Regularity and Decay of Lattice Green's Functions[*]

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Abstract. In the paper we study a class of lattice, covariant Laplace operators with external gauge fields. We prove that these operators are positive and that their Green's functions decay exponentially. They also have regularity properties similar to continuous space Green's functions. All the bounds are uniform in the lattice spacing.

1. Introduction. Formulations of Theorems

In this paper we prove some properties of Green's functions for difference Laplace operators. They imply all the properties of Green's functions (propagators) and Gaussian actions used in the papers [1–3], thereby completing the proof of ultraviolet stability of (Higgs)$_{2,3}$ models. But the range of applicability of these properties is much wider and a need of them appears in many mathematical problems of statistical physics and quantum field theory. In fact, the mathematical estimates have an intrinsic interest of their own, so we present them in a self contained paper.

The properties of Green's functions we are interested in are regularity properties and exponential decay. The difference Laplace operators are lattice approximations to second order elliptic differential operators, so the regularity properties of lattice Green's functions are similar to the properties of continuous space Green's functions. For example, if G is a lattice Green's function, then $\|Gf\|_2$, $\|\partial_\mu Gf\|_2$, $\|\partial_\mu \partial_\nu Gf\|_2 \leq c\|f\|_2$, where ∂_μ is a difference derivative and $\|\cdot\|_2$ denotes L^2-norm on the lattice. Similar estimates hold for other norms, and we are especially interested in Hölder norms. Exponential decay is interpreted in physics as the existence of an "effective mass". In mathematical terms this means that there is a strictly positive lower bound for the inverse Green's operators. The simplest example is the operator $-\Delta + m^2$. A Green's function $C_{m^2} = (-\Delta + m^2)^{-1}$ for this operator, on continuous space and on a lattice, satisfies a bound of the form $|C_{m^2}(x, y)| \leq O(1)|x - y|^{-d+2} e^{-m|x-y|}$. We are interested in proving similar bounds, or bounds derived from these, for more general operators.

Let us formulate some basic definitions. We consider operators on subsets of the lattice ηZ^d, $\eta = L^{-k}$. Here L is a small positive integer > 1, e.g. $L = 2$ or 3, and k is an arbitrary positive integer. Another common case is to consider operators on subsets of a torus T_η which we identify with a rectangular parallelepiped in ηZ^d with periodic conditions. The lattice ηZ^d is divided into unit cubes called *blocks* and parametrized by points y of the unit lattice Z^d:

$$B^k(y) = \Delta(y) = \{x \in \eta Z^d : y_\mu \leqq x_\mu < y_\mu + 1, \ \mu = 1, \ldots, d\}, \qquad y \in Z^d. \tag{1.1}$$

We consider also a second division of ηZ^d into cubes of size M called *big blocks*. They are parametrized by points of the lattice $M Z^d$ and defined in the same way as unit blocks with M instead of 1 in (1.1). Here M is a large positive integer defined later in this paper. We consider subsets Ω which are unions of big blocks. In order to define the class of Laplace operators, the so-called covariant Laplace operators, we have to introduce a notion of vector fields. These are real-valued functions defined on pairs $\langle x, x' \rangle$ of nearest neighbour points of the lattice, called bonds. We identify them with vector-valued functions defined on points of the lattice by the identity $A_{\langle x, x + \eta e_\mu \rangle} = A_\mu(x)$, where e_μ is a unit vector of μth axis. The operators we are going to define depend on A through the function

$$U(A) = e^{q e \eta A}, \quad q \text{ is an antisymmetric } N \times N \text{ matrix, where } e \text{ is a real parameter}. \tag{1.2}$$

Now the covariant Laplace operator $-\Delta_{A,\Omega}^{\eta,N}$ on a domain Ω with Neumann boundary conditions on $\partial \Omega$ is given by the following quadratic form defined on functions $\phi : \Omega \to R^N$

$$\langle \phi, (-\Delta_{A,\Omega}^{\eta,N})\phi \rangle = \sum_{b \subset \Omega} \eta^d |(D_A^\eta \phi)(b)|^2 = \sum_{b \subset \Omega} \eta^d |\eta^{-1}(U(A_b)\phi(b_+) - \phi(b_-))|^2, \tag{1.3}$$

where the summation is over the set of all bonds $b = \langle b_-, b_+ \rangle$ with end-points b_-, b_+ in Ω.

Next let us define some projection operators in the space of functions ϕ. At first we define a covariant averaging operator

$$(Q_k(A)\phi)(y) = \sum_{x \in B^k(y)} \eta^d U(A(\Gamma_{y,x}^{(k)}))\phi(x), \qquad y \in Z^d. \tag{1.4}$$

Here $\Gamma_{y,x}^{(k)}$, $x \in B^k(y)$, are oriented contours in $B^k(y)$ connecting the initial point y with a final point x. For an arbitrary contour Γ in the lattice ηZ^d (considered as a sum of bonds) we define $A(\Gamma) = \sum_{b \subset \Gamma} A_b$, where orientations of the bonds b agree with orientation of the contour Γ. The projection operator $P_k(A)$ is given by

$$P_k(A) = Q_k^*(A)Q_k(A). \tag{1.5}$$

Our fundamental Green's function is a kernel of the operator

$$G_k(\Omega, A) = (-\Delta_{A,\Omega}^{\eta,N} + m^2 + a P_k(A))^{-1}, \tag{1.6}$$

where $m^2 \geqq 0$ and a is a positive constant close to 1. We consider all these operators under the assumption that the vector field A is regular on Ω in the sense that

$$|(\partial_\mu^\eta A)(x)| \leqq c e^{\beta - 1}, \qquad x \in \Omega, \qquad \mu = 1, \ldots, d, \qquad \beta > 0, \tag{1.7}$$

and c is some universal constant. [Of course ∂_μ^η is a difference derivative defined by $(\partial_\mu^\eta A)(x) = \eta^{-1}(A(x + \eta e_\mu) - A(x))$.]

The operator defining the Green's function (1.6) has a strictly positive lower bound. More exactly we prove that there exists a positive constant γ_0 such that for e sufficiently small and for a regular vector field A

$$-\Delta_{A,\Omega}^{\eta,N} + a P_k(A) \geq \gamma_0 I. \tag{1.8}$$

The constant γ_0 is independent of the lattice spacing η, as well as of Ω and of A. This bound justifies the definition (1.6) and explains exponential decay properties.

Finally for an arbitrary pair of points $x, x' \in \eta Z^d$, let us denote by $\Gamma_{x,x'}$ a shortest contour connecting these points. Now we can formulate the fundamental theorem of the paper.

Theorem (Proposition 2.1 of [1]). *For $\alpha < 1$ there exist positive constants δ_0, c_0, R_0 independent of A, k, Ω and depending on d, M only, c_0 on α also, such that for e sufficiently small and for an arbitrary function $f : \Omega \to R^N$, we have*

$$\frac{1}{|x - x'|^\alpha} |U(A(\Gamma_{x,x'}))(D_{A,\mu}^\eta G_k(\Omega, A)f)(x') - (D_{A,\mu}^\eta G_k(\Omega, A)f)(x)|$$

$$\leq c_0 \exp(-\delta_0 \operatorname{dist}(\{x, x'\}, \operatorname{supp} f))\|f\|_\infty \tag{1.9}$$

for $x, x' \in \Omega$, and satisfying the condition $\operatorname{dist}(\{x, x'\}, \Omega^c) \geq R_0$. Similarly

$$|(D_{A,\mu}^\eta G_k(\Omega, A)f)(x)|, |(G_k(\Omega, A)(x)| \leq c_0 \exp(-\delta_0 \operatorname{dist}(x, \operatorname{supp} f))\|f\|_\infty \tag{1.10}$$

for $x \in \Omega$, $\operatorname{dist}(x, \Omega^c) \geq R_0$. If $\Omega \subset \Omega_0$, then for $\delta G_k(\Omega, \Omega_0, A)$ defined by the equality

$$\delta G_k(\Omega, \Omega_0, A) = G_k(\Omega, A) - G_k(\Omega_0, A), \tag{1.11}$$

we have the inequalities (1.5) and (1.6) (with the same restrictions on x, x') with the additional factor

$$\exp(-\delta_0 \operatorname{dist}(\operatorname{supp} f, \Omega^c) - \delta_0 \operatorname{dist}(\operatorname{supp} f, \Omega^c)) \tag{1.12}$$

on the right hand sides. For some simple sets Ω, e.g. for rectangular parallelepipeds, the inequalities hold without any restrictions on the points x, x', i.e. for all $x, x' \in \Omega$.

This theorem implies in particular the Proposition 2.3 of [1]. It concerns unit lattice Green's functions $C_A^{(k)}(\Omega, A)$ of [1]. Let us recall that for operators X defined on the unit lattice functions we define the operator $X|_A$ restricted to a subset A by $X|_A = A X A$, where A also denotes the characteristic function of the set A. We have

$$C_A^{(k)}(\Omega, A) = ((\Delta^{(k)}(\Omega, A) + a L^{-2} P(A))|_A)^{-1}. \tag{1.13}$$

Here $\Delta^{(k)}(\Omega, A)$ is an operator of the effective Gaussian action after k renormalization transformations and can be defined by

$$\Delta^{(k)}(\Omega, A) = a_k I - a_k^2 Q_k(A) G_k(\Omega, A) Q_k^*(A), \tag{1.14}$$

where a_k is a constant proportional to a. Properties of the propagators $C_A^{(k)}(\Omega, A)$ are described in

Proposition 2.3 of [1]. *There exist positive constants* $\delta_0, c_0, \gamma_0, \gamma_1$ *dependent on d and M only and such that for arbitrary* $\Lambda \subset \Omega^{(k)} = \Omega \cap Z^d$, Λ *being a sum of big blocks and for e sufficiently small, we have*

$$\gamma_0 I \leqq \Delta^{(k)}(\Omega, A) + aL^{-2} P(A) \leqq \gamma_1 I, \tag{1.15}$$

$$|C_\Lambda^{(k)}(\Omega, A; x, x')| \leqq c_0 \exp(-\delta_0|x - x'|), \quad x, x' \in \Lambda. \tag{1.16}$$

In particular the above inequality holds for $C^{(k)}(\Omega, A)$. *Putting*

$$\delta C_\Lambda^{(k)}(\Omega, A) = C_\Lambda^{(k)}(\Omega, A) - C^{(k)}(\Omega, A), \tag{1.17}$$

we have also

$$|\delta C_\Lambda^{(k)}(\Omega, A; x, x')| \leqq c_0 \exp(-\delta_0(|x - x'| + \mathrm{dist}(x, \Lambda^c) + \mathrm{dist}(x', \Lambda^c))), \quad x, x' \in \Lambda. \tag{1.18}$$

Finally, for $\Omega \subset \Omega_0$ *and*

$$\delta C_\Lambda^{(k)}(\Omega, \Omega_0, A) = C_\Lambda^{(k)}(\Omega, A) - C_\Lambda^{(k)}(\Omega_0, A), \tag{1.19}$$

we have

$$|\delta C_\Lambda^{(k)}(\Omega, \Omega_0, A; x, x')|$$
$$\leqq c_0 \exp(-\delta_0(|x - x'| + \mathrm{dist}(x, \Omega^{(k)c}) + \mathrm{dist}(x', \Omega^{(k)c}))), \quad x, x' \in \Lambda. \tag{1.20}$$

The last theorem we have to prove is the bound from below for the operator $\Delta^{(k)}(\Omega, A)$.

Proposition 3.1' of [2]. *Let* Ω *be a sum of unit blocks (i.e.* $\Omega^{(k)}$ *is an arbitrary subset of* Z^d*) and let A satisfies the condition*

$$|(\partial_\mu^\eta A)(x)| \leqq O(1)p(e) \quad (p(e) = a_0(1 + \log e^{-1})^p), \tag{1.21}$$

then there exists a positive constant γ_0 *depending on d only, such that for e sufficiently small*

$$\langle \phi, \Delta^{(k)}(\Omega, A)\phi \rangle \geqq \gamma_0 \left(\sum_{\langle x, x' \rangle \subset \Omega^{(k)}} |U(A(\langle x, x' \rangle))\phi(x') - \phi(x)|^2 + m^2 \sum_{x \in \Omega^{(k)}} |\phi(x)|^2 \right)$$
$$- O(1)e^{2-\alpha} \sum_{x \in \Omega^{(k)}} |\phi(x)|^2 \tag{1.22}$$

for arbitrary $\alpha > 0$ *and a constant* $O(1)$ *depending on* α *and the other constants, but independent of* Ω, k, A, *and for an arbitrary function* ϕ.

This theorem implies (3.29) of [2].

The theorems will be proven in Sects. 2–4 in the order in which they were written above. The fifth section will be devoted to a general theorem concerning operators on the unit lattice Z^d. There we have abstracted some basic features of our method and we have proven a theorem which, if applied to operators (1.14), gives another proof of Proposition 2.3.

2. Generalized Random Walk Representation. Proof of the Theorem

The proof will be done in several steps; in each step a reduction to a simpler problem will be achieved. Let us notice only that it is sufficient to prove the

Proposition for a function f with support in a unit cube $\Delta_0 = B^k(y_0)$, $y_0 \in \Omega^{(k)}$; the general formulation is obtained by taking the decomposition $f = \sum_{A \subset \Omega} \Delta f$ and summing the inequalities.

