(2\pi i \sin t)^{n/2}}$$

where

$$S(t,x,y) = \frac{1}{\sin t}\left(\cos t\left(\frac{x^2+y^2}{2}\right) - xy\right)$$

is the oscillator action and $a \in IS^0(\mathbb{R}^n_x \times \mathbb{R}^n_y)$.

Lemma 2.II. *With the same hypotheses as above, now let*

$$t = m\pi, \text{ then } k_V(t,x,y) = \int e^{-i(x-(-1)^m y)\cdot\xi}\sigma(x,\xi,y)d\xi,$$

where $\sigma \in IS^0(\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_y)$.

Lemma 2.III. *If we assume only that $V \in \mathscr{B}(\mathbb{R}^n)$, then the same conclusions hold except that $a \in \mathscr{B}(\mathbb{R}^n_x \times \mathbb{R}^n_y)$ and $\sigma \in \mathscr{B}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi \times \mathbb{R}^n_y)$.*

We now proceed to the proofs. There is a good deal of overlap with Sect. 1, but we feel the differences make a separate exposition desirable.

Proof of Lemma 2.I. Start again from the Dyson expansion

$$U_V(t) = U(t) + \sum_{l=0}^{\infty}(-i)^l \int_0^t \dots \tag{2.1}$$

$$\int_0^{s_{l-1}} U(t)[U(s_1)^{-1}VU(s_1)\dots U(s_l)^{-1}VU(s_l)]ds_1\dots ds_l, \tag{2.2}$$

where $U(t)$ is now the oscillator group. For $t \neq m\pi$, the kernel of $U(t)$ is well known

$$k(t,x,y) = \frac{e^{iS(t,x,y)}}{(2\pi i \sin t)^{n/2}}, \tag{2.3}$$

where

$$S(t,x,y) = \frac{1}{\sin t}\left(\cos t\left(\frac{x^2+y^2}{2}\right) - xy\right).$$

$U(s_j)^{-1}VU(s_j)$ is again a ψDO:

$$U(s_j)^{-1}VU(s_j)\phi(z_j) = \int\int \frac{\exp(i[S(s_j,w_j,z_{j+1})] - S(s_j,w_j,z_j))}{(2\pi i \sin s_j)^n} \cdot V(w_j)\phi(z_{j+1})dw_j dz_{j+1}.$$

Writing $S(s_j,w_j,z_{j+1}) - S(s_j,w_j,z_j) = (z_{j+1} - z_j)\xi_j$

$$\xi_j = \frac{1}{\sin s_j}\left(\cos s_j\left(\frac{z_{j+1}+z_j}{2}\right) - w_j\right),$$

changing variables in the integral to ξ_j, we get

$$U(s_j)^{-1} V U(s_j)\phi(z_j) = \int p(s_j,z_j,z_{j+1})\phi(z_{j+1})dz_{j+1}, \tag{2.4}$$

where

$$p(s_j,z_j,z_{j+1}) = \int \frac{\exp(i(z_{j+1}-z_j)\xi_j)}{(-2\pi i)^n} \cdot V\left(\cos s_j\left(\frac{z_{j+1}+z_j}{2}\right) - \sin s_j \xi_j\right)d\xi_j. \tag{2.5}$$

Taking kernels in the Dyson expansion, we get

$$k_V(t,x,y) = k(t,x,y) + \sum_{l=0}^{\infty} (-i)^l \int_0^{s_{l-1}} \overbrace{\int \cdots \int}^{l} k(t,x,z_1)p(s_1,z_1,z_2)\ldots p(s_l,z_l,y)d^l z d^l s. \tag{2.6}$$

Concentrate on the l^{th} term. Substituting in (2.5), we get

$$\int_0^t \cdots \int_0^{s_{l-1}} \overbrace{\int \cdots \int}^{2l} e^{i\Phi_l}(t,x,y,\vec{z},\vec{\xi})\, b_l(s_1,\ldots,s_l,t,x,y,\vec{z},\vec{\xi})d^l z d^l \xi d^l s, \tag{2.7l}$$

where

$$\phi_l = S(t,x,z_1) + (z_2 - z_1)\cdot\xi_1 + \ldots + (y - z_l)\cdot\xi_l, \tag{2.8l}$$

and

$$b_l = \left(\frac{1}{2\pi i}\right)\frac{1}{(2\pi i \sin t)^{n/2}} \sum_{j=1}^{l} V\left(\cos s_j\left(\frac{z_{j+1}-z_j}{2}\right) - \sin s_j \xi_j\right).$$

Then

$$C_{\phi_l} = \{(x,y,\vec{z},\vec{\xi}) | z_1 = \cdots = z_l = y, \xi_1 = \cdots = \xi_l = \frac{1}{\sin t}(\cos y - x)\}. \tag{2.9l}$$

Write $\xi = (1/\sin t)(\cos t\, y - x)$. Taking the Taylor expansion of Φ_l about its critical point, we get

$$\Phi = S(t,x,y) + \frac{1}{2}\frac{\cos t}{\sin t}(z_1 - y)^2 + ((z_2 - y) - (z_1 - y))(\xi_1 - \bar{\xi}) + \ldots + -(z_l - y)(\xi_l - \xi).$$

Changing variables in the integral $\bar{z}_j = z_j - y$, $\xi_j = \xi_j - \bar{\xi}$ and dropping the bars, we get for the l^{th} term

$$\frac{e^{iS(t,x,y)}}{(2\pi i\sin t)^{n/2}}\int_0^t\cdots\int_0^{s_{l-1}}\overbrace{\int\cdots\int}^{2l} e^{i\Phi_l'}b_l'(s,x,y,z,\xi)d^ls^ld^lzd^l\xi, \tag{2.10l}$$

where

$$\Phi_l'=\frac{1}{2}\frac{\cos t}{\sin t}z_1^2+(z_2-z_1)\xi_1+\cdots+(-z_l)\cdot\xi_l, \tag{2.11l}$$

and

$$b_l'=\left(\frac{1}{2\pi}\right)\prod_{j=1}^{l}V\left(\cos s_j\left(\frac{z_{j+1}+z_j}{2}\right)-\sin s_j\xi_j+\frac{\sin(t-s_j)}{\sin t}y+\frac{\sin s_j}{\sin t}x\right).$$

Then

$$k_V(t,x,y)=\frac{e^{iS(t,x,y)}}{(2\pi i\sin t)^{n/2}}\sum_{l=0}^{l}a_l(t,x,y),$$

where $a_0=1$ and for $l>0$,

$$a_l(t,x,y)=\int_0^t\cdots\int_0^{s_{l-1}}\overbrace{\int\cdots\int}^{l} e^{i\Phi_l'}(b_l'(s,x,y,\vec{z},\vec{\xi})\,d^lsd^lzd^l\xi. \tag{2.12l}$$

We now need to show that a_l is a bi-symbol. Again we will break up the proof into a sequence of four claims

Claim I.

$$\partial_x^\alpha\partial_y^\beta a(t,x,y)=\int_0^t\cdots\int_0^{s_{l-1}}\overbrace{\int\cdots\int}^{2l}\frac{d^lsd^lzd^l\xi}{(2\pi i)^l}e^{i\Phi'}$$

$$\cdot\left(\langle D_{z_1}\rangle^2\left(1+\frac{n\cos t}{i\sin t}+\left(\frac{\cos t}{\sin t}z_1-\xi_1\right)^2\right)^{-1}\right)^{n_0}$$

$$\cdot\prod_{j+2}^{l}\langle\xi_{j-1}-\xi_j\rangle^{-2n_0}\langle D_{z_1}\rangle^{2n_0}$$

$$\cdot\prod_{j=1}^{l}\langle z_{j+1}-z_j\rangle^{-2n_0}\langle D_{\xi_1}\rangle^{2n_0}$$

$$\cdot\sum_{\substack{|\alpha_1|+\cdots+|\alpha_l|=|\alpha|\\|\beta_1|+\cdots+|\beta_l|=|\beta|}}\binom{\alpha}{\vec{\alpha}}\binom{\beta}{\vec{\beta}}\prod_{j=1}^{l}\partial_x^{\alpha_j}\partial_y^{\beta_j}V_j, \tag{2.13l}$$

where $V_j=V(\cos s_j((z_{j+1}+z_j)/2)-\sin s_j\xi_j+(\sin(t-s_j)/\sin t)y+(\sin s_j/\sin t)x)$.

Proof. As before, we have rid the phase of dependence on (x,y) so may apply Leibniz laws directly to the amplitude. Then we integrate by parts as before.

