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# Nonlinear Schrödinger Equations and Simple Lie Algebras

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**Abstract.** We associate a system of integrable, generalised nonlinear Schrödinger (NLS) equations with each Hermitian symmetric space. These NLS equations are considered as reductions of more general systems, this time associated with a reductive homogeneous space. The nonlinear terms are related to the curvature and torsion tensors of the appropriate geometrical space. The Hamiltonian structure is investigated using "$r$-matrix" techniques and shown to be "canonical" for all these equations. Throughout the reduction procedure this Hamiltonian structure does not degenerate. Each of the above systems of equations is gauge equivalent to a generalised ferromagnet. Reductions of the latter are discussed in terms of the corresponding NLS type equations.

## 1. Introduction

In the past few years it was discovered how to generalise the Toda lattice equations and their generalisations [1] to two dimensions. These $2-D$ generalised Toda lattices take the form of a multicomponent, Lorentz invariant, Lagrangian field theory. To each simple Lie algebra $\mathscr{g}$, there corresponds one such field theory [2–5]. The number of field components is equal to the rank of $\mathscr{g}$. When $\mathscr{g}=\mathrm{sl}(2,\mathbb{C})$, the single component satisfies the sine-Gordon equation.

This is just one of the "group theoretic" generalisations of the sine-Gordon equation. Others include the chiral model and the nonlinear sigma model, each of which can be associated with a given semi-simple Lie group. Various reductions of these occur by restricting the field components to lie on some homogenous space.

In this paper we carry out a similar generalisation and classification of the nonlinear Schrödinger (NLS) equation. Furthermore, to each such generalised NLS equation we give the corresponding generalised ferromagnet.

As is well known, the system of equations

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$$\begin{aligned} iq_{mt} &= q_{mxx} - 2\sum_{j=1}^{n} q_j p_j q_m, \\ -ip_{mt} &= p_{mxx} - 2\sum_{j=1}^{n} q_j p_j p_m, \end{aligned} \tag{1.1}$$

where $m = 1, \ldots, n$ can be solved exactly with the aid of an $\mathrm{sl}(n+1, \mathbb{C})$ linear problem:

$$\begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{n+1} \end{pmatrix}_x = \left(\begin{array}{c|ccc} n\lambda & q_1 & \cdots & q_n \\ \hline p_1 & -\lambda & & \\ \vdots & & \ddots & \\ p_n & & & -\lambda \end{array}\right) \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_{n+1} \end{pmatrix}. \tag{1.2}$$

Furthermore, there exists a restriction $p_i = -q_i^*$, associated with the compact real form $\mathrm{su}(n+1)$ of $\mathrm{sl}(n+1, C)$. This is the well known vector NLS equation and has internal symmetry $\mathrm{U}(n)$. This generalises the original NLS equation, obtained by putting $n = 1$. The number of independent (complex) $q$'s here is equal to the rank of the algebra, as was the case for the generalised $2-D$ Toda lattices. However, this is a coincidence and will not be true in the generalisations considered below. The potentials of the generalised Toda lattices lie within the Cartan subalgebra of $\mathscr{g}$ whereas those of the generalised NLS equations lie within the tangent space of a symmetric space. The vector NLS equations are associated with

$$\frac{\mathrm{SU}(n+1)}{\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(n))}.$$

Some other generalisations considered by Kulish and Sklyamin [6] are associated with

$$\frac{\mathrm{SU}(p+q)}{\mathrm{S}(\mathrm{U}(p) \times \mathrm{U}(q))};$$

these will be found in Sect. 3. The number of independent real fields is equal to the dimension of the symmetric space.

These particular symmetric spaces are very special, being Hermitian. This is necessary in order to equate $p_i$ with $-q_i^*$. However, for the purposes of our calculation, it is certain special algebraic properties of Hermitian symmetric spaces which are important.

For each Hermitian symmetric space $G/K$ there is a very special element $A$ of the Cartan subalgebra $\mathscr{h}$ of $\mathscr{g}$. The Lie algebra $\mathscr{k}$ of $K$ is given by $\mathscr{k} = C_{\mathscr{g}}(A) = \{B \in g\colon\ [A, B] = 0\}$ and the complex structure $J$ is realised by ad A. This gives us a canonical way of generalising the linear problem (1.2)

$$\phi_x = (\lambda A + Q(x, t))\phi, \tag{1.3}$$

where $Q(x, t) \in T_{p_0}(G/K) =$ tangent space to $G/K$ at point $p_0$. The properties (2.6) of a symmetric Lie algebra, together with the property $\mathscr{k} = C_{\mathscr{g}}(A)$, give rise to a simple form for the recursion operator associated with (1.3). The second order flow has cubic interaction term and the coupling coefficients are just the components of the Riemann tensor of $G/K$. The equations have Hamiltonian form (3.17) given in terms of invariant quantities associated with the symmetric space.