In the first step we will introduce a generalization of random walk representations of [4], [5]. We have assumed that $\Omega^{(k)}$ is a sum of large blocks, i.e. blocks of the size M on the unit lattice, thus Ω is a sum of the corresponding large blocks of the size M on η-lattice T_η. For each $j \in Z^d$, let us define the set

$$\square_j = \Omega \cap \{\text{a sum of large blocks for which the point } Mj \text{ is one of the vertices}\}.$$

Let us observe that if the point Mj is not a boundary point of Ω, then $\square_j$ is a cube of the size $2M$ and with center in Mj. For Mj lying on the boundary the set $\square_j$ is a sum of several ($<2d$) large blocks.

Next let us define a partition of unity $\{h_j\}_{j \in Z^d}$ on T_η. For each $j \in Z^d$, we take

$$h_j(x) = \prod_{\mu=1}^{d} h_{j_\mu}(x_\mu),$$ and functions $h_j(x)$, $j \in Z$, of one real variable x are defined as

$$h_j(x) = h\left(\frac{x}{M} - j\right), \quad h \in C_0^\infty(]-\tfrac{2}{3}, \tfrac{2}{3}[), \quad h(x) = 1 \text{ for } x \in [-\tfrac{1}{3}, \tfrac{1}{3}],$$ and it is chosen in such a way that $\sum_{j \in Z} h_j^2 = 1$.

Let us define an operator G_0 by the formula

$$G_0 = \sum_j h_j G_k(\square_j, \tilde{A}_j) h_j, \tag{2.2}$$

where the configurations $\tilde{A}_j$ are constructed in the following way: if $\square_j$ intersects the boundary of Ω, then $\tilde{A}_j = A$; if $\square_j$ is an interior cube of Ω, then we take $\tilde{A}_j$ as equal to A on the cube $\{x : |x - Mj| \leq \tfrac{3}{4}M\}$, and changing regularly to a constant function in a neighbourhood of a boundary of $\square_j$. For example using the regularity condition (1.4) we can write the configuration A on $\square_j$ as $A = A_0 + A'$, where A_0 is a constant configuration, e.g. $A_0 = A(Mj)$, and A' is regular and small, i.e. $|A'|$, $|\partial_\mu^\eta A'| \leq c'e^{\beta-1}$, with c' depending on c, d, and M, more exactly $c' = dMc$. We take a function $\theta \in C_0^\infty(]-1, 1[)$, $\theta = 1$ on $[-\tfrac{3}{4}, \tfrac{3}{4}]$, and we define $\theta_j(x)$

$$= \prod_{\mu=1}^{d} \theta\left(\frac{x_\mu}{M} - j_\mu\right), \quad \tilde{A}_j = A_0 + \theta_j A'.$$ In the sequel we will use this definition of $\tilde{A}_j$. Of course it satisfies the regularity condition (1.4) with another constant c.

We will prove that G_0 is a very good approximation of the operator $G_k(\Omega, A)$. Let us calculate $(-\Delta_{A,\Omega}^{\eta, N} + m_k^2 + a_k P_k)G_0$. We have

$$(D_A^\eta h\phi)(b) = h(b_+)(D_A^\eta \phi)(b) + (\partial^\eta h)(b)\phi(b_-), \tag{2.3}$$

where h is a real valued function

$$(D_A^{\eta*} g\theta)(x) = \sum_{\mu=1}^{d} \eta^{-1}(g(\langle x - \eta e_\mu, x\rangle) - g(\langle x, x + \eta e_\mu\rangle))\theta(\langle x, x + \eta e_\mu\rangle)$$

$$+ \sum_{\mu=1}^{d} g(\langle x - \eta e_\mu, x\rangle)\eta^{-1}(U(A_{\langle x, x - \eta e_\mu\rangle})\theta(\langle x - \eta e_\mu, x\rangle)$$

$$- \theta(\langle x, x + \eta e_\mu\rangle))$$

$$= \sum_{\mu=1}^{d} (\partial_\mu^{\eta*} g)(x)\theta(\langle x, x + \eta e_\mu\rangle) + \sum_{\mu=1}^{d} g(\langle x - \eta e_\mu, x\rangle)(D_{A,\mu}^{\eta*}\theta)(x), \tag{2.4}$$

where g, θ are defined on bonds, g is real valued and θ is a vector valued function. We assume also that $A_{-b} = -A_b$ for an arbitrary bond b. These formulas imply

$$(-\Delta_{A,\Omega}^{\eta,N} h\phi)(x) = (-\Delta_{\Omega}^{\eta,N}h)(x)\phi(x) - \sum_{b\in st(x),\, b\subset\Omega} (\partial^\eta h)(b)(D_A^\eta\phi)(b) + h(x)(-\Delta_{A,\Omega}^{\eta,N}\phi)(x).$$

(2.5)

Special care has to be taken in considering boundary terms. Let us remark now that

$$(-\Delta_{A,\Omega}^{\eta,N}h_j G(\square_j, \tilde{A}_j) = (-\Delta_{\tilde{A}_j,\square_j}^{\eta,N})h_j G(\square_j, \tilde{A}),$$

(2.6)

because the function h_j can be ± 0 only on the part of the boundary of $\square_j$ which is contained in the boundary of Ω. From (2.5), (2.6) we have

$$(-\Delta_{A,\Omega}^{\eta,N}G_0)(x,x') = \sum_j [h_j(x)(-\Delta_{\tilde{A}_j,\square_j}^{\eta,N}G_k(\square_j, \tilde{A}_j))(x,x')h_j(x')$$
$$- \sum_{b\in st(x)} (\partial^\eta h_j)(b)(D_{\tilde{A}_j}^\eta G_k(\square_j, \tilde{A}_j))(b,x')h_j(x')$$
$$+ (-\Delta^\eta h_j)(x)G_k(\square_j, \tilde{A}_j; x, x')h_j(x')],$$

(2.7)

where we have used the fact that the normal derivative of h_j to the boundary of $\square_j$ is equal to 0.

For the operator $P_k(A)$, we have

$$(P_k(A)h\phi)(x) = h(x)(P_k(A)\phi)(x) + \sum_{x'\in B^k(y^k(x))} \eta^d U(A(\Gamma_{x,y^k(x),x'}^{(k)}))(\partial^\eta h)(\Gamma_{x,y^k(x),x'}^{(k)})\phi(x')$$
$$= :h(x)(P_k(A)\phi)(x) + (R_k(A,\partial^\eta h)\phi)(x).$$

(2.8)

Thus we get finally

$$((-\Delta_{A,\Omega}^{\eta,N} + m_k^2 + P_k(A))G_0)(x,x')$$
$$= \sum_j [h_j(x)((-\Delta_{\tilde{A}_j,\square_j}^{\eta,N} + m_k^2 + P_k(\tilde{A}_j))G_k(\square_j, \tilde{A}_j))(x,x')h_j(x')$$
$$- \sum_{b\in st(x)} (\partial^\eta h_j)(b)(D_{\tilde{A}_j}G_k(\square_j, \tilde{A}_j))(b,x')h_j(x')$$
$$+ (-\Delta^\eta h_j)(x)G_k(\square_j, \tilde{A}_j; x, x')h_j(x')$$
$$+ (R_k(\tilde{A}_j, \partial^\eta h_j)G_k(\square_j, \tilde{A}_j))(x,x')h_j(x')$$
$$= \delta^\eta(x-x') - \sum_j \Big[\sum_{b\in st(x)} (\partial^\eta h_j)(b)(D_{\tilde{A}_j}^\eta G_k(\square_j, \tilde{A}_j))(b,x')h_j(x')$$
$$+ (\Delta^\eta h_j)(x)G_k(\square_j, \tilde{A}_j; x, x')h_j(x') - (R_k(\tilde{A}_j, \partial^\eta h_j)G_k(\square_j, \tilde{A}_j))(x,x')h_j(x')\Big]. \text{ (2.9)}$$

Let us define the operators

$$(K_j\phi)(x) = \square_j(x)\Big[\sum_{b\in st(x)} (\partial^\eta h_j)(b)(D_{\tilde{A}_j}^\eta\phi)(b) + (\Delta^\eta h_j)(x)\phi(x)$$
$$- \sum_{x'\in B^k(y^k(x))} \eta^d(\partial^\eta h_j)(\Gamma_{x,y^k(x),x'}^{(k)})U(\tilde{A}_j(\Gamma_{x,y^k(x),x'}^{(k)}))\phi(x')\Big],$$

(2.10)

$$R = \sum_j K_j G_k(\square_j, \tilde{A}_j)h_j.$$

(2.11)

In the sequel we will see that R is a small operator in reasonable norms because $|\partial^\eta h_j| \leq O(M^{-1})$, $|\Delta^\eta h_j| \leq O(M^{-2})$, so we have the representations

$$G_k(\Omega, A) = G_0(I - R)^{-1} = \sum_{n=0}^{\infty} G_0 R^n. \tag{2.12}$$

They can be written in a very convenient form of "random walk" representation. Let us consider a space of paths ω, each path is a sequence of points $\omega = \{\omega_0, \omega_1, ..., \omega_n\}$, $\omega_i \in Z^d$, satisfying the following condition: the points ω_i, ω_{i+1} are vertices of a unit cube of the lattice. It can be formulated also that ω_{i+1} is of the form $\omega_{i+1} = \omega_i + \sum_{\mu=1}^{d} \varepsilon_\mu e_\mu$, where ε_μ is one of the numbers $-1, 0, 1$. We can write

$$G_k(\Omega, A) = \sum_{\omega} h_{\omega_0} G_k(\square_{\omega_0}, \tilde{A}_{\omega_0}) h_{\omega_0} K_{\omega_1} G_k(\square_{\omega_1}, \tilde{A}_{\omega_1}) h_{\omega_1} \dots K_{\omega_n} G_k(\square_{\omega_n}, \tilde{A}_{\omega_n}) h_{\omega_n}, \tag{2.13}$$

and this representation follows from (2.12) and the obvious fact that

$$G_k(\square_j, \tilde{A}_j) h_j K_{j'} G_k(\square_{j'}, \tilde{A}_{j'}) = 0,$$

if $|j - j'| = \max_\mu |j_\mu - j'_\mu| > 1$. It is a basic representation and all the statements of Proposition I.2.1 follow from it and from some properties of the propagators $G_k(\square_j, \tilde{A}_j)$. Let us now formulate these properties in separate lemmas. We will need Hölder norms:

$$\|f\|_{1,\alpha} = \max \left\{ \sup_x |f(x)|, \sup_{x,\mu} |(D^\eta_{A,\mu} f)(x)|, \sup_{x,x',\mu} \frac{1}{|x'-x|^\alpha} |U(A(\Gamma_{x,x'}))| \right.$$

$$\left. \cdot (D^\eta_{A,\mu} f)(x') - (D^\eta_{A,\mu} f)(x)| \right\}, \tag{2.14}$$

where the suprema are taken on a domain of the function f. This norm will be applied to expressions depending on some vector field, e.g. to $G_k(\square_j, \tilde{A}_j)$, and it will be understood that the same vector field is in the norm (2.14) as in these expressions.

Lemma 2.1. *Let a set $\square$ be an arbitrary sum of unit blocks, but such that it is a sum of at most few large blocks, and let A be as in the theorem. Then for e sufficiently small we have*

$$\|G_k(\square, A)f\|_2, \quad \|D^\eta_{A,\mu} G_k(\square, A)f\|_2, \quad \|G_k(\square, A)D^{\eta*}_{A,\mu} f\|_2,$$

$$\|D^\eta_{A,\mu} G_k(\square, A)D^{\eta*}_{A,\nu} f\|_2 \leq c_2 \|f\|_2. \tag{2.15}$$

Let us notice that Lemma 2.1 implies that the L^2-norm of the operator R given by (2.11) is small for M large enough, so the series in the representation (2.12) is convergent in this norm.

Lemma 2.2. *Let a rectangular parallelepiped $\square$ be a sum of few large blocks (e.g., as in the case of the cubes $\square_j$), and let $\tilde{A}$ be a regular vector field configuration in the sense of Proposition I.2.1, constant in a neighbourhood of the boundary of $\square$. Then for e sufficiently small and $\alpha < 1$, there exists a constant c_1 depending on d, α only, such that*

$$\|G_k(\square, \tilde{A})f\|_{1,\alpha} \leq c_1 \|f\|_\infty, \tag{2.16}$$

and a constant c_2 depending on d, p_1, such that

$$\|G_k(\square, \tilde{A})f\|_q, \quad \|D^\eta_{\tilde{A},\mu}G_k(\square, \tilde{A})f\|_q, \quad \|G_k(\square, \tilde{A})D^{\eta*}_{\tilde{A},\mu}f\|_q \leqq c_2\|f\|_p \quad (2.17)$$

for $1\leqq p, q\leqq\infty$, satisfying the condition $\dfrac{1}{p}-\dfrac{1}{p_1}\leqq\dfrac{1}{q}\leqq\dfrac{1}{p}$ with $p_1>d$.

These two lemmas together with the representation (2.13) imply the theorem.