Claim II. $\partial_x^\alpha \partial_y^\beta a_l$ is a sum of terms of the form

$$\int_0^t \dots \int_0^{s_{l-1}} \overbrace{\int \dots \int}^{2l} e^{i\Phi'} \left(1 + \frac{n \cos t}{i \sin t} + \left(\frac{\cos t}{\sin t} z_1 - \xi_1\right)^2\right)^{-n_0}$$

$$\cdot \prod_{j=2}^{l} \langle \xi_{j-1} - \xi_j \rangle^{-2n_0} \times \prod_{j=1}^{l} \langle z_{j+1} - z_j \rangle^{-2n_0}$$

$$\cdot P_l(s,t,z,\xi) \times \left(\prod_{j=1}^{l} V_j^{(|\alpha_j| + |\beta_j| + \leqq 6n_0)}\right.$$

$$\left. \cdot \left(\frac{\sin s_j}{\sin t}\right)^{|\alpha_j|} \left(\frac{\sin (t - s_j)}{\sin t}\right)^{|\beta_j|}\right), \tag{2.14l}$$

where $\| P_l \|_\infty \leqq C^l_{n_0, \alpha, \beta}(t)$.

Proof. Same as before; the statement about $\| P_l \|_\infty$ is deferred to Sect. 4.

Claim III. Each term in this sum is bounded by $C_{\alpha,\beta,n_0,r}(t)^l \times \| V \|^l_{|\alpha| + |\beta| + 6n_0} (t^l/l!) \cdot \langle x \rangle^{-r} \langle y \rangle^r$, where $0 \leqq r \leqq |\alpha|$, and similarly if roles of x and y are switched. Here $\| V \|_m$ is the max of the first m 0-order symbol semi-norms of V, and $n_0 > [n/2]$.

Proof. The product

$$\prod_{j=1}^{l} V_j^{(|\alpha_j| + |\beta_j| + \leqq 6n_0)} \times \left|\frac{\sin s_j}{\sin t}\right|^{|\alpha_j|} \left|\frac{\sin (t - s_j)}{\sin t}\right|^{|\beta_j|}$$

is bound by

$$\| V \|^l_{|\alpha| + |\beta| + 6n_0} \cdot \prod_{j=1}^{l} \left\langle \cos s_j \left(\frac{z_{j+1} + z_j}{2}\right) - \sin s_j \xi_j \right.$$

$$\left. + \frac{\sin (t - s_j)}{\sin t} y + \frac{\sin s_j}{\sin t} x \right\rangle^{-(|\alpha|_j + |\beta_j|)} \left|\frac{\sin s_j}{\sin t}\right|^{|\alpha_j|} \times$$

$$\cdot \left|\frac{\sin (t - s_j)}{\sin t}\right|^{|\beta_j|},$$

Since V is 0-symbol, and since any extra bracket factors[5] may be bounded by 1. Now apply the inequality

$$\langle \eta + \xi \rangle^{-1} \leqq \langle \eta \rangle^{-1} \langle \xi \rangle \cdot \sqrt{2}$$

5 By extra, we mean those from the $\leqq 6n_0$ differentiations

with

$$\xi = \cos s_j\left(\frac{z_{j+1}+z_j}{2}\right) - \sin s_j\xi_j + \frac{\sin(t-s_j)}{\sin t}y$$

and $\eta = (\sin s_j/\sin t)x$. Next use $|\sin s_j/\sin t|^{|\alpha_j|}$ to cancel the coefficient of x in $\langle(\sin s_j/\sin t)x\rangle^{-|\alpha_j|}$. Bound the bracket factors to the power $-|\beta_j|$ by 1.

Finally, apply $\langle\eta_1+\eta_2\rangle \leqq \sqrt{2}\langle\eta_1\rangle\langle\eta_2\rangle$,
with

$$\eta_1 = \cos s_j\left(\frac{z_{j+1}+z_j}{2}\right) - \sin\xi_j, \eta_2 = \frac{\sin(t-s_j)}{\sin t}y.$$

Summing up, we get the product bounded by

$$C_{\alpha,\beta}(t)\|V\|^l_{(|\alpha|+|\beta|+6n_0)}\prod_{j=1}^{l}\langle y\rangle^{|\alpha_j|}\langle x\rangle^{-|\alpha_j|}$$
$$\cdot\left\langle\cos s_j\frac{(z_{j+1}+z_j)}{2} - \sin s_j\xi_j\right\rangle^{|\alpha_j|},$$

where we have absorbed factors of $\sqrt{2}$, $t/\sin t$, etc. into $C_{\alpha\beta}(t)$.

Remarks (i). We may of course reverse the roles of x and y in this argument, i.e. bound the bracket factors to the $(-|\alpha_j|)$ power by 1, put x in the numerator, y in the denominator, and cancel the coefficient of y.

(ii) Cancelling coefficients seems necessary to get the decay laws we want. Hence differentiations in x do not produce decays in y or vice versa. However, differentiations in any x-component will produce decay in all x-components because all have the same coefficient.[6]
This is responsible for isotropicity of the symbol.

(iii) We may bound any inverse bracket factor by 1. Hence in estimating x-decay, e.g., we may ignore some of the factors of

$$\left\langle\cos s_j\left(\frac{z_{j+1}+z_j}{2}\right) - \sin s_j\xi_j + \frac{\sin(t-s_j)}{\sin t} + \frac{\sin s_j}{\sin t}x\right\rangle^{-|\alpha_j|}$$

Then going through the steps above with fewer factors, we can bound the product by

$$C_{\alpha,\beta,\vec{r}}(t)\|V\|^l_{(|\alpha|+|\beta|+6n_0)}\prod_{j=1}^{l}\langle y\rangle^{r_j}\langle x\rangle^{-r_j}\times$$
$$\cdot\left\langle\cos s_j\left(\frac{z_{j+1}+z_j}{2}\right) - \sin s_j\xi_j\right\rangle^{r_j},$$

where $0\leqq r_j\leqq|\alpha_j|$. This remark, which will be important later, is responsible for the definition of 0 order bi-symbol given earlier.

6 I.e. $(\sin s_j/\sin t)$

Resuming the proof of Claim III, we have now succeeded in bounding

$$\prod_{j=1}^{l} V_j^{(|\alpha_j|+|\beta_j|+\leqq 6n_0)}\left|\frac{\sin s_j}{\sin t}\right|^{|\alpha_j|}\left|\frac{\sin(t-s_j)}{\sin t}\right|^{|\beta_j|}$$

by

$$C_{\alpha,\beta}(t)\|V\|^{l}_{(|\alpha|+|\beta|+6n_0)}\langle x\rangle^{-|\alpha|}\langle y\rangle^{|\alpha|}$$

$$\times\prod_{j=1}^{l}\left\langle\cos s_j\left(\frac{z_{j+1}+z_j}{2}\right)-\sin s_1\xi_j\right\rangle^{|\alpha_0|},$$

or with $|\alpha_j|$ replaced by $0\leqq r_j\leqq|\alpha_j|$ and $|\alpha|$ replaced by $r=r_1+\cdots+r_l\leqq|\alpha|$. Thus we must only shown that

$$\int_0^t\cdots\int_0^{s_l=1}\overbrace{\int\cdots\int}^{l}\left|\left(1+\frac{n\cos t}{i\sin t}+\left(\frac{\cos t}{\sin t}z_1-\xi_1\right)^2\right)^{-n_0}\right.$$

$$\cdot\prod_{j=2}^{l}\langle\xi_{j-1}-\xi_j\rangle^{-2n_0}\times\prod_{j=1}^{l}\langle z_{j+1}-z_j\rangle^{-2n_0}P_l(s,t,z,\xi).$$

$$C_{\alpha,\beta}(t)\prod_{j=1}^{l}\left\langle\cos s_j\left(\frac{z_{j+1}+z_j}{2}\right)-\sin s_j\xi_j\right\rangle^{|\alpha_j|}$$

$$\leqq C^{l}_{\alpha,\beta,n_0}(t)\frac{t^l}{l!}.$$

As a result of Claim **II** and the fact that $C_{\alpha,\beta}(t)$ is independent of l, we may pull $\|P_l(s,t,\vec{z},\vec{\xi})C_{\alpha,\beta}(t)\|_\infty\leqq C^{l}_{\alpha,\beta,n_0}(t)$ outside the integral. Then we change variables as before, setting

$$\eta_1=\frac{\cos t}{\sin t}z_1-\xi_1,$$

$$\eta_j=\xi_{j-1}-\xi_j,\quad j\leqq 2,$$

$$w_j=z_{j+1}-z_j,\quad z_{l+1}=0.$$

Letting J^{-1} be this linear change of variables, we have $|\det J|=1$ and $|J|\leqq\left|\frac{\cos t}{\sin t}\right|$. Writing

$$z_j=\sum_{1=m}^{l}J_{j,m}w_m \text{ and } \xi_j=\sum_{1=m}^{l}J_{l+j,m}w_m+\sum_{l+1=m}^{2l}J_{l+j,m}\eta_m,$$