Each of these generalised NLS equations is a reduction of a generalised second order, "$N$-wave" equation. The latter is associated with the homogeneous space (no longer symmetric) $G/H$, where $H$ is the toral subgroup corresponding to the Cartan subalgebra $\mathscr{h}$. Since $H \subset K$ for all $K$ defined above, the symmetric space $G/K$ is a subspace of the homogeneous space $G/H$ and thus the (second order) "$N$-wave" equations can reduce to the NLS equations. There exist various intermediate reductions. When $K=H$ the eigenvalues of $A$ are all distinct. As the eigenvalues of $A$ coallesce $C_{\mathscr{g}}(A)$ grows larger and the homogeneous space $G/K$ reduces in size. When $A$ is its "most" degenerate, we have a symmetric space. We refer to equations associated with symmetric spaces as NLS equations. Those associated with the more general homogeneous spaces as $N$-wave equations. This much is no surprise.

What is, perhaps, a surprise is that the Hamiltonian structure survives this reduction. Hamiltonian structures often degenerate on reduction. This is best seen in terms of the associated classical $r$-matrix [7, 8]. The appropriate $r$-matrix is given in Sect. 4 in terms of the Cartan-Weyl basis of $\mathscr{g}$. For each $\mathscr{g}$ there is one $r$-matrix which corresponds to the $N$-wave equation for $K=H$ and all its reductions. It will be seen in Sect. 3 that for any homogeneous space (symmetric or otherwise)

$$Q=\sum_{\alpha\in\theta^+}(q^\alpha e_\alpha+p^\alpha e_{-\alpha}),$$

where $\theta^+$ is the subset of $\Phi^+$, the positive roots, for which $\alpha(A)\neq 0$. From Eq. (4.3) corresponding to the $N$-wave equation we see that the Poisson bracket satisfies $\{q^\alpha(x), p^\beta(y)\}=\alpha(A)\delta_{\alpha\beta}\delta(x-y)$, which is consistent with $q^\alpha\equiv p^\alpha\equiv 0$ whenever $\alpha(A)=0$. Thus we have two different principles of reduction which lead to the same results and are thus mutually consistent.

Another well known feature of the NLS equations is its relationship with the Heisenberg ferromagnet [9]. Corresponding to each $r$-matrix (one for each $\mathscr{g}$) we present a generalised, isotropic ferromagnet which is an isospectral deformation of:

$$f_x=\frac{1}{\lambda}Sf, \tag{1.4}$$

where $S$ is given by (4.8). The matrix $S$ has $l$ independent Casmir invariants $\operatorname{tr} S^{i+1}$ $(i=1,\dots,l)$, where $l=\operatorname{rank}\mathscr{g}$. These are invariant under isospectral deformations of (1.4), so can each be held constant. For $\mathscr{g}=\mathrm{su}(2)$ this corresponds to setting $S^2=1$. The resulting "magnet" is gauge equivalent to the above $N$-wave equation associated with $\mathscr{g}$. In the magnet co-ordinates, the reduction procedure is not so transparent. Nevertheless, to each reduction of the $N$-wave equation, discussed in Sects. 3 and 4, there is a corresponding reduced magnet. In particular, each generalised NLS equation is gauge equivalent to a particular magnet.

Section 2 is devoted to some mathematical preliminaries concerning Lie algebras and homogeneous and symmetric spaces. In Sect. 3 we discuss the linear eigenvalue problem (1.3) associated with an arbitrary Hermitian symmetric space. We derive the recursion operator and the second order flow. We then give the full list [10] of Hermitian symmetric spaces and present some lower dimensional examples explicitly. Section 4 reviews some facts about the $r$-matrix and the classical Yang-Baxter equations and their relationship to Hamiltonian structure.

We present some generalised ferromagnets and show their relationship to the NLS equations of Sect. 3. Some speculations about developments and generalisations of the results of the present paper are discussed in the conclusions.

## 2. Mathematical Preliminaries

In this section we state a number of relevant facts concerning simple Lie algebras, homogeneous and symmetric spaces. Irreducible symmetric spaces are classified in terms of simple Lie algebras, so we have no need of anything more general in this paper. We give the barest of details. The full theory can be found in [10–12].

*2.1. Simple Lie Algebras; Cartan-Weyl Basis.* In terms of the Cartan-Weyl basis a complex, simple Lie algebra $\mathfrak{g}$ has the following commutation relations [10, 11]

$$
\begin{aligned}
&\text{(i)} && [h_i, h_j] = 0, \quad \forall h_i, h_j \in \mathfrak{h}, \\
&\text{(ii)} && [h, e_\alpha] = \alpha(h) e_\alpha, \quad \forall h \in \mathfrak{h}, \quad \alpha \in \Phi, \\
&\text{(iii)} && [e_\gamma, e_{-\gamma}] = h_\gamma = \sum_{i=1}^{l} d_{\gamma_i} h_i, \\
&\text{(iv)} && [e_\gamma, e_\beta] = \begin{cases} N_{\gamma,\beta} e_{\gamma+\beta}, & 0 \neq \gamma+\beta \in \Phi \\ 0, & \gamma+\beta \notin \Phi. \end{cases}
\end{aligned}
\tag{2.1}
$$

It will be necessary to explain some of the terms:

(a) $\mathfrak{h}$ is the Cartan subalgebra, which is the maximal abelian subalgebra of diagonalisable elements of $\mathfrak{g}$. $\mathfrak{h}$ has basis $\{h_i\}_i^l$ and $d_{\gamma_i}$ are the components of $[e_\gamma, e_{-\gamma}] \in \mathfrak{h}$ with respect to this basis. The number $l$ is the rank of the albegra.