Let us prove the inequality (1.9). We take an arbitrary pair of points $x, x'\in\Omega$ with distances to the boundary of Ω greater than R_0. If $|x'-x|>1$, then this inequality is a simple consequence of the corresponding inequality for the derivative only, hence we can assume $|x'-x|\leqq 1$. If any of the points x, x' belongs to $\mathrm{supp}\, h_j$, then both belong to $\square_j$ and in the representation (2.13) for each term we have that either both points belong to $\square_{\omega_1}$, or none. Of course the first situation occurs for at most 2^d cubes $\square_j$. We restrict the summation in (2.13) to paths starting in the corresponding j's and we will prove the inequality (1.9) using the representation (2.13):

(the left hand side of (1.9))

$$\leqq\sum_\omega{}' \|h_{\omega_0}G_k(\square_{\omega_0}, \tilde{A}_{\omega_0})h_{\omega_0}K_{\omega_1}G_k(\square_{\omega_1}, \tilde{A}_{\omega_1})h_{\omega_2}\cdot\ldots\cdot K_{\omega_n}G(\square_{\omega_n}, \tilde{A}_{\omega_n})h_{\omega_n}f\|_{1,\alpha},$$

$$(2.18)$$

where ω_0 and ω_n are each restricted to 2^d possible values of j by the conditions $x, x'\in\square_{\omega_0}$, $\mathrm{supp}\, f\subset\square_{\omega_n}$. We do not know yet if the series on the right side of (2.18) is convergent, and it will be one of the consequences of our estimates. Let us now take a positive integer n_0, we will fix it later, and let us define R_0 as

$$R_0=\left(\text{diameter of} \bigcup_{\substack{\omega=(\omega_0,\ldots,\omega_{n_0})\\ \omega_0=0}}\bigcup_{i=1}^{n_0}\square_{\omega_i}\right)+2M. \quad (2.19)$$

Here $\square_j$ denotes a cube with center Mj and size equal to $2M$. From this definition it follows that R_0 depends on M and n_0. We divide the sum in (2.18) into two subsums: one with $n\leqq n_0$ and the other with $n>n_0$. The first is finite and the condition $\mathrm{dist}(\{x, x'\}, \Omega^c)\geqq R_0$, together with the definition (2.19) of R_0 imply that all $\square_{\omega_i}$ are cubes contained in Ω. The same of course holds for the first n_0 elements of the sequences ω in the second sum. We estimate the first sum using Lemma 2.2 by

$$\sum_{\omega:n\leqq n_0}{}' c_1\|K_{\omega_1}G_k(\square_{\omega_1}, \tilde{A}_{\omega_1})h_{\omega_1}\|_\infty\cdot\ldots\cdot\|K_{\omega_n}G_k(\square_{\omega_n}, \tilde{A}_{\omega_n})h_{\omega_n}\|_\infty\|f\|_\infty$$

$$\leqq\sum_{\omega:n\leqq n_0}{}' c_1(c_2 O(1)M^{-1})^n\|f\|_\infty. \quad (2.20)$$

We apply Lemma 2.2 to the terms of the second sum also, more exactly we apply (2.17) with $\dfrac{1}{p_1}=\dfrac{1}{2n_0}$ and we fix n_0 such that $p_1=2n_0>d+1$, e.g. $n_0=d$. We estimate the second sum by

$$\sum_{\omega:n>n_0}{}' c_1\|K_{\omega_1}G_k(\square_{\omega_1}, \tilde{A}_{\omega_1})h_{\omega_1}\|_{\infty,p_1}\prod_{i=2}^{n_0}\|K_{\omega_i}G_k(\square_{\omega_i}, \tilde{A}_{\omega_i})h_{\omega_i}\|_{\frac{p_1}{i-1},\frac{p_1}{i}}$$

$$\cdot\prod_{i=n_0+1}^{n}\|K_{\omega_i}G_k(\square_{\omega_i}, \tilde{A}_{\omega_i})h_{\omega_i}\|_{2,2}\|f\|_2\leqq\sum_{\omega:n>n_0}{}' c_1(c_2 O(1)M^{-1})^n\|f\|_\infty. \quad (2.21)$$

Here $\|T\|_{q,p}$ denotes a norm of an operator $T:L^p\to L^q$. Let us remark that the basic reason for the restriction $\mathrm{dist}(\{x,x'\},\Omega^c)\geqq R_0$ is that we do not have the inequalities (2.16) for the norms other than L^2-norms if $\Box_j$ is not a cube (or rectangular parallelepiped). The summation in (2.18) is restricted to paths ω satisfying $x,x'\in\Box_{\omega_0}$, $\mathrm{supp} f\subset\Box_{\omega_n}$, so the length n satisfies

$$n\geqq M^{-1}\mathrm{dist}(\{x,x'\},\mathrm{supp} f)-2.$$

There are at most $2^d(3^d)^{n-1}2^d$ of such paths, so finally we get the inequality

$$\text{(the left hand side of (1.9))}\leqq\sum_n 2^d c_1(3^d c_2 O(1)M^{-1})^n\|f\|_\infty$$
$$\leqq\frac{2^{d+1}}{e^2}e^{-M^{-1}\mathrm{dist}(\{x,x\},\mathrm{supp} f)}\|f\|_\infty,\tag{2.22}$$

where $n\geqq M^{-1}\mathrm{dist}(\{x,x'\},\mathrm{supp} f)-2$, and if M is fixed such that $3^d c_2 O(1)M^{-1}\leqq e^{-1}$. Thus the inequality (1.9) is proved, similarly the inequalities (1.10).

To prove the corresponding inequalities for $\delta G_k(\Omega,\Omega_0,A)=G_k(\Omega,A)-G_k(\Omega_0,A)$ with $\Omega\subset\Omega_0$, we take the representations (2.13) for both propagators. The terms with ω such that $\Box_{\omega_t}$ are interior cubes of Ω are the same in both representations, so they cancel in the difference, and for δG_k we get a representation similar to (2.13) with the additional restriction that at least one $\Box_{\omega_t}$ intersects the boundary $\partial\Omega$. We estimate the terms of the representation as above and we get the first inequality in (2.22) with 2^{d+1} instead of 2^d and with the restriction

$$n\geqq M^{-1}\sup_{x_1\in\Omega^c}(\mathrm{dist}(\{x,x'\},x_1)+\mathrm{dist}(x_1,\mathrm{supp} f))-3$$

$$\geqq(2M)^{-1}(\mathrm{dist}(\{x,x'\},\mathrm{supp} f)+\mathrm{dist}(\{x,x'\},\Omega^c)+\mathrm{dist}(\mathrm{supp} f,\Omega^c))-3.$$

It implies all the inequalities we need. Finally let us notice that if Ω is a rectangular parallelepiped, then all $\Box_j$ in the representation (2.13) are cubes and we can apply Lemma 2.2 to all operators in it, so the restriction $\mathrm{dist}(\{x,x'\},\Omega^c)\geqq R_0$ is unnecessary. Thus we have proved the theorem, or rather reduced it to Lemmas 2.1, 2.2.

Proof of Lemma 2.1. The first part of the proof is the same for Lemma 2.2 and is based on the ideas of the proof of Lemma 2.4 in [2]. We can write the configuration A as a sum A_0+A', where A_0 is a constant configuration, e.g. it is a value of A at some point of $\Box$, and A' is a small and regular configuration, i.e. we have

$$A=A_0+A',\qquad |A'|,|\partial_\mu^\eta A'|\leqq c'e^{\beta-1},\tag{2.23}$$

c' depends on c and M. Now we expand the propagator $G_k(\Box,A)$ with respect to A'. Using formula I.3.16 we have

$$G_k^{1/2}(\Box,A_0)(-\Delta_{A,\Box}^{\eta,N}+m_k^2+a_k P_k(A))G_k^{1/2}(\Box,A_0)$$
$$=I-G_k^{1/2}(\Box,A_0)[F_{1,k}(-A')^* D_{A_0}^\eta+D_{A_0}^{\eta*}F_{1,k}(-A')$$
$$-F_{1,k}(-A')^* F_{1,k}(-A')-a_k F_{2,k}(A',A_0)^* Q_k(A_0)$$
$$-a_k Q_k^*(A_0)F_{2,k}(A',A_0)-a_k F_{2,k}(A',A_0)^* F_{2,k}(A',A_0)]G_k^{1/2}(\Box,A_0)$$
$$=:I-G_k^{1/2}(\Box,A_0)V_k G_k^{1/2}(\Box,A_0),\tag{2.24}$$

where $F_{1,k}(A) = \eta^{-1}(U(A) - 1)$,

$$(F_{2,k}(A', A_0)\phi)(y) = \sum_{x \in B^k(y)} \eta^d F'_{1,k}(A'(\Gamma^{(k)}_{y,x})) U(A_0(\Gamma^{(k)}_{y,x}))\phi(x),$$

and $F'_{1,k}(A) = U(A) - 1$. Taking into account the bound (2.23) and the inequalities

$$\|G_k^{1/2}(\square, A_0)\|_{2,2}, \qquad \|D_{A_0} G_k^{1/2}(\square, A_0)\|_{2,2}, \qquad \|G_k^{1/2}(\square, A_0)D_{A_0}^*\|_{2,2} \le c_0, \tag{2.25}$$

holding for some absolute constant c_0, we can easily prove that

$$\|G_k^{1/2}(\square, A_0)V_k G_k^{1/2}(\square, A_0)\|_{2,2} \le O(1)e^\beta,$$

with a constant $O(1)$ depending on M only. For e sufficiently small, we get the representation

$$G_k(\square, A) = G_k^{1/2}(\square, A_0)(I - G_k^{1/2}(\square, A_0)V_k G_k^{1/2}(\square, A_0))^{-1} G_k^{1/2}(\square, A_0), \tag{2.26}$$

and a similar representation for derivatives of $G_k(\square, A)$, decomposing them on the right hand side according to the formula $D_A^\eta = U(A')D_{A_0}^\eta + F_{1,k}(A')$. From these representations and (2.25) we get easily the inequalities (2.15). Thus we have to prove the bounds (2.25). Using the same gauge transformation as in the proof of Lemma 2.4, we reduce them to the case $A_0 = 0$. Now these bounds are consequences of quadratic form considerations. At first let us notice that $\square$ is a sum of unit cubes Δ and we have

$$\langle \phi, (-\Delta_\square^{\eta,N} + m_k^2 + a_k P_k)\phi \rangle \ge \sum_{\Delta \subset \square} \langle \phi, (-\Delta_\Delta^{\eta,N} + a_k P_k)\phi \rangle. \tag{2.26}$$

The operator $-\Delta_\Delta^{\eta,N}$ is bounded from below by π^2 on a subspace of functions on Δ orthogonal to constant functions, which are its eigenvectors corresponding to eigenvalue 0. The operator P_k is an orthogonal projection on a subspace of constant functions, thus

$$\langle \phi, (-\Delta_\Delta^{\eta,N} + a_k P_k)\phi \rangle \ge \min\{\pi^2, a_k\}\|\phi\|_{L^2(\Delta)}, \tag{2.27}$$

and from this and (2.26) we get

$$0 < G_k(\square, 0) \le c_0 I, \qquad c_0^{-1} = \min\{\pi^2, a_k\}, \quad \text{hence} \quad \|G_k(\square, 0)f\|_2 \le c_0\|f\|_2. \tag{2.28}$$

Further we have

$$\|\partial_\mu^\eta G_k^{1/2}(\square, 0)f\|_2^2 = \langle f, G_k^{1/2}(\square, 0)\partial_\mu^{\eta*}\partial_\mu^\eta G_k^{1/2}(\square, 0)f \rangle$$
$$\le \langle f, G_k^{1/2}(\square, 0)(-\Delta_\square^{\eta,N} + m_k^2 + a_k P_k)G_k^{1/2}(\square, 0)f \rangle = \|f\|_2^2, \tag{2.29}$$

hence $\|\partial_\mu^\eta G_k^{1/2}(\square, 0)\|_{2,2} = \|G_k^{1/2}(\square, 0)\partial_\mu^{\eta*}\|_{2,2} \le 1$, and this together with (2.28) imply the bounds (2.25). Thus the Lemma 2.1 is proved.

Remark. Let us notice that this lemma alone implies a weaker version of Proposition I.2.1 with L^2-norms. More exactly we have

Corollary 2.3. *If Ω and A are as in Proposition I.2.1, then there exist positive constants c_0, δ_0 such that for arbitrary scalar field configurations f, f' defined on Ω, we have*

$$|\langle f, G_k(\Omega, A)f'\rangle|, \quad |\langle f, D_{A,\mu}^\eta G_k(\Omega, A)f'\rangle|, \quad |\langle f, G_k(\Omega, A)D_{A,\nu}^{\eta*}f'\rangle|,$$
$$|\langle f, D_{A,\mu}^\eta G_k(\Omega, A)D_{A,\nu}^{\eta*}f'\rangle| \le c_0 e^{-\delta_0 \operatorname{dist}(\operatorname{supp} f, \operatorname{supp} f')}\|f\|_2\|f'\|_2. \tag{2.30}$$

The same inequalities hold for $\delta G_k(\Omega, \Omega_0, A)$ *with the additional factor* $e^{-\delta_0(\text{dist}(\text{supp} f, \Omega^c) + \text{dist}(\text{supp} f', \Omega^c))}$.

Of course it is enough to prove it for f, f' with supports in unit cubes, and the proof proceeds as before using only the L^2-bounds of Lemma 2.1. Let us notice also that now there are no restrictions on supports of f, f', so in this aspect the Corollary is a little bit stronger than Proposition I.2.1.

Now we begin a

Proof of Lemma 2.2. It is much longer and will be reduced to some other lemmas again. We start with the same operations as in the proof of Lemma 2.2 and we have the representation (2.26) for e sufficiently small, which we can write in the following form

$$G_k(\square, \tilde{A}) = \sum_{n=0}^{\infty} G_k(\square, A_0)(V_k G_k(\square, A_0))^n. \tag{2.31}$$

Now we have to choose A_0 more carefully. According to our assumptions $\tilde{A}$ is constant in a neighbourhood of boundary of $\square$, and we choose A_0 equal to this constant. Thus in the decomposition $\tilde{A} = A_0 + A'$, the configuration A' is regular and has a compact support in $\square$. The series in (2.31) is convergent in $L^2(\square)$, but we will prove that it is in fact convergent in stronger norms appearing in formulation of Lemma 2.2.