we get

$$\prod_{j=1}^{l}\left\langle\cos s_j\left(\frac{z_{j+1}+z_j}{2}\right)-\sin s_j\xi_j\right\rangle^{|\alpha_j|}$$

$$=\prod_{j=1}^{l}\left\langle\sum_{1=m}^{l}\left(\frac{\cos s_j}{2}(J_{j+1,m}+.J_{j,m})+\sin s_jJ_{j+l,m}\right)w_m\right.$$

$$+\sum_{m=l+1}^{2l}\langle\eta_m(\sin s_j J_{l+j,m})\rangle_m{}^{|\alpha_j|}$$

$$\leqq\prod_{j=1}^{l}\left(2^l\cdot\left(\frac{2}{\sin t}\right)^l\left(\frac{1}{\sin t}\right)\langle w_1\rangle\ldots\langle w_l\rangle\langle\eta_1\rangle..\langle\eta_l\rangle\right)^{|\alpha_j|},$$

where we have repeatedly used $\langle u+v\rangle\leqq\sqrt{2}\langle u\rangle\ \langle v\rangle$ and $\langle\lambda u\rangle\leqq|\lambda|\langle u\rangle$ for $|\lambda|>1$. Taking the product over j and recalling $|\alpha_1|+\ldots+|\alpha_l|=|\alpha|$, we get $=(|2/\sin t|^{2|\alpha|})^l\prod_{j=1}^{l}\langle w_j\rangle^{|\alpha|}\prod_{j=1}^{l}\langle\eta_j\rangle^{|\alpha|}$. Absorbing $(|2/\sin t|^{2|\alpha|})$ into C_{α,β,n_0}, and noting that what is left is

$$\int_0^t\ldots\int_0^{s_{l-1}}\prod_{j=1}^{2l}\int\langle\rho_j\rangle^{-2n_0}\langle\rho_j\rangle^{|\alpha|}d\rho_j\leqq C^l_{\alpha,\beta,n_0}\frac{t^l}{l!}$$

for $2n_0-|\alpha|>n$, we can finally conclude the proof of Claim **III**, when $n_0\geqq[n/2]+\max([|\alpha|/2,|\beta|/2])+1$. (Replacing $|\alpha|$ by $r<|\alpha|$ only simplifies the proof.)

Claim IV. The number of terms in the expression for $\partial_x^\alpha\partial_y^\beta a_l(t,x,y)$ is bounded by $C^l_{n_0,\alpha,\beta}$.

Proof. This consequence of Leibniz laws will be checked in Sect. 4. The details are identical to those in Sect. 1.

Summing up, Claims **I–IV** imply that

$$|\partial_x^\alpha\partial_y^\beta a(t,x,y)\leqq\sum_1^\infty|\partial_x^\alpha\partial_y^\beta a_l(t,x,y)|\leqq\exp(tC_{\alpha,\beta,n_0,r}(t))\ \langle x\rangle^{-r}\langle y\rangle^r,$$

where $0\leqq r\leqq|\alpha|, n_0\geqq[n/2]+\max\{[|\alpha|/2],[|\beta|/2]\}+1$. And likewise for y. Thus $a\in IS^0(\mathbb{R}^n_x\times\mathbb{R}^n_y)$.

Proof of Lemma 2.**II**. Simply write

$$k_V(m\pi,x,y)=\int d\xi\, k_V\left(\frac{\pi}{2},x,\xi\right)k_V\left(m\pi-\frac{\pi}{2},\xi,y\right)$$

$$=\int d\xi\, a\left(\frac{\pi}{2},x,\xi\right)a\left(m\pi-\frac{\pi}{2},\xi,y\right)\exp\left\{i\left[\left(S\frac{\pi}{2},x,\xi\right)+S\left(m\pi-\frac{\pi}{2},\xi,y\right)\right]\right\}$$

by Lemma 2.1.

Since $a(\pi/2,x,\xi)$, $a(m\pi-\pi/2,\xi,y)\in IS^0(\mathbb{R}^n\times\mathbb{R}^n)$ their product σ is *a fortiori* in $IS^0(\mathbb{R}^n_x\times\mathbb{R}^n_\xi\times\mathbb{R}^n_y)$. In fact, of course, differentiations in x produce decay in x independently of y; however, this observation plays no essential role, so we ignore it. Then note that the phase is $-\xi\cdot(x-(-1)^m y)$. This concludes the proof.

Proof of Lemma 2.**III**. Identical to the proofs of Lemma 2.**I** and 2.**II** except for *Claim* **III**. Now we only assert the analogue of that in Sect. 1, namely that each term is bounded by $C^l_{\alpha,\beta,n_0}(t)\|V\|^l_{|\alpha|+|\beta|+6n_0(t^l/l!)}$, where $\|\ \|_k$ is the C^k norm rather than a symbol norm. Here $n_0>[n/2]$ as in Sect. 1.

Theorems **II** and **III** follows easily from these lemmas.

Proof of Theorem **II**. By *Lemma* 1.**I**, sing supp $k_V(t,\cdot,y)=\phi$ if $t\neq m\pi$. If $t=m\pi$, we may write by *Lemma* 2.**II**

$$k_V(m\pi,x,y)=\int e^{i(x-(-1)^m y)\cdot\xi}\sigma(x,\xi,y)d\xi,$$

with $\sigma\in IS^0(\mathbb{R}^n_x\times\mathbb{R}^n_\xi\times\mathbb{R}^n_y)$. For $x\neq(-1)^m y$ we may integrate by parts using

$$L=\frac{(1/i)(x-(-1)^m y)\cdot\partial_\xi}{|x-(-1)^m y|^2},\xi$$

whence for any r

$$k_V(m\pi,x,y)=\int e^{-i(x-(-1)m_y)\cdot\xi}(L^t)^r\sigma(x,\xi,y)d\xi.$$

L^t has two nice effects on σ: since σ is isotropic, L lowers its order in ξ; and $\|L^t\sigma\|_\infty=0(1/|x-(-1)^m y|)$, so for fixed y, L^t introduces decay in x. However these effects compete, since to estimate decay in ξ one must compensate with growth in x. However we only need enough decay in ξ to render the integral absolutely convergent. So we apply $(L^t)^{n+1-k}$ and set $\rho=n+1$ in Definition 2.1 to get

$$|(L^t)^{n+1+k}\sigma(x,\xi,y)|\leqq|x-(-1)^m y|^{-k-(n+1)}\cdot\langle\xi\rangle^{-n+1}\langle x\rangle^{n+1}\langle y\rangle^{n+1},$$

which for fixed y is $0(|x|^{-k})$. Since x is arbitrary, and the integral converges we have $|k_V(m\pi,x,y)|=0(|x|^{-k})$ for all k, as desired.

Finally we note that $k_V(m\pi,\cdot,y)$ cannot be locally L^2 near $(-1)^m y$ else it would be globally L^2 in x. But then $U_V(-m\pi)k_V(m\pi,\cdot,y)$ would be L^2, a contradiction since it is $\delta(x-y)$. Hence sing supp $k_V(m\pi,\cdot,y)=\{(-1)^m y\}$.

Proof of Theorem **III.** It is well known that $S(t)=\operatorname{tr}U(t)$ is a temperate distribution on $\mathbb{R}$. We briefly recall this proof. For $\theta\in\mathscr{S}(\mathbb{R}^n)$, define $U_\theta=\int_{\mathbb{R}}\theta(t)u(t)dt$. Since $i\partial_t U=HU$ one has by partial integrations that $U_\theta=\int(H^{-k})U(t)\cdot(i\partial_t)^k\theta(t)dt$. Since this holds for any k, one knows that $U_\theta:\mathscr{S}'\to\mathscr{S}$ is continuous and so its kernel $U_\theta(x,y)$ is $\mathscr{S}(\mathbb{R}^n_x\times\mathbb{R}^n_y)$, therefore $\theta\in\mathscr{S}(\mathbb{R}^n)\to\operatorname{tr}(U_\theta)=\int_{\mathbb{R}^n}U_\theta(x,x)\,dx=\langle S(t),\ \theta(t)\rangle$ defines a continuous linear functional. Then

$$S(t)=\int k(t,x,x)dx=\int\int a\left(\frac{t}{2},x,z\right)a\left(\frac{t}{2},z,x\right)e^{i\Phi(t,x,z)}dx\,dz,$$

where by *Lemma* 2.**II** the amplitude is $\mathscr{B}(\mathbb{R}^n_x\times\mathbb{R}^n_z)$ if $t=2m\pi$ and $\Phi=(1/\sin(t/2))(\cos(t/2))(x^2+z^2)-2xz)$. Then.