(b) $\alpha: \mathfrak{h} \to \mathbb{C}$ are linear functionals, called roots, on $\mathfrak{h}$ and their values on given $h \in \mathfrak{h}$ are the eigenvalues of the matrix $\operatorname{ad} h$. The corresponding eigenvectors, $e_\alpha$, are called root vectors.

(c) The coefficients $N_{\alpha,\beta}$ are the most complicated part of these commutation relations. They satisfy various identities which we use later in the paper:

$$N_{-\alpha,-\beta} = -N_{\alpha,\beta}, \qquad N_{\alpha,\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha}. \tag{2.2}$$

*2.2. Homogeneous and Symmetric Spaces.* A homogeneous space of a Lie group $G$ is any differentiable manifold $M$ on which $G$ acts transitively ($\forall p_1, p_2 \in M$, $\exists g \in G/g \cdot p_1 = p_2$). The subgroup of $G$ which leaves a given point $p_0 \in M$ fixed, is called the isotropy group at $p_0$ and is defined by:

$$K \equiv K_{p_0} = \{g \in G : \; g \cdot p_0 = p_0\}.$$

It is a theorem that each such $M$ can be identified with a coset space $G/K$ for some subgroup $K$ and that this $K$ plays the role of isotropy group of some point. There are many topological and differential geometric subtleties, but we have no need of them in this paper. We are only interested in the decompositions of the corresponding Lie algebras.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of $G$ and $K$ respectively, and let $\mathfrak{m}$ be the vector space complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}, \qquad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \tag{2.3}$$

and $\mathfrak{m}$ is identified with the tangent space $T_{p_0}M$ of $M = G/K$ at point $p_0$. At the moment we have $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, but know nothing of $[\mathfrak{k}, \mathfrak{m}]$ and $[\mathfrak{m}, \mathfrak{m}]$.

When $\mathscr{g}$ satisfies the more stringent conditions:

$$\mathscr{g}=\mathscr{k}\oplus\mathscr{m}, \quad [\mathscr{k},\mathscr{k}]\subset\mathscr{k}, \quad [\mathscr{k},\mathscr{m}]\subset\mathscr{m}, \tag{2.4}$$

then $G/K$ is called a reductive homogeneous space. These spaces possess canonically defined connections with curvature and torsion. Evaluated at fixed point $p_0$, the curvature and torsion tensors are given purely in terms of the Lie bracket operation on $\mathscr{m}$:

$$\begin{aligned}(R(X,Y)Z)_{p_0}&=-[[X,Y]_{\mathscr{k}},Z], \quad X,Y,Z\in\mathscr{m},\\ T(X,Y)_{p_0}&=-[X,Y]_{\mathscr{m}}, \quad X,Y\in\mathscr{m},\end{aligned} \tag{2.5}$$

where subscript $\mathscr{k}$ and $\mathscr{m}$ refer to the components of $[X,Y]$ in those vector subspaces.

When $\mathscr{g}$ satisfies the conditions[1]:

$$\begin{aligned}&\mathscr{g}=\mathscr{k}\oplus\mathscr{m}, \quad [\mathscr{k},\mathscr{k}]\subset\mathscr{k},\\ &[\mathscr{k},\mathscr{m}]\subset\mathscr{m}, \quad [\mathscr{m},\mathscr{m}]\subset\mathscr{k},\end{aligned} \tag{2.6}$$

then $\mathscr{g}$ is called a symmetric algebra and $G/K$ is a symmetric space. For these spaces the above mentioned canonical connection is derived from a metric, which is itself given by the restriction of the Killing form to $\mathscr{m}$. This connection is torsion free. Evaluated at fixed point $p_0$, the curvature tensor is given as in (2.5):

$$(R(X,Y)Z)_{p_0}=-[[X,Y],Z], \quad X,Y,Z\in\mathscr{m}, \tag{2.7}$$

where we now automatically have $[X,Y]\in\mathscr{k}$.

The components $R^i{}_{jkl}$ and $T^i{}_{jk}$ of the curvature and torsion tensors with respect to a basis $X_i$ of $T_{p_0}M$ are defined by:

$$R(X_k,X_l)X_j=R^i{}_{jkl}X_i, \quad T(X_j,X_k)=T^i{}_{jk}X_i. \tag{2.8}$$

For a symmetric space, the corresponding metric is given by the Killing form:

$$g(X,Y)=\operatorname{tr}\operatorname{ad}X\operatorname{ad}Y, \quad g_{ij}=g(X_i,X_j). \tag{2.9}$$

Tensorial indices are lowered and raised in the usual way by means of the metric tensor and its inverse.

For spaces of constant curvature, the Riemann curvature tensor is related to the metric tensor in a simple way:

$$R^i{}_{jkl}=K(\delta^i{}_k g_{jl}-\delta^i{}_l g_{jk}), \tag{2.10}$$

where $K$ is the constant Gaussian curvature.

We are particularly interested in those homogeneous and symmetric spaces which have a complex structure. This is a linear endomorphism $J:\mathscr{m}\to\mathscr{m}$ satisfying $J^2=-1$. The vector subspace $\mathscr{m}$ must have even real dimension. Hermitian symmetric spaces are very special. For this paper, the most useful properties are algebraic:

i) $\exists A\in\mathscr{k}$ such that $\mathscr{k}=C_{\mathscr{g}}(A)=\{B\in\mathscr{g}: \ [B,A]=0\}$.