Let us assume that Lemma 2.2 holds for $G_k(\square, A_0)$ with constant configurations A_0. We will prove it for a general case using (2.31).

At first let us observe that V_k can be interpreted as a first order differential operator acting on a function on the right hand side of it. The only trouble is with the term $D_{A_0}^{\eta*} F_{1,k}(-A')$, for which it is not a natural interpretation. Using the formula (2.4) and the fact that A' has a compact support in $\square$, we have

$$(D_{A_0}^{\eta*} F_{1,k}(-A') f)(x) = (\partial^{\eta*} F_{1,k}(-A'))(x) f(x) + \sum_{\mu=1}^{d} F_{1,k}(-A'_{\langle x - \eta e_\mu, x \rangle})(D_{A_0, \mu}^{\eta*} f)(x)$$

$$= - \sum_{\mu=1}^{d} (\partial_\mu^\eta F_{1,k}(-A'_\mu))(x - \eta e_\mu) f(x)$$

$$- \sum_{\mu=1}^{d} F_{1,k}(-A'_\mu(x - \eta e_\mu)) U(-A_{0,\mu})(D_{A_0, \mu}^\eta f)(x - \eta e_\mu). \tag{2.32}$$

Thus it is a first order differential operator with small coefficients. Only here we needed the assumption that $\tilde{A}$ is constant in a neighbourhood of $\partial\square$.

Now using Lemma 2.2 for $G_k(\square, A_0)$ and the decomposition $D_{\tilde{A}}^\eta = U(A') D_{A_0}^\eta + F_{1,k}(A')$, we have

$$\|G_k(\square, \tilde{A}) f\|_{1,\alpha} \leq O(1) c_1 \sum_{n=0}^{\infty} \|(V_k G_k(\square, A_0))^n f\|_\infty \leq O(1) c_1 \sum_{n=0}^{\infty} (O(1) e^\beta c_2)^n \|f\|_\infty. \tag{2.33}$$

The series on the right hand side is convergent for e sufficiently small, and we get the inequality (2.16). The inequality (2.17) is proved in the same way.

Lemma 2.2 in the case of a constant configuration A_0 is equivalent to the case of configuration $A_0 = 0$ by the same argument with the gauge transformation as before. Thus we have reduced the proof of this lemma to a proof of the

corresponding properties for the propagator $G_k(\square, 0)$, or to the one-component propagator $G_k(\square)$ $[G_k(\square, 0) = G_k(\square)1$, 1 is identity operator on $R^N]$. This proof is based on renormalization group equations (2.43) of [1] rescaled to the η-lattice:

$$G_k(\square) = C^{(0), \eta}(\square) + \sum_{j=1}^{k-1} a_j^2 (L^j \eta)^{-4} G_j^\eta(\square) Q_j^* C^{(j), L^j \eta}(\square) Q_j G_j^\eta(\square). \qquad (2.34)$$

The expressions on the right hand side above are built with the help of the operators $G_j^\eta(\square) Q_j^*$, $C^{(j), L^j \eta}(\square)$ and we need some of their basic properties. They are gathered in the following

Lemma 2.4. *There exist positive constants c_0, δ_0, and for $\alpha < 1$, there exists a constant c_1, such that*

$$|(G_j(\square) Q_j^*)(x, y)|, |(\partial_\mu^{L^{-j}} G_j(\square) Q_j^*)(x, y)| \leqq c_0 e^{-\delta_0 |x-y|}, \qquad (2.35)$$

$$\frac{1}{|x-x'|^\alpha} |(\partial_\mu^{L^{-j}} G_j(\square) Q_j^*)(x, y) - (\partial_\mu^{L^{-j}} G_j(\square) Q_j^*)(x', y)|$$

$$\leqq c_1 e^{-\delta_0 \operatorname{dist}(\{x, x'\}, y)}, \qquad (2.36)$$

$$|C^{(j)}(\square; y, y')| \leqq c_0 e^{-\delta_0 |y-y'|}, \qquad (2.37)$$

for arbitrary non-negative integer j, arbitrary, rectangular parallelepiped $\square \subset L^{-j} Z^d$ built of large blocks, and $x, x' \in \square$, $y, y' \in \square^{(j)} = \square \cap Z^d$.

Using this lemma we will prove Lemma 2.2 for the propagator $G_k(\square)$. Let us start with the proof of the inequality (2.16). We use (2.34) with j^{th} term rescaled to the L^{-j}-lattice:

$$\frac{1}{|x-x'|^\alpha} ((\partial_\mu^\eta G_k(\square) f)(x) - (\partial_\mu^\eta G_k(\square) f)(x'))$$

$$= \frac{\eta^{1-\alpha}}{|\eta^{-1} x - \eta^{-1} x'|^\alpha} ((\partial_\mu^1 C^{(0)}(\eta^{-1} \square) f)(\eta^{-1} x) - (\partial_\mu^1 C^{(0)}(\eta^{-1} \square) f)(\eta^{-1} x'))$$

$$+ \sum_{j=1}^{k-1} a_j^2 \sum_{y \in (L^j \eta)^{-1} \square^{(j)}} \frac{(L^j \eta)^{1-\alpha}}{|(L^j \eta)^{-1} x - (L^j \eta)^{-1} x'|^\alpha}$$

$$\cdot ((\partial_\mu^{L^{-j}} G_j((L^j \eta)^{-1} \square) Q_j^*)((L^j \eta)^{-1} x, y) - (\partial_\mu^{L^{-j}} G_j((L^j \eta)^{-1} \square) Q_j^*)((L^j \eta)^{-1} x', y))$$

$$\cdot (C^{(j)}((L^j \eta)^{-1} \square) Q_j G_j((L^j \eta)^{-1} \square) f)(y), \qquad x, x' \in \square, \qquad (2.38)$$

hence Lemma 2.4 implies

$$\frac{1}{|x-x'|^\alpha} |(\partial_\mu^\eta G_k(\square) f)(x) - (\partial_\mu^\eta G_k(\square) f)(x')|$$

$$\leqq \eta^{1-\alpha} 4 c_0 O(1) \|f\|_\infty$$

$$+ \sum_{j=1}^{k-1} a_j^2 (L^j \eta)^{1-\alpha} c_1 \sum_{y \in (L^j \eta)^{-1} \square^{(j)}} e^{-\delta_0 \operatorname{dist}(\{(L^j \eta)^{-1} x, (L^j \eta)^{-1} x'\}, y)}$$

$$\cdot |(C^{(j)}((L^j \eta)^{-1} \square) Q_j G_j((L^j \eta)^{-1} \square) f)(y)|$$

$$\leqq O(1) \sum_{j=0}^{k-1} (L^j \eta)^{1-\alpha} \|f\|_\infty \leqq c_1' \|f\|_\infty, \qquad (2.39)$$

where the constant c_1' is built of $c_0, c_1, \sum_{x \in Z^d} e^{-\delta_0|x|}, \sum_{j=1}^{\infty} (L^{-j})^{1-\alpha}$. We estimate in the same way $|(G_k(\square)f)(x)|, |(\partial_\mu^\eta G_k(\square)f)(x)|$, and we get (2.16).

Remark. Let us mention here that the above method can be used also to prove pointwise estimates of $G_k(\square), G_k(\square, A)$, and even $G_k(\Omega, A)$, their derivatives and "Hölder-derivatives".

Now we will prove the inequality (2.17) for $G_k(\square)$. For $q = p = \infty$, it is a special case of (2.16), but for lack of dependence on α, and was proved above. For $q = p = 1$ we get it by duality argument, i.e. using the fact that the space $L^\infty(\square)$ is adjoint to $L^1(\square)$. The Riesz-Thorin Theorem implies it for arbitrary $q = p$ from $[1, \infty]$. (For Riesz-Thorin Theorem see Ref. [6].)

Now we will prove that the operators $G_k(\square), \partial_\mu^\eta G_k(\square), G_k(\square)\partial_\mu^{\eta*}$ are bounded operators from $L^{p_1}(\square)$ with $p_1 > d$ to $L^\infty(\square)$. We will use again the representation (2.34). Let us consider for example the operator $G_k(\square)\partial_\mu^{\eta*}$:

$$(G_k(\square)\partial_\mu^{\eta*}f)(x) = \eta(C^{(0)}(\eta^{-1}\square)\partial_\mu^{1*}f)(\eta^{-1}x)$$
$$+ \sum_{j=1}^{k-1} a_j^2 L^j \eta(G_j((L^j\eta)^{-1}\square)Q_j^*C^{(j)}((L^j\eta)^{-1}\square)Q_jG_j((L^j\eta)^{-1}\square)\partial_\mu^{L-j*}f)$$
$$\cdot ((L^j\eta)^{-1}x). \tag{2.40}$$

Hence

$$|(G_k(\square)\partial_\mu^{\eta*}f)(x)| \leq \eta 2c_0 O(1)\eta^{-d/p_1}\|f\|_{p_1}$$
$$+ \sum_{j=1}^{k-1} O(1)L^j\eta \sum_{y \in (L^j\eta)^{-1}\square^{(j)}} e^{-\delta_0|(L^j\eta)^{-1}x-y|}|(Q_jG_j((L^j\eta)^{-1}\square)\partial_\mu^{L-j*}f)(y)|$$
$$\leq O(1)\eta^{1-d/p_1}\|f\|_{p_1} + \sum_{j=1}^{k-1} O(1)(L^j\eta)^{1-d/p_1}\|f\|_{p_1} \leq c_2'\|f\|_{p_1} \tag{2.41}$$

for $x \in \square, p_1 > d$, where the constant c_2' is built of $c_0, \sum_{x \in Z^d} e^{-\delta_0|x|}, \sum_{j=1}^{\infty} (L^{-j})^{1-d/p_1}$. Again by the duality argument we get that the operator $G_k(\square)$ and its first order derivatives are bounded operators from $L^1(\square)$ to $L^{p_1'}(\square), p_1'^{-1} + p_1^{-1} = 1$. The Riesz-Thorin Theorem gives us finally (2.17) for $G_k(\square)$ and for all p, q described in the figure
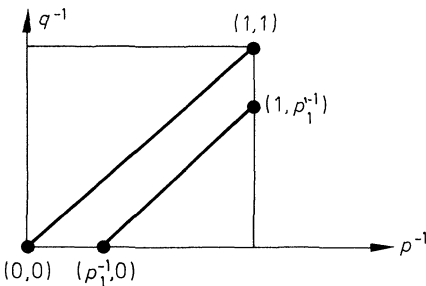
$$\frac{1}{p} - \frac{1}{p_1} \leq \frac{1}{q} \leq \frac{1}{p}, \quad 1 \leq p, q \leq \infty.$$

Thus Lemma 2.2 is proved, or rather reduced to Lemma 2.4.

Proof of Lemma 2.4. At first let us notice that the inequality (2.37) concerning $C^{(j)}(\square)$ is a special case of Proposition 2.3. The proof of this proposition will be given in the next section and is based on Corollary 2.3 only. Thus we can assume that (2.37) is proved.

The proof of (2.35), (2.36) can be reduced again to a simpler case. This part of the argument is valid for an arbitrary rectangular parallelepiped $\square$ built of unit blocks, so the inequalities are valid for all such sets. Let us denote $\xi = L^{-j}$. We represent $G_j(\square)$ with the help of the propagator G_j with free boundary conditions on ξZ^d using the multiple reflection method. If $\square$ is written as

$$\square = \{x \in \xi Z^d : 0 \leqq x_\mu \leqq M_\mu, \mu = 1, \ldots, d\}, \text{ then}$$

$$G_j(\square; x, x') = G_j(x, x') + \sum_{\mu=1}^{d} G_j(x, (x'_1, \ldots, -x'_\mu - \xi, \ldots, x'_d))$$

$$+ \sum_{\mu=1}^{d} G_j(x, (x'_1, \ldots, 2M_\mu - \xi - x'_\mu, \ldots, x'_d)) + \ldots. \quad (2.42)$$

Using this representation it is enough to prove (2.35), (2.36) for the propagator G_j.