$$\begin{cases}(1-\Delta x)e^{i\Phi}=\left(1+\dfrac{2n}{i}+\dfrac{4}{\sin^2(t/2)}(\cos t/2x-z)^2\eta e^{i\Phi}=\rho(x,z)e^{i\Phi}\right.\\[2ex](1-\Delta z)e^{i\Phi}=\left(1+\dfrac{2n}{i}+\dfrac{4}{\sin^2(t/2)}(\cos t/2z-x)^2\right)e^{i\Phi}=\rho(z,x)e^{i\Phi}\end{cases}$$

So $S(t)=\int\int e^{i\Phi}((1-\Delta_x)\rho(x,z)^{-1})^{n_0}((1-\Delta_z)\rho(z,x)^{-1})^{n_0}\times a(t/2,x,z)a(t/2,z,x)$ and as in the proof of Lemmas 2.**I**, 2.**II** this is bounded by $C_{n_0}\|a\|^2_{4n_0}\times$

$\int\int \rho(x,z)^{-n_0}\rho(z,x)^{-n_0}dxdz$. Changing variables to

$$\begin{cases} \xi_1 = \cos t/2x - z, \text{ with Jacobian } \cos^2 t/2 - 1 \neq 0 \\ \xi_2 = \cos t/2z - x \end{cases}$$

for $t \neq 2m\pi$, the integral is bounded by $C(t)\int\int \langle \xi_1 \rangle^{-n_0} \langle \xi_2 \rangle^{-n_0} d\xi_1 d\xi_2 < \infty$ for $n_0 > [n/2]$. But a and Φ are continuous in t away from $t = 2m\pi$, and the estimates are uniform in t in compact sets away from $2\pi\mathbb{Z}$, so $S(t)$ is continuous there as well.[7] So sing supp $S(t) \subseteq \{2m\pi\}$. Q.E.D.

Section 3. Reconstruction of Singularities for Perturbed Anisotropic Oscillator Propagators

In this section we wish to explain the modifications of Sect. 2 needed to handle anisotropic oscillators. In particular, the amplitudes of the perturbed propagators will now be anisotropic symbols, and the locus of singularities will lose the isotropicity of Sect. 2.

We will prove:

Theorem IV. *Let* $V \in S^0(\mathbb{R}^n)$, $H = -1/2\Delta + \sum_{k=1}^{n} \omega_k^2 x_k^2 + V(x)$, *and* $k_V(t,x,y)$ *be the Schwartz kernel for* $\exp(-itH)$. *Assume that the* $\{\omega_i\}$ *are irrationally related, then* $WF(k_V(t,\cdot,y)) \subseteq WF(k(t,\cdot,y))$.

Remarks. We assume $\{\omega_i\}$ are irrationally related for simplicity. If some are equal, and the rest irrationally related, the conclusion would still follow. If some are unequal but rationally related, it seems we cannot describe the wave front set as precisely as in Theorem **IV**. This will be explained in remarks during the proof.

First, we summarize how the amplitudes and phases change when the oscillations are anisotropic.

a) The phase is now $S(t,x,y) =$

$$\sum_{k=1}^{n} \frac{\omega_k}{\sin \omega_k t}\left[\cos \omega_k t \frac{x_k^2 + y_k^2}{2} - x_k y_k\right].$$

b) The unperturbed propagator $k(t,x,y) =$

$$\left(\sum_{k=1}^{n} \sqrt{\frac{\omega_k}{2\pi i \sin \omega_k t}}\right) e^{iS(t,x,y)}.$$

c) We now define ordinary bi-symbols and multi-symbols by a component-by-component rewording of (Definition 2.1*a*). Definition 3.1*a*): Let $a(x,y)$ (respectively (x,ξ,y)) be a complex-valued function on $\mathbb{R}_x^n \times \mathbb{R}_y^n$ (respectively $\mathbb{R}_x^n \times \mathbb{R}_\xi^n \times \mathbb{R}_y^n$); then $a \in S^0(\mathbb{R}_x^n \times \mathbb{R}_y^n)(S^0(\mathbb{R}_x^n \times \mathbb{R}_\xi^n \times \mathbb{R}_y^n))$ if

7 Smoothness is proved in the same way as continuity

(i) $|\partial_x^\alpha \partial_y^\beta a| \leqq A^{\vec{\rho}}_{(\alpha,\beta)} \langle x_1 \rangle^{-\rho_1} \langle y_1 \rangle^{\rho_1} \cdots \langle x_n \rangle^{-\rho_n} \langle y_n \rangle^{\rho_n}$,

with $0 \leqq \rho_k < |\alpha_k|$. Analogously for y.

(ii) $|\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma a| \leqq A^{\vec{\rho}}_{(\alpha,\beta,\gamma)} \prod_{k=1}^{n} \langle x_k \rangle^{-\rho_k} \langle y_k \rangle^{\rho_k} \langle \xi_k \rangle^{\rho_k}$

with $0 \leqq \rho_k \leqq |\alpha_k|$; analogously for y and ξ.

Then the analogue of Lemma 3.I is:

Lemma 3.1. *With H and k_V as above, let us assume $t \neq m\pi/\omega_k$ for $k = 1, \ldots, n$. Then $k_V = a(t,x,y)k(t,x,y)$ with $a \in S^0(\mathbb{R}^n_x \times \mathbb{R}^n_y)$.*

Proof. All goes the same as in Lemma 2.I up to (2.1.0l). We now get:
The lth term in the Dyson expansion is

$$k(t,x,y)\cdot \int_0^t \ldots \int_0^{s_{l-1}} \int \ldots \int e^{i\Phi'_l} b'_l(s,x,y,z,\xi) d^l s\, d^l z^l d\xi, \tag{3.1.0l}$$

where

$$\Phi'_l = \tfrac{1}{2} \sum_{k=1}^{n} \frac{\cos \omega_k t}{\sin \omega_k t} (z_1^k)^2 + \sum_{j=1}^{l} (z_{j+1} - z_j) \cdot \xi_j$$

and

$$b'_l = \prod_{j=1}^{l} V\left(\cos \omega_k s_j \left(\frac{z_{j+1}^k + z_j^k}{2} \right) - \sin \omega_k s_j \xi_j^k \right.$$
$$\left. + \frac{\sin \omega_k (t - s_j)}{\sin t} y^k + \frac{\sin \omega_k s_j}{\sin \omega_k t} x^k \right), \tag{3.1.1l}$$

where $k = 1, \ldots, n$ and the arrow denotes the vector with those components, e.g. $(\vec{z}_j) = (z_j^1, \ldots, z_j^n)$.

We integrate by parts exactly as before except that now

$$\langle D_{z_1} \rangle^2 e^{i\Phi'_l} = \left(1 + \frac{1}{i} \left(\sum_{k=1}^{n} \frac{\cos \omega_k t}{\sin \omega_k t} \right) + \left| \left(\overrightarrow{\frac{\cos \omega_k t}{\sin \omega_k t} z_1^k - \xi_1^k} \right) \right|^2 \right) \times e^{i\Phi'_l}.$$

(Write the parenthetical expression on the right as ρ.) All else is as before.

For Claim **II** we now need to change the chain rule factors to

$$\prod_{k=1}^{n} \left| \frac{\sin \omega_k s_j}{\sin \omega_k t} \right|^{\alpha_j^k} \left| \frac{\sin \omega_k (t - s_j)}{\sin \omega_k t} \right|^{\beta_j^k},$$

where $\alpha_j = (\alpha_j^1, \ldots, \alpha_j^n)$, etc.

Claim **III** is where a real change is needed. Indeed, let us now do the cancellations last. Bounding each $V_j^{(|\alpha_j| + |\beta_j| + \leqq 6n_0)}$ by its norm times its bracket[8] and using $\langle u + v \rangle^{-1} \leqq \sqrt{2} \langle u \rangle \langle v \rangle^{-1}$ to put (ξ, z) dependence in the numerator, we may now write

8 I.e. the bracket of its argument

Claim III. Each term is bounded by

$$\int_0^t \cdots \int_0^{s_{l-1}} \prod_{j=1}^{l} \left\langle \left(\frac{\sin \omega_k (t - s_j)}{\sin \omega_k t} \overrightarrow{y^k} + x^k \frac{\sin \omega_k s_j}{\sin \omega_k t} \right) \right\rangle^{-|\alpha|_j}$$

$$\cdot \left(\frac{\sin \vec{\omega}_k s_j}{\sin \omega_k t} \right)^{\alpha_j} \times \overbrace{\int \cdots \int}^{2l} P_l(s, t, \vec{z}, \vec{\xi})$$

$$\cdot \rho^{-n_0} \prod_{j=2}^{l} \langle \xi_{j-1} - \xi_j \rangle^{-2n_0} \times \prod_{j=1}^{l} \langle z_{j+1} - z_j \rangle^{-2n_0}$$

$$\cdot \prod_{j=1}^{l} \left\langle \left(\cos \omega_k s_j \left(\frac{z_{j+1}^k + z_j^k}{2} \right) - \overrightarrow{\sin \omega_k s_j \xi_j^k} \right) \right\rangle^{|\alpha_j|}$$

$$\leqq C^l_{\alpha, \beta, n_0}(t) \int_0^t \cdots \int_0^{s_{l-1}} \prod_{j=1}^{l} \left\langle \frac{\sin \omega_k (t - s_j)}{\sin \omega_k t} y^k + \frac{\sin \omega_k s_j}{\sin \omega_k t} x^k \right\rangle^{-|\alpha_j|} \times \left| \frac{\sin \omega_k s_j}{\sin t} \right|^{|\alpha_j|},$$

where $(\vec{u}_j)^{\alpha_j} = (u_j^1)^{\alpha_j^1} \dots (u_j^n)^{\alpha_j^n}$.