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1 Helgason demands that $\mathscr{k}$ be compact. This corresponds to the metric being positive definite

ii) For a particular scaling of $A$, $J = \operatorname{ad} A$.

iii) $\exists$ *a subset* $\theta^+ \subset \Phi^+$ *of the positive root system such that* $m = \operatorname{span}\{e_{\pm\alpha}\}_{\alpha\in\theta^+}$ and $\alpha(A)$ is constant on $\theta^+$. (2.11)

iv) Following from (iii) $[e_\alpha, e_\beta] = 0$ if $\alpha, \beta \in \theta^+$ or $\alpha, \beta \in \theta^-$.

## 3. NLS and $N$-Wave Equations

We choose an element $A$ of the Cartan subalgebra $h$ of $g: A \in h \subset g$. If $A$ is regular, $C_g(A) = h$. Otherwise $C_g(A) \supset h$. Let $k = C_g(A)$ and $m$ be the complementary vector subspace of $k$ in $g: g = k \oplus m$. As in (2.3), $m$ is identified with the tangent space $T_{p_0}(G/K)$ of the homogeneous space $G/K$. In this paper we use representations in which all elements of $h$ are diagonalised. Because of the construction, this homogeneous space is automatically reductive.

Consider the linear equations

$$\begin{aligned} \phi_x &= (\lambda A + Q(x,t))\phi \equiv L(\lambda)\phi, \\ \phi_t &= P(x,t;\lambda)\phi, \end{aligned} \tag{3.1}$$

where $Q(x,t) \in m$ and $P(x,t) \in g$. We may decompose $P$ in terms of (2.3):

$$P(x,t;\lambda) = P_k(x,t;\lambda) + P_m(x,t;\lambda). \tag{3.2}$$

The integrability conditions of (3.1) are:

$$Q_t = P_x - [Q, P] - \lambda[A, P_m], \tag{3.3}$$

where we have used $[A, P_k] = 0$, since $k = C_g(A)$. We treat first the simplest case, which is the symmetric space.

*3.1. NLS Equations Associated with Symmetric Spaces.* In this case $g$ satisfies (2.6) and the integrability conditions (3.3) decouple to give:

$$\begin{aligned} Q_t &= P_{mx} - [Q, P_k] - \lambda[A, P_m], \\ P_{kx} &= [Q, P_m]. \end{aligned} \tag{3.4}$$

Note that in general the second of these equations would contain a term $\lambda[A, P_k]$. However, in our case we can immediately integrate the second equation:

$$P_k = \partial^{-1}[Q, P_m], \tag{3.5}$$

so that the first equation takes the form:

$$Q_t = (\partial - \operatorname{ad} Q \cdot \partial^{-1} \cdot \operatorname{ad} Q - \lambda \operatorname{ad} A) P_m, \tag{3.6}$$

where $(\operatorname{ad} X)Y \equiv [X, Y]$. Notice that the recursion operator appears on the right

$$(\partial - \operatorname{ad} Q \cdot \partial^{-1} \cdot \operatorname{ad} Q) P_m^{(j)} = (\operatorname{ad} A) P_m^{(j-1)}, \quad j = 1, \ldots, N+1, \quad \text{if} \quad P = \sum_{j=0}^{N} P^{(j)} \lambda^j. \tag{3.7}$$

Since $\operatorname{ad} A$ is nonsingular on $m$, we can solve for $P_m^{(j-1)}$ at each step. At each step $P_k^{(j)}$ is obtained by integration and the process terminates with the $N^{\text{th}}$ order flow:

$$Q_t=(\partial-\operatorname{ad}Q\cdot\partial^{-1}\cdot\operatorname{ad}Q)P_m{}^{(0)}. \tag{3.8}$$

The second order flow is particularly easy to construct[2].

$$[A,P_m{}^{(2)}]=0 \Rightarrow P_m{}^{(2)}=0, \quad \text{since} \quad C_{\mathscr{g}}(A)=\mathscr{k}. \tag{3.9}$$

The integration (3.5) implies that $P_k{}^{(2)}$ is a constant element of $\mathscr{k}$. We choose

$$P_k{}^{(2)}=A. \tag{3.10}$$

Such a constant arises at each integration. However, all subsequent ones just correspond to the addition of lower order flows, so we set them to zero:

$$[A,P_m{}^{(1)}]=[P_k{}^{(2)},Q]=[A,Q] \Rightarrow P_m{}^{(1)}=Q \quad \text{and} \quad P_k{}^{(1)}=0. \tag{3.11}$$

*Remark.* Had we chosen $P_k{}^{(2)}$ to be anything other than a multiple of $A$, $P_m{}^{(1)}$ would have been different from $Q$ and $[Q,P_m{}^{(1)}]$ would not have been an exact derivative. As a result, our equation would have been nonlocal.

For the next step we use some details of the root space decomposition (2.1) of $\mathscr{g}$. We use the specific properties (2.11) of those symmetric algebras associated with Hermitian symmetric spaces. Recall that there exists a subset $\theta^+$ of the positive roots $\Phi^+$ such that:

$$B\in\mathscr{m} \Rightarrow B=\sum_{\alpha\in\theta^+}(B^\alpha e_\alpha+B^{-\alpha}e_{-\alpha}),$$

and that $\alpha(A)$ is a nonzero constant on $\theta^+$; set $\alpha(A)=a$ $\forall\alpha\in\theta^+$.