Now we will construct an explicit representation for G_j. Let us introduce a Fourier transform on ξZ^d by the formulas

$$\tilde{f}(p) = \sum_x \xi^d e^{-ip \cdot x} f(x), \qquad f(x) = (2\pi)^{-d} \int_{|p| \leqq \pi/\xi} e^{ix \cdot p} \tilde{f}(p). \quad (2.43)$$

We apply it to the basic equation

$$(-\Delta^\xi + m_j^2 + a_j Q_j^* Q_j)\phi_0 = f. \quad (2.44)$$

Defining the propagator G_j, $\phi_0 = G_j f$, we get

$$\Delta^\xi(p)\tilde{\phi}_0(p) + a_j u_j(p) \sum_{l'} \overline{u_j(p' + l')} \tilde{\phi}_0(p' + l') = \tilde{f}(p),$$

$$u_j(p) = \prod_{\mu=1}^{d} \frac{e^{-ip_\mu} - 1}{\dfrac{e^{-i\xi p_\mu} - 1}{\xi}}, \qquad \Delta^\xi(p) = \sum_{\mu=1}^{d} \left|\frac{e^{-i\xi p_\mu} - 1}{\xi}\right|^2 + m_j^2,$$

$$p' \in [-\pi, \pi[^d, \qquad l' = (l'_1, \ldots, l'_d), \qquad l'_\mu = 2\pi m'_\mu, \quad (2.45)$$

$$m'_\mu \text{ is an integer}, \quad -\frac{L^j - 1}{2} \leqq m'_\mu \leqq \frac{L^j - 1}{2} \quad \text{for } L \text{ odd},$$

$$\frac{-L^j}{2} \leqq m'_\mu < \frac{L^j}{2} \quad \text{for } L \text{ even}.$$

Solving this equation we obtain the following formula:

$$\tilde{\phi}_0(p' + l) = \frac{1}{\Delta^\xi(p' + l)} \tilde{f}(p' + l) - \frac{u_j(p' + l)}{\Delta^\xi(p' + l)}$$

$$\cdot \frac{a_j}{a_j \sum_{l'} \dfrac{|u_j(p' + l')|^2}{\Delta^\xi(p' + l')} + 1} \sum_{l'} \frac{\overline{u_j(p' + l')}}{\Delta^\xi(p' + l')} \tilde{f}(p' + l'). \quad (2.46)$$

To calculate $G_j Q_j^*$, we take $f = Q_j^* g$ in this formula, g is a function on the unit lattice, $\tilde{f}(p) = u_j(p)\tilde{g}(p')$, $\tilde{g}(p') = \sum_y e^{-ip' \cdot y} g(y)$, and we get

$$\widetilde{(G_j Q_j^* g)}(p' + l) = \frac{u_j(p' + l)}{\Delta^\xi(p' + l)} \frac{1}{a_j \sum_{l'} \dfrac{|u_j(p' + l')|^2}{\Delta^\xi(p' + l')} + 1} \tilde{g}(p'). \tag{2.47}$$

From this we obtain finally the following formula for $(G_j Q_j^*)(x, y)$,

$$(G_j Q_j^*)(x, y) = (2\pi)^{-d} \int\limits_{|p'| \leq \pi} dp'$$

$$\cdot \sum_l \frac{e^{i(p' + l) \cdot (x - y)}}{a_j \sum_{l'} \dfrac{|u_j(p' + l')|^2}{\Delta^\xi(p' + l')} + 1} \frac{u_j(p' + l)}{\Delta^\xi(p' + l)}. \tag{2.48}$$

We are now ready to prove the inequalities (2.35), (2.36); for example, let us take the expression on the left hand side of (2.36):

$$\frac{1}{|x - x'|^\alpha} ((\partial_\mu^\xi G_j Q_j^*)(x, y) - (\partial_\mu^\xi G_j Q_j^*)(x', y))$$

$$= (2\pi)^{-d} \int\limits_{|p'| \leq \pi} dp'$$

$$\cdot \sum_l \frac{\dfrac{1}{|x - x'|^\alpha}(e^{i(p' + l) \cdot (x - x')} - 1)e^{i(p' + l)(x' - y)}}{a_j \sum_{l'} \dfrac{|u_j(p' + l')|^2}{\Delta^\xi(p' + l')} \Delta^1(p') + \Delta^1(p')}$$

$$\cdot \frac{\partial_\mu^\xi(p' + l)u_j(p' + l)\Delta^1(p')}{\Delta^\xi(p' + l)}, \tag{2.49}$$

where

$$\Delta^1(p') = \sum_{\mu = 1}^d |e^{-ip_\mu'} - 1|^2 + m_j^2, \qquad \partial_\mu^\xi(p) = \frac{e^{i\xi p_\mu} - 1}{\xi},$$

and we have assumed that $|x' - y| \leq |x - y|$. It is easy to prove that the underintegral expression is bounded because

$$O(1) \prod_{\mu = 1}^d \frac{|p_\mu'|}{|p_\mu' + l_\mu|} \leq |u_j(p' + l)| \leq O(1) \prod_{\mu = 1}^d \frac{|p_\mu'|}{|p_\mu' + l_\mu|}$$

$$\leq O(1) \prod_{\mu = 1}^d \frac{1}{1 + |l_\mu|},$$

$$O(1) \frac{|p'|^2 + m_j^2}{|p' + l|^2 + m_j^2} \leq \frac{\Delta^1(p)}{\Delta^\xi(p' + l)} \leq O(1) \frac{|p'|^2 + m_j^2}{|p' + l|^2 + m_j^2},$$

$$|\partial_\mu^\xi(p' + l)| \leq O(1)|p_\mu' + l_\mu|,$$

$$\frac{1}{|x - x'|^\alpha} |e^{i(p' + l) \cdot (x - x')} - 1| \leq O(1)|p' + l|^\alpha,$$

hence

$$a_j \sum_{l'} \frac{|u_j(p'+l')|^2}{\Delta^\xi(p'+l')} \Delta^1(p') + \Delta^1(p') \geqq a_j \frac{|u_j(p')|}{\Delta^\xi(p')} \Delta^1(p')$$

$$\geqq O(1) > 0, \qquad (2.50)$$

and we have

|(the expression under the integral in (2.49))|

$$\leqq \sum_l O(1)|p'+l|^\alpha |p'_\mu + l_\mu| \frac{|p'|^2 + m_j^2}{|p'+l|^2 + m_j^2} \prod_{\mu=1}^d \frac{1}{1+|l_\mu|}$$

$$\leqq O(1) + \sum_{l \neq 0} O(1) \frac{1}{|p'+l|^{1-\alpha}} \prod_{\mu=1}^d \frac{1}{1+|l_\mu|}$$

$$\leqq O(1) + O(1) \left(\sum_{l \in Z} \frac{1}{(1+|l_\mu|)^{1+(1-\alpha)/d}} \right)^d \leqq O(1),$$

the constant depends on $\quad \alpha < 1$. $\qquad (2.51)$

The expression is a function of p' and can be extended as an analytic function to some neighbourhood of $[-\pi, \pi]^d$. It is more troublesome, but equally elementary, to prove that this neighbourhood can be chosen independently of j and that the expression is bounded also in this neighbourhood. Shifting the domain of integration in (2.49) into a complex domain in the direction of the vector $x' - y$, we can bound the left hand side of (2.49) by a constant depending on α multiplied by the exponential factor $e^{-\delta_0 |x'-y|} = e^{-\delta_0 \text{dist}(\{x, x'\}, y)}$. We get the inequality (2.36). The inequalities (2.35) are proved in the same way and constants depend on d only.

Thus we have proved Lemma 2.4, hence also the Theorem.

3. Proof of Proposition 2.3 of [1]

The proof will be based on explicit representation for the propagators $C_A^{(k)}(\Omega, A)$ in terms of $G_k(\Omega, A)$ or related propagators. We have the following equality

$$C_A^{(k)}(\Omega, A; y, y') = ((\Delta^{(k)}(\Omega, A) + aL^{-2}P(A))|_A)^{-1}(y, y')$$

$$= (Z_A^{(k)}(\Omega, A))^{-1} \int d\psi|_A \left(\frac{aL^{d-2}}{2\pi} \right)^{\frac{N|A'|}{2}}$$

$$\cdot \exp\left[-\frac{1}{2} aL^{d-2} \sum_{z \in A'} |(Q(A)\psi)(z)|^2 - \frac{1}{2} \langle \psi, \Delta^{(k)}(\Omega, A)\psi \rangle \right] \psi(y)\psi^*(y')$$

$$= (Z_A^{(k)}(\Omega, A))^{-1} \left(\frac{aL^{d-2}}{2\pi} \right)^{\frac{N|A'|}{2}} \int d\psi|_A \psi(y)\psi^*(y')(Z_k(\Omega, A))^{-1} \int d\phi|_\Omega \left(\frac{a_k}{2\pi} \right)^{\frac{N|\Omega^{(k)}|}{2}}$$

$$\cdot \exp\left[-\frac{1}{2} aL^{d-2} \sum_{z \in A'} |(Q(A)\psi)(z)|^2 \right.$$

$$\left. -\frac{1}{2} a_k \sum_{y \in \Omega^{(k)}} |(A\psi)(y) - (Q_k(A)\phi)(y)|^2 - \frac{1}{2} \langle \phi, (-\Delta_{A,\Omega}^{\eta,N} + m_k^2)\phi \rangle \right]$$

$$= (Z_A^{(k)}(\Omega, A) Z_k(\Omega, A))^{-1} \left(\frac{aL^{d-2}}{2\pi} \right)^{\frac{N|A'|}{2}} \left(\frac{a_k}{2\pi} \right)^{\frac{N|\Omega^{(k)}|}{2}}$$

$$
\cdot \int d\phi|_{\Omega} \left\{ \int d\psi|_{\Lambda} (\psi(y) + (Q_k(A)\phi)(y))(\psi^*(y') + (Q_k(A)\phi)^*(y')) \right.
$$

$$
\cdot \exp\left[-\tfrac{1}{2} a_k \sum_{y \in \Lambda} |\psi(y)|^2 - \tfrac{1}{2} a L^{d-2} \sum_{z \in \Lambda'} |(Q(A)\psi)(z)|^2 \right.
$$

$$
\left. - a L^{d-2} \sum_{z \in \Lambda'} (Q(A)\psi)(z)(Q_{k+1}(A)\phi)(z) \right] \right\}
$$

$$
\cdot \exp\left[-\tfrac{1}{2} a L^{d-2} \sum_{z \in \Lambda} |(Q_{k+1}(A)\phi)(z)|^2 \right.
$$

$$
\left. - \tfrac{1}{2} a_k \sum_{y \in \Omega^{(k)} \setminus \Lambda} |(Q_k(A)\phi)(y)|^2 - \tfrac{1}{2} \langle \phi, (-\Delta_{A,\Omega}^{\eta,N} + m_k^2)\phi \rangle \right]. \tag{3.1}
$$

The integral in the curly bracket {...} factorizes into integrals with respect to $\psi|_{B(z)}$. In these we make a gauge transformation $\psi(y) \to U(A(\Gamma_{y,z}))\psi(y)$, and we get integrals which can be factorized with respect to components of ψ. After this we have to compute integrals of the form

$$
\int d\psi|_{B(z)} \{1, \psi(y), \psi(y)\psi(y')\}
$$

$$
\cdot \exp\left[-\tfrac{1}{2} a_k \sum_{y \in B(z)} \psi^2(y) - \tfrac{1}{2} a L^{d-2} \left(L^{-d} \sum_{y \in B(z)} \psi(z) \right)^2 - a L^{d-2} L^{-d} \sum_{y \in B(z)} \psi(y) C \right], \tag{3.2}
$$

where C is a component of $(Q_{k+1}(A)\phi)(z)$.

We apply the general formula

$$
\int \prod_{j=1}^{N} dx_j \{1, x_j, x_j x_k\} \exp\left[-\tfrac{1}{2} A \left(\sum_{j=1}^{N} x_j \right)^2 - \tfrac{1}{2} B \sum_{j=1}^{N} x_j^2 + C \sum_{j=1}^{N} x_j \right]
$$

$$
= \left\{ 1, \frac{C}{NA+B}, \left(\frac{C}{NA+B} \right)^2 - \frac{A}{B(NA+B)} + \frac{\delta_{j,k}}{B} \right\}
$$

$$
\cdot (2\pi)^{N/2} ((NA+B)B^{N-1})^{-1/2} \exp\left(\frac{1}{2} \frac{NC^2}{NA+B} \right), \tag{3.3}
$$

and using the recursive relation $a_{k+1} = \dfrac{a a_k}{a L^{-2} + a_k}$, we get

$$
C_{\Lambda}^{(k)}(\Omega, A; y, y') = Z^{-1} \int d\phi|_{\Omega} \left\{ \left[-\frac{a_{k+1}}{a_k} L^{-2} (Q^*(A)Q_{k+1}(A)\phi)(y) + (Q_k(A)\phi)(y) \right] \right.
$$

$$
\cdot \left[-\frac{a_{k+1}}{a_k} L^{-2} (Q^*(A)Q_{k+1}(A)\phi)^*(y') + (Q_k(A)\phi)^*(y') \right]
$$

$$
\left. - \frac{a_{k+1}}{a_k^2} L^{-2} P(A; y, y') + \frac{\delta_{y,y'}}{a_k} \right\}
$$

$$
\cdot \exp\left[-\tfrac{1}{2} a_k \sum_{y \in \Omega^{(k)} \setminus \Lambda} |(Q_k(A)\phi)(y)|^2 \right.
$$

$$
\left. - \tfrac{1}{2} a_{k+1} L^{d-2} \sum_{z \in \Lambda'} |(Q_{k+1}(A)\phi)(z)|^2 - \tfrac{1}{2} \langle \phi, (-\Delta_{A,\Omega}^{\eta,N} + m_k^2)\phi \rangle \right], \tag{3.4}
$$

Z is of course a normalization factor. Denoting

$$
G_k(\Omega, \Lambda, A) = (-\Delta_{A,\Omega}^{\eta,N} + m_k^2 + a_k P_k(A)(\Omega \setminus B^k(\Lambda)) + a_{k+1} L^{-2} P_{k+1}(A)B^k(\Lambda))^{-1}, \tag{3.5}
$$

we have finally

$$
\begin{aligned}
C_A^{(k)}(\Omega, A; y, y') = & \left(\left[-\frac{a_{k+1}}{a_k} L^{-2} Q^*(A) Q_{k+1}(A) + Q_k(A)\right] G_k(\Omega, \Lambda, A) \right. \\
& \left. \cdot \left[-\frac{a_{k+1}}{a_k} L^{-2} Q_{k+1}^*(A) Q(A) + Q_k^*(A)\right]\right)(y, y') \\
& - \frac{a_{k+1}}{a_k^2} L^{-2} P(A; y, y') + \frac{\delta_{y,y'}}{a_k}.
\end{aligned}
\tag{3.6}
$$