The proof is exactly as before, as is the proof of Claim **IV**. Again, n_0 depends only on the dimension, and $|\alpha|$.

Summing up,

$$\partial_x^\alpha \partial_t^\beta a = \sum_1^\infty \partial_x^\alpha \partial_y^\beta a_l,$$

where

$$|\partial_x^\alpha \partial_y^\beta a_l| \leqq C^l_{\alpha, \beta}(t) \int_0^t \cdots \int_0^{s_{l-1}} d^l s \prod_{k=1}^{l}$$

$$\cdot \left\langle \frac{\sin \omega_k (t - s_j)}{\sin \omega_k t} \overrightarrow{y^k} + \frac{\sin \omega_k s_j}{\sin \omega_k t} x^k \right\rangle^{-\alpha_j^k} \left| \frac{\sin \omega_k s_j}{\sin \omega_k t} \right|^{\alpha_j^k}.$$

Now the chain rule factors $|\sin \omega_k s_j / \sin \omega_k t|$ can only cancel the coefficients of x^k; the remaining components in the corresponding bracket factors do not go to zero as $|x| \to \infty$ uniformly in s_j after cancellation. However we apparently require this uniformity to get a $1/l!$ in the estimate on this term. So it appears the best we can do is (i) use $(1 + |u|^2)^{-1} \leqq (1 + |u_k|^2)^{-1}$ to ignore badly behaved components, then (ii) use

$$\left\langle \frac{\sin \omega_k (t - s_j)}{\sin \omega_k t} y^k + \frac{\sin \omega_k s_j}{\sin \omega_k t} x^k \right\rangle^{-\alpha_j^k} \left| \frac{\sin \omega_k s_j}{\sin \omega_k t} \right|^{\alpha_j^k}$$

$$\leqq C(t) \langle x^k \rangle^{-\alpha_j^k} \langle y^k \rangle^{\alpha_j^k}$$

(or more generally $\leqq C(t) \langle x^k \rangle^{-\rho_j^k}$ with $\rho_j^k \leqq \alpha_j^k$) and finally, (iii) integrate in $d^l s$ to get

$$\leqq C^l_{\alpha, \beta}(t) \cdot \prod_{j=1}^{l} \prod_{k=1}^{n} \langle x^k \rangle^{-\rho_j^k} \langle y^k \rangle^{\rho_j^k} \frac{t^l}{l!}$$

$$= C^l_{\alpha,\beta}(t) \prod_{k=1}^{n} \langle x \rangle^{-\rho^k} \langle y \rangle^{\rho^k} \frac{t^l}{l!}, \quad \text{where} \quad \rho^k \leqq |\alpha^k|.$$

Summing in l then gives the desired conclusion of Lemma 3.**I.**

Remarks. To see that our method requires this cancellation of coefficients, consider $a_1(t,x,y)$ in dimension 2. We bound $\partial^m_{x_1} a_1$ by the function

$$\int_0^t dx \left\langle \left(\frac{\sin \omega_1(t-s)}{\sin \omega_1 t} y^1 + \frac{\sin \omega_1 s}{\sin \omega_1 t} x^1 \frac{\sin \omega_2(t-s)}{\sin \omega_2 t} y^2 + \frac{\sin \omega_2 s}{\sin \omega_2 t} x^2 \right) \right\rangle^{-m} \left| \frac{\sin \omega_1 s}{\sin \omega_1 t} \right|^m.$$

Fix y and x^1 and consider decay in x^2. By dominated Convergence, the integral goes to zero; however we have asked in the lemmas for a high rate of decrease. We may estimate this rate by $\int_0^t ds \langle \sin \omega_2 s x^2 \rangle^{-m} |\sin \omega_1 s|^m$. Now assume the anisotropicity condition that ω_1 and ω_2 are irrationally related. We then claim that $\int_0^t ds \langle \sin \omega_2 s x^2 \rangle^{-m} |\sin \omega_1 s|^m$ can decay no more rapidly than $\int_0^t \chi(\sin \omega_2 s x^2) ds$, where χ is the characteristic function of $[-1, 1]$. Indeed, $\sin \omega_1 s$ is bounded above zero on some fixed intervals around those $\{m\pi/\omega_2\}$ in $[0, t]$. But for large enough x^2, $\chi(\sin \omega_2 x^2)$ will be zero off those intervals anyway. Thus the $|\sin \omega_1 s|$ can't affect the decay rate, and of course χ decays more rapidly than $\langle \cdot \rangle^{-m}$ for any m.

However $\int_0^t \chi(\sin \omega_2 s x^2) ds$ just counts the amount of time that $\sin \omega_2 s x^2$ spends in $[-1, 1]$, and if any $m\pi/\omega_2 \in (0,t)$ this is $\sim$ const $1/|x^2|$.

So our bound function for $\partial^m_{x_1} a_1$ cannot decay more rapidly than $\langle x \rangle^{-1}$ as $|x| \to \infty$, which is not good enough to allow our analysis of singularities.

(2) Of course if some of the ω_i are equal, one gets an isotropic decay in their respective directions. If two are pairwise rationally related, there are some obvious relations between differentiations in one of the directions and decay in the other. We ignore these possibilities for simplicity, and assume the frequencies are irrationally related.

Now let us prove Theorem **IV**:

Proof. For $t \neq m\pi/\omega_i, i = 1, \ldots, n$ we know from Lemma 3.1 that $WF(k(t, \cdot, y)) = \phi$.

Now let $t = m\pi/\omega_1$, say. We need to show

$$WF\left(k_V \frac{m\pi}{\omega_1}, \cdot, y\right) = \{((-1)^m y_1, {*}\ldots, {*}\xi_1, 0, \ldots, 0)\},$$

where * denotes a free entry.

Write $k_V(t, x, y) = \int k_V(t - \pi/2\omega_1, x, z) k(\pi/2\omega_1, z, y) dz$.

Also write the action function S as $\sum_{k=1}^{n} S_k(t, x_k, y_k)$.

From Lemma 3.**I**,

$$k_V = \int \exp\left(i\left\{S\left(t - \frac{\pi}{2\omega_1}, x, z\right) + S\left(\frac{\pi}{2\omega_1}, z, y\right)\right\}\right) a\left(t - \frac{\pi}{2\omega_1}, x, z\right) a\left(\frac{\pi}{2\omega_1}, z, y\right) dz.$$

Now $S(m\pi/\omega_1 - \pi/2\omega_1, x, z) + S(\pi/2\omega_1, z, y) = -\omega_1(x_1 - (-1)y_1) + \Phi$, where $\Phi = \sum_{k \neq 1} S_k(m\pi/\omega_1 - \pi/2\omega_1, x_k, z_k) + S(\pi/2\omega_1, z_k, y_k)$. Our first observation is that if we integrate out the $(z_2, \ldots, z_n)$ variables, we will be left with a symbol in z_1.
Namely, let $\sigma(x, z_1, y) = \int \ldots \int dz_2 \ldots dz_n e^{i\Phi} a((2m-1)\pi/2\omega_1, x, z) a(\pi/2\omega_1, z, y)$. Then Φ is independent of z_1, so

$$\partial_{z_1}^r \sigma = \int \ldots \int dz_2 \ldots dz_n e^{i\Phi} \sum_{r_1 + r_2 = r} \binom{r}{r_1} \partial_{z_1}^{r_1} a\left(\frac{(2m-1)}{2\omega_1}\right) \partial_{z_1}^{r_2} \times a\left(\frac{\pi}{2\omega_1}\right).$$