Returning to our calculation: set

$$Q=\sum_{\alpha\in\theta^+}(q^\alpha e_\alpha+p^\alpha e_{-\alpha}), \tag{3.12}$$

so that $P_m{}^{(1)}{}_x=[A,P_m{}^{(0)}]$ ensures

$$P_m{}^{(0)}=\frac{1}{a}\sum_{\alpha\in\theta^+}(q^\alpha{}_x e_\alpha-p^\alpha{}_x e_{-\alpha}), \qquad P_k{}^{(0)}=-\frac{1}{a}\sum_{\alpha,\beta\in\theta^+}q^\alpha p^\beta[e_\alpha,e_{-\beta}]. \tag{3.13}$$

We have used the property $[e_\alpha,e_\beta]=0$ $\forall\alpha,\beta\in\theta^+$. We can now read off the equation from:

$$Q_t=P_m{}^{(0)}{}_x-[Q,P_k{}^{(0)}]. \tag{3.14}$$

First note that either $\alpha+\beta-\gamma$ is not a root or $\alpha+\beta-\gamma\in\theta^+$ $\forall\alpha,\beta,\gamma\in\theta^+$ since $(\alpha+\beta-\gamma)(A)=a$. This allows us to decouple (3.14) somewhat:

$$\begin{aligned}
\sum_{\alpha\in\theta^+}q^\alpha{}_t e_\alpha&=\frac{1}{a}\Big(\sum_{\alpha\in\theta^+}q^\alpha{}_{xx}e_\alpha+\sum_{\beta,\gamma,\delta\in\theta^+}q^\beta q^\gamma p^\delta[e_\beta,[e_\gamma,e_{-\delta}]]\Big)\\
\sum_{\alpha\in\theta^+}p^\alpha{}_t e_{-\alpha}&=-\frac{1}{a}\Big(\sum_{\alpha\in\theta^+}p^\alpha{}_{xx}e_{-\alpha}+\sum_{\beta,\gamma,\delta\in\theta^+}p^\beta p^\gamma q^\delta[e_{-\beta},[e_{-\gamma},e_\delta]]\Big).
\end{aligned} \tag{3.15}$$

These equations can be decoupled even further if we use the definition (2.7) of the

2 Most of this calculation is identical to the first few iterations for general $N$

Riemann curvature tensor. We use the components (2.8) with respect to the basis $\{e_{\pm\alpha}\}_{\alpha\in\theta^+}$:

$$\begin{aligned} aq^\alpha{}_t &= q^\alpha{}_{xx} + \sum_{\beta,\gamma,\delta\in\theta^+} R^\alpha{}_{\beta\gamma-\delta} q^\beta q^\gamma p^\delta , \\ -ap^\alpha{}_t &= p^\alpha{}_{xx} + \sum_{\beta,\gamma,\delta\in\theta^+} R^{-\alpha}{}_{-\beta-\gamma\delta} p^\beta p^\gamma q^\delta . \end{aligned} \tag{3.16}$$

These equations have Hamiltonian form.

$$aq^\alpha{}_t = g^{\alpha-\beta}\frac{\delta H}{\delta p^\beta}, \qquad -ap^\alpha{}_t = g^{-\alpha\beta}\frac{\delta H}{\delta q^\beta}, \tag{3.17}$$

with $H = -g_{\alpha-\beta} q^\alpha{}_x p^\beta{}_x + \frac{1}{2} g_{\varepsilon-\alpha} R^\varepsilon{}_{\beta\gamma-\delta}\, p^\alpha q^\beta q^\gamma p^\delta$. The summation convention has been used here. To derive (3.16) from (3.17) it is necessary to use some of the algebraic symmetries of the Riemann tensors.

*Remark.* The term $R_{-\alpha\beta\gamma-\delta} p^\alpha q^\beta q^\gamma p^\delta$ appearing in the Hamiltonian is just the sectional curvature in the $2-D$ surface spanned by the vectors $p^\alpha$ and $q^\alpha$.

Since in the corresponding Hermitian symmetric space the Riemann tensor has the property:

$$(R^\alpha{}_{\beta\gamma-\delta})^* = R^{-\alpha}{}_{-\beta-\gamma\delta}, \tag{3.18}$$

we can set $p^\alpha = \pm(q^\alpha)^*$ with $a=i$. The minus and plus signs correspond to the compact and noncompact real forms respectively.

*3.2. The Classification and Examples.* On p. 518 of Helgason's book [10] there is a table of symmetric spaces. Directly beneath this table those spaces which are Hermitian are listed. We now give the linear problem and associated NLS equations for lower dimensional examples of these spaces. It is an exercise for the reader to calculate the matrix $P$ from the formulae of Sect. 3.1.

*A.III.* $\dfrac{SU(p+q)}{S(U(p)\times U(q))}$.