A composition of an operator T defined on the η-lattice with $Q_k^*(A)$ or $Q_{k+1}^*(A)Q(A)$ can be interpreted as the action of the operator T on properly defined functions on the η-lattice. More exactly if we define the functions $q_k(y)$, $q_{k+1}(y)$ by the formulas

$$q_k(y, x) = \text{a column of } U(A(\Gamma_{x,y}^{(k)})) \text{ for } x \in B^k(y), \text{ 0 otherwise},$$

$$q_{k+1}(y, x) = \text{a column of } L^{-d} U(A(\Gamma_{x,z}^{(k+1)} \cup \Gamma_{z,y})) \text{ for } x \in B^{k+1}(z),$$

$$z \text{ such that } y \in B(z), \text{ 0 otherwise}, \tag{3.7}$$

then we have

$$
\begin{aligned}
& C_A^{(k)}(\Omega, A; y, y') \\
& = \left\langle -\frac{a_{k+1}}{a_k} L^{-2} q_{k+1}(y) + q_k(y), G_k(\Omega, \Lambda, A) \left(-\frac{a_{k+1}}{a_k} L^{-2} q_{k+1}(y') + q_k(y')\right) \right\rangle \\
& \quad - \frac{a_{k+1}}{a_k^2} L^{-2} P(A; y, y') + \frac{\delta_{y,y'}}{a_k}. \tag{3.8}
\end{aligned}
$$

The L^2-norms of the functions $q_k(y)$, $q_{k+1}(y)$ are equal to 1, $L^{-d/2}$ and their supports are in $B^k(y)$, $B^{k+1}(z(y))$ respectively, so the inequalities (1.11), (1.12), and (1.16) are simple consequences of Corollary 2.3. At this moment let us mention that the corollary was proved for the propagator $G_k(\Omega, A)$, not for $G_k(\Omega, \Lambda, A)$, but the only difference between these operators is that $G_k(\Omega, \Lambda, A)$ is, according to (3.5), defined by L-blocks on unit scale Λ, i.e. blocks $B^{k+1}(z)$, $z \in \Lambda'$, instead of unit blocks $B^k(y)$, $y \in \Lambda$, for $G_k(\Omega, A)$. This difference is unessential and the whole proof of Corollary 2.3 goes on with only slight changes, the most important is that in the inequality (2.27) for $\Delta = B^{k+1}(z)$, $z \in \Lambda'$, we have to replace $\min\{\pi^2, a_k\}$ by $\min\{\pi^2 L^{-2}, a_{k+1} L^{-2}\}$, but this is also some positive constant only depending additionally on L. Generally we have to expect that bounds on $C_A^{(k)}(\Omega, A)$ will depend on L.

Now to prove the inequality (1.14) we have to estimate the difference $G_k(\Omega, \Lambda, A) - G_{k+1}^\eta(\Omega, A)$. This is easy because on the basis of (3.5) we have

$$
\begin{aligned}
& G_k(\Omega, \Lambda, A) - G_{k+1}^\eta(\Omega, A) \\
& = G_k(\Omega, \Lambda, A)[a_k P_k(A) - a_{k+1} L^{-2} P_{k+1}(A)](\Omega \backslash B^k(\Lambda)) G_{k+1}^\eta(\Omega, A), \tag{3.9}
\end{aligned}
$$

and for $f, f' \in L^2(B^k(\Lambda))$, we get

$$|\langle f, (G_k(\Omega, \Lambda, A) - G_{k+1}^\eta(\Omega, A)) f' \rangle| \leqq c_0 \|f\|_2 \|f'\|_2$$
$$\cdot \exp[-\delta_0(\text{dist}(\text{supp}\, f, \text{supp}\, f') + \text{dist}(\text{supp}\, f, B^k(\Lambda^c)) + \text{dist}(\text{supp}\, f', B^k(\Lambda^c)))].$$

$$(3.10)$$

The function $\delta C_\Lambda^{(k)}(\Omega, A)$ is given by (3.8) with the operator (3.9) instead of $G_k(\Omega, \Lambda, A)$ and the inequality (3.10) implies (1.14). This ends the proof of Proposition I.2.3. Another proof will be given in the last section.

4. Proof of the Lower Bound for the Quadratic Form $\Delta^{(k)}$ (Proposition 3.1′)

We will estimate the quadratic form on the left hand side of (1.18) in several ways, finally getting all the terms on the right hand side.

At first we have $G_k(\Omega, A) \leqq ((a_k P_k(A) + m^2)|_\Omega)^{-1}$, and the left side of (1.18) can be estimated from below by

$$\sum_{x \in \Omega^{(k)}} [a_k |\phi(x)|^2 - a_k^2 \phi(x) \cdot (Q_k(A)((a_k P_k(A) + m^2)|_{B^k(x)})^{-1} Q_k^*(A))(x, x) \phi(x)]. \quad (4.1)$$

Each term of the above sum is determined by the integral

$$\int d\phi'|_{B^k(x)} \exp\left[-\tfrac{1}{2} a_k |\phi(x) - (Q_k(A)\phi')(x)|^2 - \tfrac{1}{2} m^2 \sum_{x' \in B^k(x)} \eta^d |\phi'(x)|^2\right]. \quad (4.2)$$

This means that the integral is equal to const $\exp[-\tfrac{1}{2}(\text{the term of the sum (4.1)}$ corresponding to the point $x)]$. In this integral we make the gauge transformation $\phi'(x') = U(A(\Gamma_{x',x}^{(k)}))\phi''(x')$, and we integrate over the subspace orthogonal to the constant functions on $B^k(x)$. Then we get

$$\text{const} \int d\phi'' \exp(-\tfrac{1}{2} a_k |\phi(x) - \phi''|^2 - \tfrac{1}{2} m^2 |\phi''|^2)$$
$$= \text{const}' \exp\left(-\frac{1}{2}\left(a_k - \frac{a_k^2}{a_k + m^2}\right)|\phi(x)|^2\right), \quad (4.3)$$

and hence

$$(\text{the left hand side of (1.18)}) \geqq \sum_{x \in \Omega^{(k)}} \frac{a_k m^2}{a_k + m^2} |\phi(x)|^2 \geqq \frac{a_k}{a_k + O(1)} \sum_{x \in \Omega^{(k)}} m^2 |\phi(x)|^2, \quad (4.4)$$

if $m^2 \leqq O(1)$. Thus a part of the inequality (1.18) is proved.

Now let us consider the set of all positively oriented bonds $\langle x, x' \rangle \subset \Omega^{(k)}$. This set can be represented as a sum of $2d$ subsets B_i with the property that each point of $\Omega^{(k)}$ belongs to at most one bond in a given subset B_i. For example we can define $B_{2\mu-1}$ as a set of bonds $\langle x, x + e_\mu \rangle \subset \Omega^{(k)}$ for which the coordinate x_μ of the point x is an odd number; similarly we define $B_{2\mu}$ with the difference that x_μ is an even number. For a given B_i there may be points which do not belong to any bond of this set. Let us denote $\Delta(x, x') = B^k(x) \cup B^k(x')$. Again separating subsets of Ω by the Neumann boundary conditions we have

(the left hand side of (1.18))

$$\geqq \sum_{\langle x,x'\rangle \in B_i} [a_k|\phi(x)|^2 + a_k|\phi(x')|^2 - a_k^2\langle\phi, Q_k(A)G_k(\Delta(x,x'),A)Q_k^*(A)\phi\rangle]$$

$$+ \sum_{x\in\Omega^{(k)},\ x \text{ does not belong to a bond in } B_i} [a_k|\phi(x)|^2 - a_k^2\phi(x)$$

$$\cdot (Q_k(A)G_k(B^k(x),A)Q_k^*(A))(x,x)\phi(x)]. \tag{4.5}$$

Each term in the second sum of the right side (4.5) is non-negative, as it follows from the proof of (4.4), so we can omit these terms. Summing the inequalities (4.5) over B_i, $i = 1, \ldots, 2d$, we get the inequality

(the left hand side of (1.18))

$$\geqq \frac{1}{2d} \sum_{\langle x,x'\rangle \subset \Omega^{(k)}} [a_k|\phi(x)|^2 + a_k|\phi(x')|^2 - a_k^2\langle\phi, Q_k(A)G_k(\Delta(x,x'),A)Q_k^*(A)\phi\rangle]. \tag{4.6}$$

Now it is easily seen that to prove (1.18) it is sufficient to prove

$$a_k|\phi(x)|^2 + a_k|\phi(x')|^2 - a_k^2\langle\phi, Q_k(A)G_k(\Delta(x,x'),A)Q_k^*(A)\phi\rangle$$

$$\geqq \gamma_0'|U(A(\langle x,x'\rangle))\phi(x') - \phi(x)|^2 - O(1)e^2p^2(e)(|\phi(x)|^2 + |\phi(x')|^2), \tag{4.7}$$

with γ_0' independent of k, $\langle x,x'\rangle$, and A. Then $\gamma_0 = \frac{1}{2}\min\left\{\frac{1}{2d}\gamma_0', \frac{a_k}{a_k + O(1)}\right\}$. We will transform the left hand side of (4.7). Using the regularity condition for A, we write $A = A_0 + A'$ on $\Delta(x,x')$ with A_0 constant and A' satisfying the bounds $|A'|, |\partial_\mu^\eta A'| \leqq O(1)p(e)$. We expand the expression on the left hand side of (4.7) with respect to A' using (I.3.15), (I.3.44), and we separate terms of first order in e. Using Lemma 2.1 the remaining terms can be easily estimated by $O(e^2p^2(e))(|\phi(x)|^2 + |\phi(x')|^2)$. Now let us consider the terms of first order. Formally we could rely on the estimates of perturbative expansions in the third paper [3], but there they were derived in a different setting, the interaction terms considered on a domain far from the boundary of basic domains for propagators, so let us repeat this part of the argument. Denoting $\phi^{(k)} = a_k G_k(\Delta(x,x'),A_0)Q_k^*(A_0)\phi$, we have the following expression

$$2\sum_{b\subset\Delta(x,x')} [(D_{A_0}^\eta \phi^{(k)})(b) \cdot q\phi^{(k)}(b_-)]eA_b'$$

$$+ 2a_k \sum_{y=x,x'} (Q_k(A_0)\phi^{(k)})(y) \cdot \sum_{z\in B^k(y)} \eta^d e\eta A'(\Gamma_{y,z}^{(k)})qU(A_0(\Gamma_{y,z}^{(k)}))\phi^{(k)}(z)$$

$$- 2a_k \sum_{y=x,x'} \phi(y) \cdot \sum_{z\in B^k(y)} \eta^d e\eta A'(\Gamma_{y,z}^{(k)})qU(A_0(\Gamma_{y,z}^{(k)}))\phi^{(k)}(z). \tag{4.8}$$

Now we have to inspect closely the proof of the Lemma II.2.4. At first let us notice that we can "gauge away" the configuration A_0 using the same gauge transformation, and we get the expression (4.8) with $A_0 = 0$ and $\phi(x)$, $\phi(x')$ replaced by $\phi(x)$, $U(A_0(\langle x,x'\rangle))\phi(x')$ respectively. Then using II.2.75 we have

$$\phi^{(k)}(z) = \frac{1}{1+m^2}\phi(x) + O(U(A_0(\langle x,x'\rangle))\phi(x') - \phi(x)), \quad z\in\Delta(x,x'),$$

$$(\partial^\eta\phi^{(k)})(b) = O(U(A_0(\langle x,x'\rangle))\phi(x') - \phi(x)), \quad b\in\Delta(x,x'),$$

$$\tag{4.9}$$

and this implies the following estimate

$$|(\text{the expression } (4.8))|$$

$$\leq \left| \frac{4a_k}{(1+m^2)^2} [\phi(x) \cdot q\phi(x)] \sum_{y=x,x'} \sum_{z \in B^k(y)} \eta^d e\eta A'(\Gamma_{y,z}) \right.$$

$$- \left. \frac{4a_k}{1+m^2} [\phi(x) \cdot q\phi(x)] \sum_{y=x,x'} \sum_{z \in B^k(y)} \eta^d e\eta A'(\Gamma_{y,z}) \right|$$

$$+ O(1)ep(e)|U(A_0(\langle x,x' \rangle))\phi(x') - \phi(x)|(|\phi(x)| + |\phi(x')|)$$

$$\leq \delta|U(A_0(\langle x,x' \rangle))\phi(x') - \phi(x)|^2 + O(\delta^{-1})e^2 p^2(e)(|\phi(x)|^2 + |\phi(x')|^2),$$

$$(\phi(x) \cdot q\phi(x) = 0 \text{ because } q \text{ is an antisymmetric matrix}),$$

for arbitrary $\delta > 0$.   (4.10)

Thus we get

(the left hand side of (4.7))

$$\geq (\text{the left hand side of (4.7) with } A \text{ replaced by } A_0)$$

$$- \delta|U(A_0(\langle x,x' \rangle))\phi(x') - \phi(x)|^2 - O(\delta^{-1})e^2 p^2(e)(|\phi(x)|^2 + |\phi(x')|^2). \quad (4.11)$$

Now it is sufficient to prove the inequality

(the left hand side of (4.7) with A replaced by A_0) $\geq \gamma_0''|U(A_0(\langle x,x' \rangle))\phi(x') - \phi(x)|^2$,

(4.12)

because this together with (4.11) with $\delta = \frac{1}{2}\gamma_0''$ and

$$|U(A_0(\langle x,x' \rangle))\phi(x') - \phi(x)|^2 \geq \frac{1}{2}|U(A(\langle x,x' \rangle))\phi(x') - \phi(x)|^2 - O(1)e^2 p^2(e)|\phi(x)|^2,$$

(4.13)

give (4.7) with $\gamma_0' = \frac{1}{4}\gamma_0''$. Let us notice that the second term on the right hand side of (4.7) does not appear if $A = 0$.