Then integrate by parts using

$$(1 - \partial_{z_k}^2) e^{i\Phi} = \left(1 + \frac{1}{i}\gamma_k + (\partial_{z_k}\Phi)^2\right) e^{i\Phi},$$

where

$$\gamma_k = \frac{\sin \frac{\omega_k}{\omega_1} m\pi}{\sin \frac{\omega_k}{\omega_1}\left(\frac{(2m-1)\pi}{2}\right) \sin \frac{\omega_k}{\omega_1}\frac{\pi}{2}},$$

so

$$\partial_{z_1}^r \sigma = \int \ldots \int dz_2 \ldots dz_n e^{i\Phi} \prod_{k=2}^{n} \{(1 - \partial_{z_k}^2)(1 + (1/i)\gamma k + (\partial_{z_k}\Phi)^2)^{-1}\}^{n_0}$$

$$\cdot \sum_{r_1 + r_2 = r} \binom{r}{r_1} \partial_{z_1}^{r_1} a_1 \partial_{z_1}^{r_2} a_2.$$

As usual we can push the derivatives past the convergence factors, eventually arriving at sums of terms of the form

$$\int \ldots \int dz_2 \ldots dz_r e^{i\Phi} \prod_{k=2}^{n} \left(1 + \frac{1}{i}\gamma_k + (\partial_{z_k}\Phi)^2\right)^{-n_0}$$

$$\cdot P(x, y, z_2 \ldots z_n) D_{z'}^{\alpha} \partial_{z_1}^{r_1} a\left(\frac{2m-1}{2\omega_1}\pi\right) \partial_{z_1}^{r_2} a\left(\frac{\pi}{2\omega_1}\right),$$

where $z' = (z_2, \ldots, z_n)$.
Then $|D_{z'}^{\alpha} \partial_{z_1}^{r_1} a_1 \partial_{z_1}^{r_2} a_2| \leqq C_{\alpha,m} \langle z_1 \rangle^{-r} \langle y_1 \rangle^r \langle x_1 \rangle^r$, where we use $\rho_k = 0$ for $k = 2, \ldots, n$. P is bounded, so each term is bounded by a constant C_r times

$$\langle z_1 \rangle^{-r} \langle x_1 \rangle^r \langle y_1 \rangle^r \int \ldots \int dz_1 \ldots dz_n \prod_{k=2}^{n} \left(1 + \frac{1}{i\gamma_k} + (\partial_{z_k}\Phi)^2\right)^{-n_0}.$$

This integral is a product of one dimensional integrals. Since $\partial_{z_k}\Phi$ is affine in z_k with a non-vanishing coefficient of z_k (due to anisotropicity), the integrals converge as long as $n_0 \geqq 1$.

Thus $|\partial_{z_1}\sigma| \leqq C_r \langle z_1 \rangle^{-r} \langle x_1 \rangle^r \langle y_1 \rangle^r$. Moreover

$$k_V(m\pi, x, y) = \int \exp(-i\omega_1(x_1 - 1)^m y_1) \cdot z_1) \sigma(x, z_1, y) dz_1.$$

Clearly, then, if $x_1 \neq (-1)^m y_1$ we may integrate by parts as before using $i/\omega_1(x_1 - (-1)^m y \partial_{z_1}$ repeatedly to render the integral absolutely convergent, and uniformly in (x, y) so that $k(m\pi, x, y)$ is regular in x for x away from $\{(-1)^m y_1, *, \ldots, *)\}$. Thus far we have determined sing supp $k_V(m\pi, \cdot, y)$. At a point $\bar{x} = ((-1)^m y_1, \bar{x}_2, \ldots, \bar{x}_m) \in$ sing supp $k(m\pi, \cdot, y)$ we must find $\Sigma_{\bar{x}}$.

Let V be the open conical set $\{\eta \in \mathbb{R}^n | \eta_j \neq 0$ for some $j = 2, \ldots, n\}$. We sketch the proof that for $\xi \in V$ there is a $\phi \in C_0^\infty$, $\phi(\bar{x}) = 1$ but

$$\widehat{\phi \cdot k_V}(m\pi/\omega_1, \cdot, y)(\tau\xi) = 0(\tau^{-N}) \text{ for all } N.$$

The left side is

$$\int\int \phi(x)\sigma(x, z_1, y) \exp(-i\{\omega_1(x_1 - (-1)^m y_1) \cdot z_1 + \tau x \cdot \xi\}) \, dz_1 \, dx.$$

Then integrate by parts with $(1 + (\omega_1 z_1 + \tau\xi_1)^2)^{-1}(1 - \partial_{x_1})^2$ once to insure convergence in dz_1. Next integrate by parts in $(\tau\xi_j)^{-1}\partial_{x_j}$ N times with all j such that $\xi_j \neq 0$. Since ϕ provides convergence in dx, the integral converges and is $0(\tau^{-N})$ for all N. Then $\xi \in V$ are all in the complement of $\Sigma_{\bar{x}}$, so $\Sigma_{\bar{x}} = \{(\xi_1, 0, \ldots, 0) | \xi_1 \in \mathbb{R}\}$ as claimed.

This concludes the proof of Theorem IV.

Remarks. The same proof works if some ω_i are identical. But if, say $\omega_1 = 1$, $\omega_2 = 2$ then the unperturbed oscillators has singular support at $(-y_1, y_2)$ at $t = \pi$. Factors of $\sin 2s$ can cancel those of $\sin s$, but not vice versa, so the ∂_{x_1} derivatives of the amplitudes decay in x_1 but not necessarily to the same rate in x_2. At $t = \pi$, one can write

$$k(\pi, x, y) = \int \exp[i((x_1 + y_1)z_1 + (x_2 - y_2)z_2)]\sigma(x, z, y) \, dz,$$

but now σ is not isotropic. So integrating by parts as in the isotropic case does not yield convergence; one has to use instead $(1/i)(1/(x_1 + y_1))\partial_{z_1}$, i.e. to assume $x_1 \neq y_1$. So *a priori* the singular support $= \{(-y_1, *)\}$. This seems unlikely, but cannot as yet be disproved.

Section 4. Details from Sects 1, 2, and 3

The purpose of this section is to fill in the gaps from Sects 1, 2, and 3.

First, we must make copious use of Leibniz's laws to settle the claims in 1 – 3. We need to show that the operators

$$L_{l,n_0} = ((1 - \Delta_{z_1})\rho^{-1})^{n_0} \prod_{j=2}^{l} \langle D_{z_j} \rangle^{2n_0} \prod_{j=1}^{l} \langle z_{j+1} - z_j \rangle^{-2n_0}$$

may be written

$$L_{l,n_0} = \sum_{|\alpha| \leqq 2n_0} \rho^{-n_0} \prod_{j=1}^{l} \langle z_{j+1} - z_j \rangle^{-2n_0} P_{\alpha,n_0} D^\alpha, \tag{4.1}$$

where

$$\rho = \left(1 + \frac{n \cos t}{i \sin t} + \left(\frac{\cos r}{\sin t} z_1 - \xi_1\right)^2\right)$$

(or analogously in other cases), $\| P_{\vec{\alpha}, n_0} \|_\infty \leqq C^l_{n_0}(t)$, and the number of terms is $\leqq C^l_{n_0}$. Here as before $C_{n_0}(t)$ is a constant depending only on (n_0, t); we do not relabel from step to step.

To see this, we apply Leibniz law in the form of [9], p. 10). The operators $\langle D_{zj} \rangle^{2n_0}$ are of the form $P(D_{zj})$ with $P(\xi) = \langle \xi \rangle^{2n_0}$. They are constant coefficient operators, so

$$\langle D_{z_k} \rangle^{2n_0} \prod_{j=1}^{l} \langle z_{j+1} - z_j \rangle^{-2n_0} = \sum_{\alpha_k} \frac{1}{\alpha_k!} D^{\alpha_k}_{z_k} \left(\prod_{j=1}^{l} \langle z_{j+1} - z_j \rangle^{-2n_0} \right) P^{(\alpha_k)}(D_{z_k}), \tag{4.2}$$

where $P^{(\alpha)}(\xi) = \partial^{|\alpha|} P(\xi) / (\partial \xi_1^{\alpha_1} \cdots \partial \xi_n^{\alpha_n})$. Of course $|\alpha_k| \leqq 2n_0$. Iterating, we get:

$$\begin{aligned} &\prod_{k=2}^{l} \langle D_{z_k} \rangle^{2n_0} \prod_{j=1}^{l} \langle z_{j+1} - z_j \rangle^{-2n_0} \\ &\quad = \sum_{(\alpha_2, \ldots, \alpha_l)} \frac{1}{\alpha_2! \ldots \alpha_1!} D^{\alpha_l}_{z_l} \cdots D^{\alpha_2}_{z_2} \left(\prod_{j=1}^{l} \langle z_{j+1} - z_j \rangle^{-2n_0} \right) \\ &\quad \cdot P^{\alpha_l}(D_l) \cdots P^{\alpha_2}(D_2). \end{aligned} \tag{4.3}$$

Next, we unravel $((1 - \Delta z_1)\rho^{-1})^{n_0}$. Again,

$$(1 - \Delta_{z_1})\rho^{-1} = \sum_{\alpha_1} \frac{1}{\alpha_1^1!} D^{\alpha_1^1}_{z_1}(\rho^{-1}) P^{\alpha_1^1}(D_{z_1}),$$

where $\bar{P}(\xi) = (1 + |\xi|^2)$ and $\bar{P}^{\alpha_1^1}(\xi) = (\partial^{|\alpha|_1^1} \bar{P} / \partial^{\alpha_{11}^1}_{\xi_1} \cdots \partial^{\alpha_{1n}^1}_{\xi_n}$.