This example is relatively well known and includes the case of the vector NLS (1.1). The linear problem is an equation in the Lie algebra $su(p+q)$, which is the compact real form associated with the root space $A_{p+q-1}$. When $p=q=2$ we have:

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}_x = \left(\begin{array}{cc|cc} \frac{1}{2}i\lambda & 0 & q_1 & q_2 \\ 0 & \frac{1}{2}i\lambda & q_4 & q_3 \\ \hline -q_1^* & -q_4^* & -\frac{1}{2}i\lambda & 0 \\ -q_2^* & -q_3^* & 0 & -\frac{1}{2}i\lambda \end{array}\right)\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}. \tag{3.19}$$

The choice of $A$ makes $\alpha(A)=i$ in the top right hand block. The first two components of the second order flow are:

$$\begin{aligned} iq_{1t} &= q_{1xx} + 2q_1(q_1q_1^* + q_2q_2^* + q_4q_4^*) + 2q_2q_4q_3^*, \\ iq_{2t} &= q_{2xx} + 2q_2(q_1q_1^* + q_2q_2^* + q_3q_3^*) + 2q_1q_3q_4^*. \end{aligned} \tag{3.20}$$

The second two are generated from (3.20) by the interchange $1\leftrightarrow 3$, $2\leftrightarrow 4$. There are then the four complex conjugate equations.

*Remark.* The choice of compact real form $\mathrm{su}(p+q)$ corresponds to setting $p_\alpha = -q_\alpha{}^*$. The noncompact real form $\mathrm{su}(p,q)$ corresponds to $p_\alpha = q_\alpha{}^*$.

When $p=1$ we are dealing with the usual vector NLS equation and the symmetric space is just complex projective space $\mathbb{C}P_q$. Since this is a space of constant curvature $K$, we use (2.10) to obtain:

$$iq^\alpha{}_t = q^\alpha{}_{xx} + K\left(\sum_{\beta,\gamma\in\theta^+} g_{\beta,-\gamma} q^\beta (q^\gamma)^*\right) q^\alpha, \tag{3.21}$$

and its complex conjugate. The metric is given by the Killing form (2.9) restricted to the symmetric space. However, the Killing form is proportional to the trace form in the fundamental representation. With respect to the root vector basis (which is *not* a co-ordinate basis) the metric is thus proportional to $\delta_{\beta,-\gamma}$, which gives the usual form of (3.21).

*C.I.* $\dfrac{\mathrm{Sp}(n)}{\mathrm{U}(n)}$

The compact group $\mathrm{Sp}(n)$ [sometimes called $\mathrm{USp}(2n)$] of $2n\times 2n$ matrices which are both symplectic and unitary is associated with the root space $C_n$. For the simplest of these $n=2$:

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}_x = \left(\begin{array}{cc|cc} \frac{1}{2}i\lambda & 0 & q_1 & q_2\\ 0 & \frac{1}{2}i\lambda & q_2 & q_3\\ \hline -q_1^* & -q_2^* & -\frac{1}{2}i\lambda & 0\\ -q_2^* & -q_3^* & 0 & -\frac{1}{2}i\lambda \end{array}\right)\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{pmatrix}. \tag{3.22}$$

Notice that this is a reduction of (3.19), with $q_4 \equiv q_2$. The NLS equations for this case are given by (3.20) with the same reduction. In general:

$$\frac{\mathrm{Sp}(n)}{\mathrm{U}(n)} \subset \frac{\mathrm{SU}(2n)}{\mathrm{S}(\mathrm{U}(n)\times\mathrm{U}(n))},$$

and corresponds to each of the off-diagonal blocks being symmetric. The noncompact real form $\mathrm{Sp}(n,R)$ corresponds to the choice $p_i = q_i{}^*$.

*D.III.* $\dfrac{\mathrm{SO}(2n)}{\mathrm{U}(n)}$.

The orthogonal algebra $\mathrm{so}(2n)$ is the compact real form associated with the root space $D_n$. The general case is exemplified by the $D_4$ linear problem.

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\\ \phi_7\\ \phi_8\end{pmatrix}_x = \left(\begin{array}{cccc|cccc} \frac{1}{2}i\lambda & 0 & 0 & 0 & 0 & q_1 & q_3 & q_6\\ 0 & \frac{1}{2}i\lambda & 0 & 0 & -q_1 & 0 & q_2 & q_5\\ 0 & 0 & \frac{1}{2}i\lambda & 0 & -q_3 & -q_2 & 0 & q_4\\ 0 & 0 & 0 & \frac{1}{2}i\lambda & -q_6 & -q_5 & -q_4 & 0\\ \hline 0 & q_1^* & q_3^* & q_6^* & -\frac{1}{2}i\lambda & 0 & 0 & 0\\ -q_1^* & 0 & q_2^* & q_5^* & 0 & -\frac{1}{2}i\lambda & 0 & 0\\ -q_3^* & -q_2^* & 0 & q_4^* & 0 & 0 & -\frac{1}{2}i\lambda & 0\\ -q_6^* & -q_5^* & -q_4^* & 0 & 0 & 0 & 0 & -\frac{1}{2}i\lambda \end{array}\right)\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\\ \phi_7\\ \phi_8\end{pmatrix} \tag{3.23}$$

There are three basic equations:

$$
\begin{aligned}
iq_{1t} &= q_{1xx} + 2q_1 \sum_{j \neq 4} q_j q_j^* + 2q_4^*(q_3 q_5 - q_2 q_6), \\
iq_{2t} &= q_{2xx} + 2q_2 \sum_{j \neq 6} q_j q_j^* + 2q_6^*(q_3 q_5 - q_1 q_4), \\
iq_{3t} &= q_{3xx} + 2q_3 \sum_{j \neq 5} q_j q_j^* + 2q_5^*(q_1 q_4 + q_2 q_6).
\end{aligned}
\tag{3.24}
$$

Another three are obtained by making the interchanges $1 \leftrightarrow 4$, $2 \leftrightarrow 6$, $3 \leftrightarrow 5$. The system is completed by complex conjugation. This is another reduction of the A III case:

$$
\frac{\mathrm{SO}(2n)}{\mathrm{U}(n)} \subset \frac{\mathrm{SU}(2n)}{\mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(n))},
$$

this time corresponding to each of the off-diagonal blocks being anti-symmetric.