Now we will prove (4.12). At first we estimate the left side of it from below by the same expression with $m^2 = 0$. Further we "gauge out" the constant field A from the expression with the simultaneous gauge transformations of $\phi : \phi(x) = \phi''(x)$, $\phi(x') = U(A(\langle x',x \rangle))\phi''(x')$, as in the proof of Lemma II.2.4. Then the inequality (4.12) is transformed into the inequality

$$a_k|\phi''(x)|^2 + a_k|\phi''(x')|^2 - a_k^2\langle \phi'', Q_k G_k(\Delta(x,x'),0)Q_k^* \phi'' \rangle \geq \gamma_0''|\phi''(x') - \phi''(x)|^2.$$

(4.14)

We use again the fact that const. $\exp[-\frac{1}{2}(\text{the left side of (4.14)})]$ is equal to the integral

$$\int d\phi'|_{\Delta(x,x')}$$
$$\cdot \exp[-\frac{1}{2}a_k|\phi''(x) - (Q_k\phi')(x)|^2 - \frac{1}{2}a_k|\phi''(x') - (Q_k\phi')(x')|^2 - \frac{1}{2}\langle \phi', (-\Delta_{\Delta(x,x')}^{\eta,N})\phi' \rangle].$$

(4.15)

Denoting $\Delta = B^k(x)$, $\Delta' = B^k(x')$, $\psi = \phi_k''(x') - \phi_k''(x)$, and making the translations $\phi'(y) = \phi(y) + \phi''(x)$ for $y \in \Delta$, $\phi'(y) = \phi(y) + \phi''(x)$ for $y \in \Delta'$ in the above integral, we get

$$\int d\phi|_{\Delta(x,x')}$$

$$\cdot \exp\left[-\tfrac{1}{2}a_k|(Q_k\phi)(x)|^2 - \tfrac{1}{2}\langle\phi,(-\Delta_\Delta^{\eta,N})\phi\rangle - \tfrac{1}{2}a_k|(Q_k\phi)(x')|^2 - \tfrac{1}{2}\langle\phi,(-\Delta_\Delta^{\eta,N})\phi\rangle\right.$$

$$\left.-\tfrac{1}{2}\sum_{b\in B(\langle x,x'\rangle)}\eta^d|(\partial^\eta\phi)(b) + \eta^{-1}\psi|^2\right] = \text{const}\,\exp(-\tfrac{1}{2}A_0|\psi|^2)\,, \tag{4.16}$$

where $B(\langle x,x'\rangle)$ denotes the set of all bonds of the η-lattice connecting the block $B^k(x)$ with the block $B^k(x')$, i.e. $B(\langle x,x'\rangle) = \{b\subset T_\eta: b_-\in B^k(x),\ b_+\in B^k(x')\}$. Now we have to show that $A_0 > 0$, more exactly $A_0 \geqq \gamma_0''$, $\gamma_0'' > 0$, and γ_0'' depends on d and a only. It is obvious that the integral on the left hand side and the function on the right hand side of (4.16) factorizes into the products of the corresponding expressions for the components of ϕ and ψ. We can assume that ϕ and ψ are real quantities. We will use the following simple formula for the Fourier transform of the Gaussian function:

$$\int d\psi\,\exp(-i\xi\psi)\exp(-\tfrac{1}{2}A_0\psi^2) = \sqrt{2\pi A_0^{-1}}\,\exp(-\tfrac{1}{2}A_0^{-1}\xi^2). \tag{4.17}$$

Now it is sufficient to prove that $0 < A_0^{-1} \leqq \gamma_0''^{-1}$, and $\gamma_0''^{-1}$ depends on d and a only. Let us calculate the Fourier transform of the integral on the left hand side of (4.16). At first we transform the last expression in the exponent:

$$\sum_{b\in B(\langle x,x'\rangle)}\eta^d|(\partial^\eta\phi)(b) + \eta^{-1}\psi|^2 = \sum_{b\in B(\langle x,x'\rangle)}\eta^d|(\partial^\eta\phi)(b)$$

$$-\sum_{b'\in B(\langle x,x'\rangle)}\eta^{d-1}(\partial^\eta\phi)(b')|^2 + \eta^{-1}|\psi + \sum_{b\in B(\langle x,x'\rangle)}\eta^d(\partial^\eta\phi)(b)|^2. \tag{4.18}$$

Let us introduce the notation

$$\langle\phi,A\phi\rangle = a_k|(Q_k\phi)(x)|^2 + \langle\phi,(-\Delta_\Delta^{\eta,N})\phi\rangle + a_k|(Q_k\phi)(x')|^2 + \langle\phi,(-\Delta_{\Delta'}^{\eta,N})\phi\rangle$$

$$+\sum_{b\in B(\langle x,x'\rangle)}\eta^d|(\partial^\eta\phi)(b) - \sum_{b'\in B(\langle x,x'\rangle)}\eta^{d-1}(\partial^\eta\phi)(b')|^2. \tag{4.19}$$

The Fourier transform with respect to ψ of the expression on the left hand side of (4.16) can be written in the form

$$\int d\psi\,\exp(-i\xi\psi)\int d\phi\,\exp\left[-\tfrac{1}{2}\eta^{-1}|\psi + \sum_{b\in B(\langle x,x'\rangle)}\eta^{d-1}(\phi(b_+) - \phi(b_-))|^2 - \tfrac{1}{2}\langle\phi,A\phi\rangle\right]$$

$$=\int d\psi\,\exp(-i\xi\psi)\exp(-\tfrac{1}{2}\eta^{-1}\psi^2)\int d\phi\,\exp i\xi\sum_{b\in B(\langle x,x'\rangle)}\eta^{d-1}(\phi(b_+) - \phi(b_-))$$

$$\cdot\exp(-\tfrac{1}{2}\langle\phi,A\phi\rangle)$$

$$=\sqrt{2\pi\eta}\,\exp(-\tfrac{1}{2}\eta\xi^2)(2\pi)^{L^k d}(\det A)^{-1/2}\exp(-\tfrac{1}{2}\langle f_{\langle x,x'\rangle},A^{-1}f_{\langle x,x'\rangle}\rangle\xi^2), \tag{4.20}$$

and we get the formula for A_0^{-1}:

$$A_0^{-1} = \eta + \langle f_{\langle x,x'\rangle},A^{-1}f_{\langle x,x'\rangle}\rangle, \tag{4.21}$$

where

$$f_{\langle x,x'\rangle}(y) = \sum_{b\in B(\langle x,x'\rangle)}\eta^{d-1}(\delta^\eta(y - b_+) - \delta^\eta(y - b_-)).$$

Of course both terms on the right side of (4.21) are $\geqq 0$, and $\eta = L^{-k}$, $0 < \eta < 1$. We

have to estimate the second term. From (4.19) and the definition of $f_{\langle x, x' \rangle}$ it follows that

$$\langle f_{\langle x, x' \rangle}, A^{-1} f_{\langle x, x' \rangle} \rangle \leqq \langle f_{\langle x, x' \rangle}, G_k(\Delta) f_{\langle x, x' \rangle} \rangle + \langle f_{\langle x, x' \rangle}, G_k(\Delta') f_{\langle x, x' \rangle} \rangle$$
$$= 2 \sum_{y, y' \in \Delta_1} \eta^{2(d-1)} G_k(\Delta; y, y'), \qquad (4.22)$$

where Δ_1 is any face of the block Δ, e.g. $\Delta_1 = \{y \in \Delta : y_1 = x_1 + 1 - \eta\}$. Of course the expression on the right side of (4.22) is a constant $\gamma_0''^{-1}$ depending on d and a only. Thus the inequality (4.14) is proved and the proof of Proposition 3.1' is completed.

Remark. The argument applied in the proof of (4.14) is an example of an application of "duality transformations". In our case this is an ordinary Fourier transform.

5. A General Theorem on Unit Lattice Operators

There is another way of proving Proposition I.2.3, relying on a general theorem concerning inverses to some unit lattice operators. This theorem is interesting enough in itself to devote a separate section to it.

At first let us prove that the bounds (1.11) are consequences of Proposition II.3.1'. The upper bound is quite elementary because

$$\Delta^{(k)}(\Omega, A) + aL^{-2} P(A) = a_k I - a_k^2 Q_k(A) G_k(\Omega, A) Q_k^*(A) + aL^{-2} P(A)$$

$$\leqq a_k I + aL^{-2} Q^*(A) Q(A) \leqq (a_k + aL^{-2}) I = a \frac{a_k}{a_{k+1}} I. \qquad (5.1)$$

The operators are considered as defined on $L^2(\Omega^{(k)})$. To prove the lower bound we apply Proposition II.3.1' to $\Delta^{(k)}(\Omega, A)$:

$$\langle \phi, (\Delta^{(k)}(\Omega, A) + aL^{-2} P(A)) \phi \rangle \geqq \gamma_0 \sum_{\langle x, x' \rangle \subset \Omega^{(k)}} |U(A(\langle x, x' \rangle)) \phi(x') - \phi(x)|^2$$

$$+ aL^{d-2} \sum_{y \in \Omega^{(k+1)}} \left| L^{-d} \sum_{x \in B(y)} U(A(\Gamma_{y,x})) \phi(x) \right|^2 - O(e^{2-\delta}) \sum_{x \in \Omega^{(k)}} |\phi(x)|^2. \qquad (5.2)$$

Now we use again the method we have applied so many times: in the expression on the right hand side above we separate the blocks by Neumann boundary conditions, in each block we decompose $A = A_0 + A'$ into a constant field A_0 and small field A' and we expand with respect to A' getting the same expression with A_0 instead of A and a bigger constant in the last sum. Next we "gauge away" the constant field, and we obtain the Laplace operator with Neumann boundary conditions for each block plus the projection operator on constant functions. This sum is bounded from below by a positive constant (more precisely by $\frac{1}{2} \gamma_0 \min \{\pi^2 L^{-2}, aL^{-2}\}$), thus we have

$$\langle \phi, (\Delta^{(k)}(\Omega, A) + aL^{-2} P(A)) \phi \rangle \geqq \gamma_0' \langle \phi, \phi \rangle - O(e^{2-\delta}) \langle \phi, \phi \rangle \geqq \gamma_0'' \langle \phi, \phi \rangle. \qquad (5.3)$$

for e sufficiently small and we get the lower bound. Finally Corollary 2.3 implies that the considered operator is short-ranged in the sense that for some $\delta_0 > 0$

$$|(\Delta^{(k)}(\Omega, A) + aL^{-2} P(A))(x, x')| \leqq c_0 e^{-\delta_0 |x - x'|}, \qquad x, x' \in \Omega^{(k)}, \qquad (5.4)$$

and a change of the domain Ω implies a change of the operator which can be estimated in the following way

$$|(\Delta^{(k)}(\Omega, A) - \Delta^{(k)}(\Omega_0, A))(x, x')| \leq c_0 e^{-\delta_0(|x-x'| + \text{dist}(x, \Omega^{(k)c}) + \text{dist}(x', \Omega^{(k)c}))},$$

$$\Omega \subset \Omega_0, \qquad x, x' \in \Omega^{(k)}. \tag{5.5}$$

From these properties it follows that Proposition I.2.3 is a consequence of the following

Theorem. *Let $\Omega \subset Z^d$ and let A be a symmetric operator defined on the space $L^2(\Omega)$ of functions $\phi : \Omega \to R^N$ and satisfying the following condition: there exist positive constants γ_0, c_0, δ_0 such that*

$$A \geq \gamma_0 I, \qquad |A(x, x')| \leq c_0 e^{-\delta_0|x-x'|}, \qquad x, x' \in \Omega. \tag{5.6}$$

Then there exist positive constants c_1, δ_1 such that for arbitrary $\Lambda \subset \Omega$ and for $C_\Lambda = A_\Lambda^{-1}$, A_Λ is an operator defined on $L^2(\Lambda)$ by $A_\Lambda = \Lambda A \Lambda$. We have

$$|C_\Lambda(x, x')| \leq c_1 e^{-\delta_1|x-x'|}, \qquad x, x' \in \Lambda, \tag{5.7}$$

$$|\delta C_\Lambda(x, x')| \leq c_1 e^{-\delta_1(|x-x'| + \text{dist}(x, \Lambda^c) + \text{dist}(x', \Lambda^c))}, \qquad \delta C_\Lambda = C_\Lambda - C_\Omega. \tag{5.8}$$

If we perturb the operator A by an operator B such that the condition (5.6) is satisfied for $A + B$, and additionally B has the property

$$|B(x, x')| \leq c_0 e^{-\delta_0(|x-x'| + \text{dist}(x, \Omega^c) + \text{dist}(x', \Omega^c))}, \qquad x, x' \in \Omega, \tag{5.9}$$

then we have also

$$|A_\Lambda^{-1}(x, x') - (A+B)_\Lambda^{-1}(x, x')| \leq c_1 e^{-\delta_1(|x-x'| + \text{dist}(x, \Omega^c) + \text{dist}(x', \Omega^c))}, \qquad x, x' \in \Lambda. \tag{5.10}$$