Iterating, we get

$$\sum_{(\alpha_1^1, \ldots, \alpha_1^{n_0})} \frac{1}{\alpha_1^1! \ldots \alpha_1^{n_0}!} \times D^{\alpha^1_{n_0}}_{z_1}(\rho^{-1} D^{\alpha^1_{n_0-1}}_{z_1} \cdots (\rho^{-1} D^{\alpha_1^1}_{z_1} \rho^{-1})) \bar{P}(D^{\alpha^1_{n_0}}_{z_1}) \ldots \bar{P}^{\alpha_1^1}(D_{z_1}). \tag{4.4}$$

Now push $\bar{P}^{\alpha^1_{n_0}}(D_{z_1}) \ldots \bar{P}^{\alpha_1^1}(D_{z_1})$ past the multiplications in the big sum (4.3) above. $\bar{P}^{\alpha_n}(D_{z_1}) \ldots \bar{P}^{\alpha_1}(D_{z_1})$ is constant, say $Q_{\tilde{\alpha}}(D_{z_1})$, where $\tilde{\alpha} = (\alpha^1_{n_0}, \ldots, \alpha^1_1)$. Applying Leibniz rule again, we get $L_{l, n_0} =$

$$\begin{aligned} &\sum_{(\alpha, \alpha_1, \ldots, \alpha_l)} \frac{1}{\tilde{\alpha}! \alpha_1! \ldots \alpha_l!} \\ &\left(\cdot D^{\alpha_{n_0}}_{z_1}(\rho^{-1}(D^{\alpha^l_{n}}_{z_1^{\circ}} \ldots (D^{\alpha_1^1}_{z_1} \rho^{-1}) \ldots) D^{\alpha_l}_{z_l} \cdots D^{\alpha_1}_{z_1} \right. \\ &\left. \cdot \prod_{j=1}^{l} \langle z_{j+1} - z_j \rangle^{-2n_0} \right) \times \tilde{Q}^{\alpha_1}_{\alpha}(D_{z_1}) P^{\alpha_2}(D_{z_2}) \ldots P^{\alpha_l}(D_{z_l}). \end{aligned} \tag{4.5}$$

This is finally in the form (4.1). We must now show

(i) $$\frac{Q^{\alpha_1}_{\alpha}(D_{z_1}) P^{\alpha_2}(D_{z_1}) \ldots P^{\alpha_l}(D_{z_l})}{\tilde{\alpha}! \alpha_1! \ldots \alpha_l!} = \sum_{\beta} C_\beta D^{\beta_1}_{z_1} \ldots D^{\beta_l}_{z_l},$$

where $\max\{C_\beta\} \leqq C^l_{n_0}$, $\#\{\beta\} \leqq C^l_{n_0}$.

(ii) $$D^{\alpha^1_{n}}_{z_1}(\rho^{-1} \ldots \rho^{-1}(D^{\alpha_1^1}_{z_1} \rho^{-1}) \ldots) D^{\alpha_l}_{z_l} \ldots D^{\alpha}_{z_1} \left(\prod_{j=1}^{l} \langle z_{j+1} - z_j \rangle^{-2n_0} \right)$$

$$= P_{\alpha,n_0}(z_1,\ldots,z_l,\xi_1,t)\rho^{-n_0}\prod_{j=1}^{l}\langle z_{j+1}-z_j\rangle^{-2n_0},$$

where $\|P_{\alpha,n_0}\| \leqq C_{n_0}(t)^l$.

(iii) The number of terms in $\sum_{(\tilde{\alpha},\ldots,\alpha_l)} \leqq C^l_{n_0}$.

Proof. (i) $|\beta_j|$ is bounded by the degree of $P^{\alpha_j}(\xi) \leqq 2n_0$. So the number of relevant terms is bounded by $^{\#}\{\beta_j\}\,|\beta_j| \leqq 2n_0\} = C^l_{n_0}$.

Let us consider $\max|C_\beta|$. Recall that $P(\xi) = (1+|\xi|^2)^{n_0}$, so that at $\xi = (1,\ldots,1)$ all derivatives of P are positive. Write $P^\alpha(\xi)/\alpha! = \sum_\beta A_\beta \xi^\beta$. Then each A_β is bounded by $P^\alpha/\alpha!(1)$, where $1 = (1,\ldots,1)$. We are writing

$$\frac{P^{\alpha_2}(\xi_2)}{\alpha_2!}\cdots\frac{P^{\alpha_l}(\xi_l)}{\alpha_l!} = \sum_{\vec{\beta}} A_{\vec{\beta}}\xi_1^{\beta_1}\cdots\xi_l^{\beta_l}.$$

Since distinct ξ_j's come from distinct factors, $A_{\vec{\beta}} = A_{\beta_2}\ldots A_{\beta_l} \leqq P^{\alpha_2}/\alpha_2!(1)\ldots P^{\alpha_l}(1)/\alpha_l!$. Finally the same argument applies to $Q^{\alpha^1}_\alpha(1)$, so

$$\max|C_\beta| \leqq Q^{\alpha_1}_\alpha(1)\cdots\frac{P^{\alpha_l}}{\alpha_l!}(1).$$

Now take $\max_{|\alpha_j|\leqq 2n_0}(P^{\alpha_j}/\alpha_j')(1) = C_{n_0}$ and the result is $C^l_{n_0}$.

(ii) First consider $D^{\alpha_l}_{z_l}\cdots D^{\alpha_1}_{z_1}\prod_{j=1}^{l}\langle z_{j+1}-z_j\rangle^{-2n_0}$.

Applying Leibniz rule, and the fact that only two bracket factors are operated on by a given D_{z_j} to get

$$\sum_{\substack{(\gamma_1\ldots,\gamma_l)\\ |\gamma_j|\leqq|\alpha_j|}}\binom{\alpha_1\alpha_2\ldots\alpha_l}{\alpha_1\gamma_2\ldots\gamma_l} D^{\gamma_2}_{z_2}D^{\alpha_1}_{z_1}\langle z_2-z_1\rangle^{-2n_0}D^{\gamma_3}_{z_3}D^{\alpha_2-\gamma_2}_{z_2}\langle z_3-z_2\rangle^{-2n_0}$$

$$\times\ldots\times D^{\gamma_l}_{z_l}D^{\alpha_{l-1}-\gamma_{l-1}}_{z_{l-1}}\langle z_l-z_{l-1}\rangle^{-2n_0}D^{\alpha_l-\gamma_l}_{z_l}\langle z_l\rangle^{-2n_0}.$$

Next note that $\langle x-y\rangle^{-2n_0}$ behaves like a symbol in $(x-y)$, in fact $|\partial^\alpha_x\partial^\beta_y\langle x-y\rangle^{-2n_0}| \leqq C_{\alpha\beta}\langle x-y\rangle^{-2n_0-|\alpha|-|\beta|}$.

Proof. Let $z = x-y$, and $\lambda(z) = \langle z\rangle^2$. Then

$$\partial^\alpha_z\lambda(z)^{-n_0} = \sum_{\substack{1\leqq\sigma\leqq|\alpha|\\ |\alpha_1|+\ldots+|\alpha_\sigma|=|\alpha|}}(-n_0)\ldots(-n_0-\sigma_{+1})\lambda(z)^{-n_0-\sigma}\prod_{j=1}^{\sigma}\partial^{\alpha_j}_z\lambda.$$

Now

$$\partial^\alpha_z j = \begin{cases} 0 & |\alpha_j|>2\\ 0 \text{ or } 2 & |\alpha_j|=2\\ \leqq|z| & |\alpha_j|=1\end{cases}$$

$$\Longrightarrow \prod_{j=1}^{\sigma}|\partial^{\alpha_j}_z\lambda| \leqq \prod_{j||\alpha_j|=1}||\partial^{\alpha_j}_z\lambda|\times\prod_{j||\alpha_j|=2}|\partial^{\alpha_j}_z\lambda| \leqq |z|^\sigma,$$

σ factors $|\partial_z^{\alpha_j}\lambda|$ bounded each by $|z|$. This bound is achieved if all $|\alpha_j|=1$ and $\sigma=|\alpha|$. Then

$$|\partial_z^\alpha \lambda^{-n_0}| \leqq \langle\lambda\rangle^{-n_0} C_{n_0,\alpha}\lambda^{-\sigma}|z|^\sigma.$$

But

$$|z|^\sigma\lambda^{-\sigma} \leqq \lambda^{-\sigma/2} = \langle z\rangle^{-\sigma} = |\partial_z^\alpha\langle z\rangle^{-2n_0} \leqq \langle z\rangle^{-2n_0-|\alpha|}.$$

Finally substituting $z=(x-y)$ and differentiating in $\partial_x^\alpha\partial_y^\beta$ produces $\partial_z^{\alpha+\beta}\lambda(z)|_{z=(x-y)}$ up to sign due to linearity of the substitute. This concludes the proof.