Notice that $q_4 \equiv q_5 \equiv q_6 \equiv 0$ is a consistent reduction. This corresponds to taking the subsystem $D_3$ of $D_4$. Furthermore, this reduction is identical to the 3-component *vector* NLS equation. This corresponds to the isomorphism $D_3 \cong A_3$, leading to

$$
\frac{\mathrm{SO}(6)}{\mathrm{U}(3)} \cong \frac{\mathrm{SU}(4)}{\mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(3))}.
$$

$$
BD.I. \quad \frac{\mathrm{SO}(p+q)}{\mathrm{SO}(p) \times \mathrm{SO}(q)}, \quad p=2.
$$

This symmetric space is only Hermitian when $p=2$. In general $\mathrm{so}(p)+\mathrm{so}(q)$ has no centre. When $p=2$ the $\mathrm{so}(2)$ subalgebra is the centre. Depending upon whether $q$ is odd or even this symmetric space is associated with either $B_{\frac{1}{2}(q+1)}$ or $D_{\frac{1}{2}(q+2)}$. The simplest nontrivial example is associated with $D_3$. Even though $D_3 \cong A_3$, $\mathrm{SO}(2) \times \mathrm{SO}(4) \cong \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(3))$. Indeed, this example has four independent components $q_i$, not just three. The eigenvalue problem is possibly best understood in terms of the skew symmetric matrix representations [10]. However, the present calculation is more easily performed in the representation given by Humphreys [11]. The eigenvalue problem is:

$$
\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{pmatrix}_x = \left(\begin{array}{ccc|ccc} i\lambda & q_1 & q_2 & 0 & q_3 & q_4 \\ -q_1^* & 0 & 0 & -q_3 & 0 & 0 \\ -q_2^* & 0 & 0 & -q_4 & 0 & 0 \\ \hline 0 & q_3^* & q_4^* & -i\lambda & q_1^* & q_2^* \\ -q_3^* & 0 & 0 & -q_1 & 0 & 0 \\ -q_4^* & 0 & 0 & -q_2 & 0 & 0 \end{array}\right) \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{pmatrix}.
\tag{3.25}
$$

There are two basic equations:

$$
\begin{aligned}
iq_{1t} &= q_{1xx} + 2q_1 \sum_{j \neq 3} q_j q_j^* - 2q_2 q_4 q_3^*, \\
iq_{2t} &= q_{2xx} + 2q_2 \sum_{j \neq 4} q_j q_j^* - 2q_1 q_3 q_4^*.
\end{aligned}
\tag{3.26}
$$

Two more equations are obtained by the interchange $1\leftrightarrow 3$, $2\leftrightarrow 4$. There are then the complex conjugates of these four.

*Exceptional Algebras.* All the examples so far given have been associated with the classical Lie algebras. There are two Hermitian symmetric spaces EIII and EVII, associated with the exceptional E-series. They possess respectively 16 and 27 independent complex potentials, $q_i$, which would satisfy a corresponding system of generalised NLS equations. We do not present these examples explicitly.

*3.3. N-Wave Hierarchies and Reductive Homogeneous Spaces.* In Sects. (3.1) and (3.2) we dealt with symmetric spaces. These were particularly simple because of the condition $[\mathscr{m},\mathscr{m}]\subset\mathscr{k}$ of (2.6). One interpretation of this condition is that a symmetric space is a reductive homogeneous space on which the canonical connection has zero torsion. In this section we deal with reductive homogeneous spaces which have non-zero torsion. Equation (3.3) decouples to give:

$$\begin{aligned} Q_t &= P_{mx} - [Q, P_k] - [Q, P_m]_m - \lambda [A, P_m], \\ P_{kx} &= [Q, P_m]_k . \end{aligned} \tag{3.27}$$

Here, as indicated, $[Q, P_m]$ has components both in $\mathscr{k}$ and in $\mathscr{m}$. We construct the second order flow. The first part of the calculation is the same as before:

$$\begin{gathered} P_m^{(2)} = 0, \quad P_k^{(2)} = A, \quad P_m^{(1)} = Q, \quad P_k^{(1)} = 0, \\ Q = \sum_{\alpha\in\theta^+} (q^\alpha e_\alpha + p^\alpha e_{-\alpha}), \quad P_m^{(0)} = \sum_{\alpha\in\theta^+} \frac{1}{\alpha(A)} (q^\alpha{}_x e_\alpha - p^\alpha{}_x e_{-\alpha}). \end{gathered} \tag{3.28}$$

A new feature is that $\alpha(A)$ is no longer constant on $\theta^+$. However, it is constant on blocks within the representation, depending upon the degeneracy of $A$. Thus $\theta^+$ can be written as the union of a number of subsets $\theta_j^+$ on each of which $\alpha(A)$ takes a constant value $a(A) = a_j$, $\forall \alpha\in\theta_j^+$.