The rest of the section will be devoted to a

Proof of the Theorem. The proof goes along the lines of the proof of Proposition I.2.1. We start with a construction of a "generalized random walk" representation. We define the sets

$$\square_j = \Lambda \cap \{x \in Z^d : -M \leq x_\mu - M j_\mu < M, \mu = 1, \ldots, d\}, \qquad j \in Z^d, \tag{5.11}$$

and we introduce the same partition of unity h_j as before. In the sequel we restrict ourselves to these j's for which $\square_j \neq \phi$. Let us define

$$C_j = C_{\square_j}, \qquad C = \sum_j h_j C_j h_j. \tag{5.12}$$

We will show again that C is a very good approximation of C_Λ in the sense that $A_\Lambda C$ is almost an identity. We have

$$A_\Lambda C = \sum_j \Lambda A h_j C_j h_j = \sum_j \square_j A h_j C_j h_j + \sum_{j, j', j \neq j'} (1 - \square_j) h_j^2 A h_j C_j h_j$$

$$= \sum_j h_j A_{\square_j} C_j h_j + \sum_j \square_j [A, h_j] C_j h_j + \sum_{j, j', j \neq j'} (1 - \square_{j'}) h_j^2 A h_{j'} C_{j'} h_{j'}$$

$$= I - R, \tag{5.13}$$

where

$$R = \sum_{j,j'} R_{j,j'} C_{j'} h_{j'}, \quad R_{j,j} = -\Box_j [A, h_j] \Box_j, \quad R_{j,j'} = -(1 - \Box_{j'}) h_j^2 A h_{j'} \quad \text{for} \quad j \neq j'.$$
(5.14)

It is natural to interpret $R_{j,j'}$ as an operator $R_{j,j'} : L^2(\Box_{j'}) \to L^2(\Box_j)$, similarly $R_{j,j'} C_{j'} h_{j'}$. We have

$$|R_{j,j}(x, x')| = |A(x, x')(h_j(x') - h_j(x))| \leq c_0 e^{-\delta_0 |x - x'|} O(1) \frac{|x - x'|}{M}, \quad x, x' \in \Box_j,$$

$$|R_{j,j'}(x, x')| = |(1 - \Box_{j'})(x) h_j^2(x) A(x, x') h_{j'}(x')| \leq (1 - \Box_{j'}(x)) c_0 e^{-\delta_0 |x - x'|} h_{j'}(x')$$

$$\leq c_0 e^{-\frac{1}{2}\delta_0 |x - x'|} e^{-\frac{1}{6}\delta_0 M} \quad \text{for} \quad x \in \Box_j, \quad x' \in \Box_{j'},$$

and these estimates imply that the norms of the operators $R_{j,j'}$ have bounds

$$\|R_{j,j'}\| \leq \alpha e^{-\delta_2 |j - j'|}$$
(5.15)

for α depending on M and arbitrarily small if M is sufficiently large, δ_2 depending on δ_0, e.g. $\delta_2 = \frac{1}{4}\delta_0$. From these bounds it follows that

$$\|R\| \leq \max \left\{ \sup_j \sum_{j'} \|R_{j,j'}\|, \sup_{j'} \sum_j \|R_{j,j'}\| \right\} \gamma_0^{-1} \leq \alpha \gamma_0^{-1} \sum_{j \in Z^d} e^{-\delta_2 |j|}$$

is small for M large and

$$C_A = C(I - R)^{-1} = \sum_{n=0}^{\infty} CR^n,$$
(5.16)

which can be given a random walk form:

$$C_A = \sum_\omega h_{\omega_0} C_{\omega_0} h_{\omega_0} R_{\omega_1, \omega_2} C_{\omega_2} h_{\omega_2} \cdot \ldots \cdot R_{\omega_{2n-1}, \omega_{2n}} C_{\omega_{2n}} h_{\omega_{2n}},$$

$\omega = (\omega_0, \omega_1, \ldots, \omega_{2n})$, ω_i are arbitrary indices j, but satisfying the restrictions

$$\max_{\mu = 1, \ldots, d} |\omega_{2i, \mu} - \omega_{2i+1, \mu}| \leq 1 \quad \text{for} \quad i = 0, 1, \ldots, n-1, \quad n \geq 0.$$
(5.17)

Let us now prove (5.7). We have

$$|C_A(x, x')| \sum_{\omega : x \in \Box_{\omega_0}, x' \in \Box_{\omega_{2n}}} |\langle \delta_x, h_{\omega_0} C_{\omega_0} \ldots R_{\omega_{2n-1}, \omega_{2n}} C_{\omega_{2n}} h_{\omega_{2n}} \delta_{x'} \rangle|$$

$$\leq \sum_\omega' \gamma_0^{-(n+1)} \alpha^n e^{-\delta_2 |\omega_1 - \omega_2|} \cdot \ldots \cdot e^{-\delta_2 |\omega_{2n-1} - \omega_{2n}|}$$

$$\leq \sum_{j, j' : x \in \Box_j, x' \in \Box_{j'}} \left[\sum_{n=1}^{\infty} \gamma_0^{-(n+1)} \alpha^n (e^{d\delta_2})^n \right.$$

$$\cdot \sum_{j_1, \ldots, j_{2n-1} \in Z^d} e^{-\delta_2 |j - j_1|} \cdot \ldots \cdot e^{-\delta_2 |j_{2n-1} - j'|} + \left. \gamma_0^{-1} e^{d\delta_2} e^{-\delta_2 |j - j'|} \right].$$
(5.18)

To investigate the sums above, it is convenient to take the absolute value $|j|$ defined by $|j| = \sum_{\mu=1} |j_\mu|$; then the sum over j's factorizes into sums over components. If we define $g_1(j) = e^{-\delta_2 |j|}$, $j \in Z$, then the sum over j's in (5.18) can be written as $\prod_{\mu=1}^{d} g_{2n}(j_\mu - j'_\mu)$, $g_{2n} = g_1 * \ldots * g_1$ is a convolution of $2n$ functions g_1. The Fourier transform of g_1 is equal to

$$\sum_{j\in Z} e^{-ipj}e^{-\delta_2|j|} = (1-e^{-ip-\delta_2})^{-1} + (1-e^{ip-\delta_2})^{-1} - 1 = \frac{1-e^{-2\delta_2}}{(1-e^{-ip-\delta_2})(1-e^{ip-\delta_2})},$$

so we have

$$g_n(j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dp e^{ijp} \frac{(1-e^{-2\delta_2})^n}{(1-e^{-ip}e^{-\delta_2})^n(1-e^{ip}e^{-\delta_2})^n}, \qquad g_n(j) = g_n(|j|), \qquad (5.19)$$

and from this we obtain

$$|g_n(j)| \leq c_2^n e^{-\frac{1}{2}\delta_2|j|}, \text{ for some } c_2. \qquad (5.20)$$

Finally we fix M such that $\gamma_0^{-1}\alpha e^{d\delta_2}c_2^{2d} < 1$, and we get

$$|C_A(x,x')| \leq \sum_{j,j':x\in\square_j, x'\in\square_{j'}} \gamma_0^{-1}e^{d\delta_2}\frac{1}{1-\gamma_0^{-1}e^{d\delta_2}c_2^{2d}\alpha}e^{-\frac{1}{2}\delta_2|j-j'|}$$

$$\leq c_1 e^{-\frac{1}{2}\delta_2 M^{-1}|x-x'|}. \qquad (5.21)$$

This inequality implies (5.7) with $\delta_1 = \frac{1}{2}\delta_2 M^{-1}$. The inequality (5.8) is proved using the representations (5.17) for both operators C_A, C_Ω and properly taking into account cancellations of some terms, exactly as in the proof of bounds for $\delta G_k(\Omega, \Omega_0, A)$ in Sect. 2.

We define $\square'_j$ by the equality (5.11) with Ω instead of Λ, and $C'_j = C_{\square'_j}$. If $\square'_j$ is disjoint with the complement Λ^c of Λ in Ω, then $\square'_j = \square_j$ and $C'_j = C_j$. Similarly if both $\square'_j$, $\square'_{j'}$ are disjoint with Λ^c, then $R'_{j,j'} = R_{j,j'}$. Let us take a difference of the representations (5.17) for C_A and C_Ω. In this difference all terms corresponding to walks ω with $\square'_{\omega_i}$ disjoint with Λ^c are canceled, and we have

$$\delta C_A(x,x')$$

$$= \sum_{\substack{\omega:x\in\square_{\omega_0}, x'\in\square_{\omega_{2n}}, \\ \text{for some } i\square_{\omega_i}\cap\Lambda^c\neq\phi}} \langle\delta_x, h_{\omega_0}C_{\omega_0}h_{\omega_0}R_{\omega_1,\omega_2}C_{\omega_2}h_{\omega_2}\cdot\ldots\cdot R_{\omega_{2n-1},\omega_{2n}}C_{\omega_{2n}}h_{\omega_{2n}}\delta_{x'}\rangle$$

$$- \sum_{\substack{\omega':x\in\square'_{\omega_0}, x'\in\square'_{\omega_{2n}}, \\ \text{for some } i\square'_{\omega_i}\cap\Lambda^c\neq\phi}} \langle\delta_x, h_{\omega_0'}C'_{\omega_0'}h_{\omega_0'}R'_{\omega_1',\omega_2'}C'_{\omega_2'}h_{\omega_2'}\cdot\ldots\cdot R'_{\omega_{2n-1}',\omega_{2n}'}C'_{2n'}h_{\omega_{2n}'}\delta_{x'}\rangle.$$

$$(5.22)$$

The range of ω_i' in the second sum might be larger because $\Omega \supset \Lambda$. Estimating (5.22) in the same way as in (5.18) and (5.21), we get

$$|\delta C_A(x,x')| \leq c_1' e^{-\frac{1}{3}\delta_2 M^{-1}(\text{dist}(x,\Lambda^c)+\text{dist}(x',\Lambda^c)+|x-x'|)}. \qquad (5.23)$$

A proof of the last inequality (5.10) is very similar, although a little bit more awkward. We form the representations (5.17) for A_Λ^{-1} and $(A+B)_\Lambda^{-1}$, and we take their difference:

$$A_\Lambda^{-1} - (A+B)_\Lambda^{-1} = \sum_\omega [h_{\omega_0}C(A)_{\omega_0}h_{\omega_0}R(A)_{\omega_1,\omega_2}$$

$$\cdot C(A)_{\omega_2}h_{\omega_2}\cdot\ldots\cdot R(A)_{\omega_{2n-1},\omega_{2n}}C(A)_{\omega_{2n}}h_{\omega_{2n}}$$

$$- h_{\omega_0}C(A+B)_{\omega_0}h_{\omega_0}R(A+B)_{\omega_1,\omega_2}C(A+B)_{\omega_2}h_{\omega_2}\cdot\ldots$$

$$\cdot R(A+B)_{\omega_{2n-1},\omega_{2n}}C(A+B)_{\omega_{2n}}h_{\omega_{2n}}]$$

$$= \sum_\omega [h_{\omega_0}(C(A)_{\omega_0} - C(A+B)_{\omega_0})h_{\omega_0}R(A)_{\omega_1,\omega_2}C(A)_{\omega_2}h_{\omega_2}\cdot\ldots$$

$$+ h_{\omega_0}C(A+B)_{\omega_0}h_{\omega_0}(R(A)_{\omega_1,\omega_2} - R(A+B)_{\omega_1,\omega_2})$$

$$\cdot C(A)_{\omega_2}h_{\omega_2}\cdot\ldots + \ldots]. \qquad (5.24)$$

From the definition (5.14) of the operators $R_{j,\,j'}$ we have

$$R(A)_{j,\,j'} - R(A+B)_{j,\,j'} = -R(B)_{j,\,j'},$$

and from the condition (5.9), we get the estimate

$$\|R(B)_{j,\,j'}\| \leqq \alpha e^{-\delta_2|j-j'|} e^{-\delta_0(\mathrm{dist}(\square_j,\Omega^c)+\mathrm{dist}(\square_{j'},\Omega^c))}. \qquad (5.25)$$

Further we have

$$C(A)_j - C(A+B)_j = A_{\square_j}^{-1} - (A+B)_{\square_j}^{-1} = A_{\square_j}^{-1} B_{\square_j}(A+B)_{\square_j}^{-1}, \qquad (5.26)$$

and

$$\|C(A)_j - C(A+B)_j\| \leqq \gamma_0^{-2} c_0' e^{-2\delta_0\,\mathrm{dist}(\square_j,\Omega^c)}. \qquad (5.27)$$

The equality (5.24) and the above estimates imply

$$|A_A^{-1}(x,x') - (A+B)_A^{-1}(x,x')|$$

$$\leqq \sum_{\omega:\,x\in\square_{\omega_0},\,x'\in\square_{\omega_{2n}}} c_0' \gamma_0^{-(n+2)} \alpha^n (e^{d\delta_2})^n e^{-\delta_2|\omega_0-\omega_1|} \cdot \ldots \cdot e^{-\delta_2|\omega_{2n-1}-\omega_{2n}|}$$

$$\cdot \left(\sum_{i=1}^n e^{-2\delta_0\,\mathrm{dist}(\square_{\omega_{2i}},\Omega^c)} + \sum_{i=1}^n e^{-\delta_0(\mathrm{dist}(\square_{\omega_{2i-1}},\Omega^c)+\mathrm{dist}(\square_{\omega_{2i}},\Omega^c))} \right)$$

$$\leqq e^{-\frac{1}{3}\delta_2 M^{-1}(\mathrm{dist}(x,\Omega^c)+\mathrm{dist}(x',\Omega^c))} \gamma_0^{-2} c_0' e^{2d\delta_2} \frac{1}{1-\gamma_0^{-1}4e^{d\delta_2}c_2^{2d}\alpha} e^{-\frac{1}{3}\delta_2 M^{-1}|x-x'|}.$$

Thus Inequality (5.10) is proved. The constants δ_1, c_1 are functions of δ_0, γ_0, c_0, and from the above proof we can get more precise estimates for them.

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