Write $D_{z_2}^{\gamma_2}D_{z_1}^{\alpha_1}\langle z_2-z_1\rangle^{-2n_0}\cdots D_l^{\alpha_l-\gamma_l}\langle z_l\rangle^{-1} = P_{\alpha,\gamma}(z)\cdot\prod_{j=1}^{l}\langle z_2-z_1\rangle^{-2n_0}$. It follows from the above that each factor $D_{z_{j+1}}^{\alpha_{j+1}}D_{z_j}^{\alpha_j-\gamma_j}\langle z_{j+1}-z_j\rangle^{-2n_0}$ may be bounded by $\langle z_{j+1}-z_j\rangle^{-2n_0}C_{n_0}$, where C_{n_0} is the symbol norm of $\langle x-y\rangle^{-2n_0}$ or order $(2n_0,2n_0)$. That is, $C_{n_0}=\max\{C_{\alpha\beta}|C_{\alpha\beta}$ is best constant in $|\partial_x^\alpha\partial_y^\beta\langle x-y\rangle^{-2n_0}| \leqq C_{\alpha\beta}\langle x-y\rangle^{-2n_0-|\alpha|-|\beta|}\}$ with $|\alpha|\leqq 2n_0$, $|\beta|\leqq 2n_0$.

Thus $\|P_{\alpha,\gamma}(z)\| \leqq C_{n_0}^l$; it does not necessarily decay. Now let

$$\tilde{P}_{\alpha,n_0}(z) = \sum_{(\gamma_1,\ldots,\gamma_l),|\gamma_j|\leqq|\alpha_j|}\begin{pmatrix}\alpha_1\ldots\alpha_l\\ \alpha_j\ldots\gamma_l\end{pmatrix}P_{\alpha,\gamma}(z).$$

Again, bounding $P_{\alpha,\gamma}$ and summing the binomial coefficients yields $|\tilde{P}_{\alpha,n_0}|\leqq C_{n_0}^l$.

Finally, ρ^{-1} is also a symbol in $((\cos t/\sin t)z_1-\xi_1)$. Defining now $\lambda(u)=(1+(1/i)(\cos t/\sin t)+|u|^2)$ yields λ^{-n_0} a symbol in u of order $2n_0$ as before without change. Substituting $u=(\cos t/\sin t)z_1-\xi_1$ only changes the previous argument by putting in factors of $\cos t/\sin t$, which makes all estimates t-dependent. However this can clearly be made continuous in t for $t\neq m\pi$, so that all estimates may be assumed uniform in t on compact sets disjoint from $\{m\pi\}$. Then

$$D_{z_1}^{\alpha'_{n_0}}(\rho^{-1}\ldots\rho^{-1}D_{z_1}^{\alpha'_1}\rho^{-1})$$

is a product of differentiations of ρ^{-1}, and multiplications by ρ^{-1} all of which only increase the order of ρ^{-1}. We then certainly get ρ^{-n_0} and may sum the other factors to get $P_{\tilde{\alpha}}(z,\xi_1,t)$, which is bounded by a C_{n_0}.

Finally let $P_{\alpha,n_0}(z_1,\ldots,z_l,\xi_1,t)=P_{\tilde{\alpha}}(z,\xi_1,t)\tilde{P}_{\alpha,n_0}(z_1\ldots,z)$. This concludes part (ii).

(iii) This is obvious since if $C_{n_0}=\{\alpha_j||\alpha_j|\leqq 2n_0\}$ then the number of terms is bounded by $C_{n_0}^l$.

This concludes the proofs of the major claims of power law growth of the bounding constants.

Finally, we show that if one of the standard classical Hamiltonian systems (i)–(iii) is perturbed by $V\in S^0(\mathbb{R}^n)$, then the associated lagrangian submanifolds Λ_y^t are asymptotic to the unperturbed ones. More precisely,

Proposition 4.1. *Let $H_0(x,\xi)$ be one of case (i)–(iii), and let $H(x,\xi)=H_0+V(x)$, $V\in S^0(\mathbb{R}^n)$. Let $(x_0(t,y,\eta),\xi_0(t,y,\eta))$ be solutions of the unperturbed initial value Hamilton's equations and let $(x(t,y,\eta),\xi(t,y,\eta))$ be the perturbed solutions. Then $(x(t,y,\eta),\ \xi(t,y,\eta))=(x_0(t,y,\eta),\ \xi_0(t,y,\eta))+(o(1),\ o(1))$ for fixed (t,y) as $|\eta|\to\infty$.*

Proof. First use the variations of constant formulas:

$$\begin{Bmatrix} x(t,y,\eta) \\ \xi(t,y,\eta) \end{Bmatrix} = \begin{Bmatrix} x_0(t,y,\eta) \\ \xi_0(t,y,\eta) \end{Bmatrix} - \int_0^t \begin{Bmatrix} G(t,s) \\ (\partial G/\partial t)(t,s) \end{Bmatrix} \cdot V'(x(s,y,\eta))\,ds,$$

where $G(t,s)$ are the initial value Greens' functions

$$\text{i) } G(t,s) = (t-s),$$

$$\text{ii) } G(t,s) = \sin(t-s),$$

$$\text{iii) } G(t,s) = \sin\omega_i(t-s)/\omega_i.$$

We then need to show

$\int_0^t G(t,s)V'(x(s,y,\eta))$ and $\int_0^t (\partial G/\partial t)\ (t,s)V'(x(s,y,\eta))ds$ are $o(1)$ for fixed (t,y) as $|\eta| \to \infty$. But $V \in S^0 \Rightarrow |V'(x(s,y,\eta))| \leqq C\langle x,s,y,\eta\rangle^{-1}$. Since $x(s,y,\eta) = x_0(s,y,\eta) + o(1)$, we have $\langle x(s,y,\eta)\rangle^{-1} \leqq G_1 \langle x_0(s,y,\eta)\rangle^{-1}$ by the $\langle u+v\rangle^{-1} \leqq \sqrt{2}\langle u\rangle^{-1}\langle v\rangle$ inequality. Now

$$x_0(s,y,\eta) = \text{i) } y + s\eta,$$

$$\text{ii) } \cos sy + \sin s\eta,$$

$$\text{iii) } \cos\omega_i sy + \sin\omega_i s\eta.$$

So, writing the coefficient of η as $\rho(s)$ and applying the same inequality to any of the sums i)–iii) we get $|V'(x(s,y,\eta))| \leqq C\langle y\rangle\langle\rho(s)\eta\rangle^{-1}$. For almost all s, $\langle\rho(s)\eta\rangle^{-1} \to 0$ as $|\eta| \to \infty$. All constants and Green's functions are bounded continuous functions of s. So by the dominated convergence the integrals

$$\int_0^t G(t,s)V'(x(s,y,\eta))\,ds \text{ and}$$

$$\int_0^t \frac{\partial G}{\partial t}(t,s)V'(x(s,y,\eta))\,ds$$

are (1) as $|\eta| \to \infty$.

Remarks. Let L_y^t be the Lagrangian manifold $\chi_0^t \Lambda_0^y$ where χ_0 is the unperturbed phase flow, and let $\Lambda_y^t = \chi^t \Lambda_0^y$, where $\chi^t(y,\eta) = (x(t,y,\eta), \xi(t,y,\eta))$. Equip both with initial value coordinates η determined by the diffeomorphisms $\chi_0^t : \Lambda_y^0 \to L_y^t$ and $\chi^t : \Lambda_y^0 \to \Lambda_y^t$. Then the Euclidean distance in $T^*\mathbb{R}^n$ between L_y^t and Λ_y^t outside the coordinate balls $|\eta| \leqq r$ is bounded by the length of the pair of integrals in the proposition. Hence this distance approaches zero as the balls increase. So we are justified in saying that L_y^t and Λ_y^t are asymptotic.

Acknowledgement. This paper is essentially the author's Ph.D. thesis. I would like to thank Alan Weinstein for his advice and encouragement. In particular, the conjectural description of singularities of wave functions given above was evolved in discussions with him, and may apply to more general situations. This research was partially supported by NSF grant No. MCS-77-23579. My thanks also go to the University of California, Berkeley and to NASA-JPL for financial support.

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Communicated by B. Simon
Received May 23, 1982; in revised form November 4, 1982