It is only at the next step of the calculation that the property $[\mathscr{m},\mathscr{m}]\cap\mathscr{m}$ nonempty plays a role. For instance, we no longer have $[e_\alpha, e_\beta] = 0$, $\forall \alpha, \beta\in\theta^+$. However, we do have $[e_\alpha, e_\beta]_{\mathscr{k}} = 0$ for $\alpha, \beta\in\theta^+$. We wish to calculate $[Q, P_m^{(0)}]$:

$$\begin{aligned} [Q, P_m^{(0)}] = &- \sum_{\alpha,\beta\in\theta^+} \left(\frac{1}{\beta(A)} q^\alpha p^\beta{}_x + \frac{1}{\alpha(A)} q^\alpha{}_x p^\beta\right) [e_\alpha, e_{-\beta}] \\ &+ \sum_{\alpha,\beta\in\theta^+} \frac{1}{\beta(A)} q^\alpha q^\beta{}_x [e_\alpha, e_\beta] - \sum_{\alpha,\beta\in\theta^+} \frac{1}{\beta(A)} p^\alpha p^\beta{}_x [e_{-\alpha}, e_{-\beta}]. \end{aligned} \tag{3.29}$$

The only part of this expression which can have a component in $\mathscr{k}$ is $[e_\alpha, e_{-\beta}]$. We have the following:

$$[e_\alpha, e_{-\beta}] = [e_\alpha, e_{-\beta}]_k \Leftrightarrow (\alpha - \beta)(A) = 0,$$

which means $\alpha, \beta\in\theta_j{}^+$ for the *same* $j$. Otherwise $[e_\alpha, e_{-\beta}]_k = 0$. When $[e_\alpha, e_{-\beta}] = [e_\alpha, e_{-\beta}]_k$, its coefficient is an exact derivative, so that

$$P_k^{(0)} = - \sum_j \sum_{\alpha,\beta\in\theta_j^+} \frac{1}{a_j} q^\alpha p^\beta [e_\alpha, e_{-\beta}]_k . \tag{3.30}$$

Thus, corresponding to (3.15) we have:

$$\sum q^\alpha{}_t e_\alpha = \sum_{\alpha\in\theta^+}\left(\frac{1}{\alpha(A)} q^\alpha{}_{xx} e_\alpha + \sum_{\alpha,\beta,\gamma\in\theta^+} \frac{1}{\beta(A)} q^\alpha q^\beta p^\gamma [e_\alpha, [e_\beta, e_{-\gamma}]_{k}]\right.$$
$$+ \sum_{\alpha-\beta\in\theta^+}\left(\frac{1}{\beta(A)} q^\alpha p^\beta{}_x + \frac{1}{\alpha(A)} q^\alpha{}_x p^\beta\right)[e_\alpha, e_{-\beta}]$$
$$- \sum_{\alpha,\beta\in\theta^+} \frac{1}{\beta(A)} q^\alpha q^\beta{}_x [e_\alpha, e_\beta], \tag{3.31a}$$

and

$$-\sum_{\alpha\in\theta^+} p^\alpha{}_t e_{-\alpha} = \sum_{\alpha\in\theta^+} \frac{1}{\alpha(A)} p^\alpha{}_{xx} e_{-\alpha} + \sum_{\alpha,\beta,\gamma\in\theta^+} \frac{1}{\beta(A)} p^\alpha p^\beta q^\gamma [e_{-\alpha}, [e_{-\beta}, e_\gamma]_k]$$
$$- \sum_{\beta-\alpha\in\theta^+}\left(\frac{1}{\beta(A)} q^\alpha p^\beta{}_x + \frac{1}{\alpha(A)} q^\alpha{}_x p^\beta\right)[e_\alpha, e_{-\beta}]$$
$$- \sum_{\alpha,\beta\in\theta^+} \frac{1}{\beta(A)} p^\alpha p^\beta{}_x [e_{-\alpha}, e_{-\beta}]. \tag{3.31b}$$

Using the definitions (2.5) of the curvature and torsion tensors, these equations can be further decoupled to give:

$$q^\alpha{}_t = \frac{1}{\alpha(A)} q^\alpha{}_{xx} + \sum_{(\gamma-\delta)(A)=0} \frac{1}{\gamma(A)} R^\alpha{}_{\beta\gamma-\delta} q^\beta q^\gamma p^\delta$$
$$- \sum_{\alpha=\beta-\gamma\in\theta^+}\left(\frac{1}{\gamma(A)} q^\beta p^\gamma{}_x + \frac{1}{\beta(A)} q^\beta_x p^\gamma\right) T^\alpha_{\beta-\gamma}$$
$$+ \sum_{\substack{\beta+\gamma=\alpha\\ \beta,\gamma\in\theta^+}} \frac{1}{\gamma(A)} q^\beta q^\gamma_x T^\alpha_{\beta\gamma}, \tag{3.32a}$$

$$-p^\alpha{}_t = \frac{1}{\alpha(A)} p^\alpha{}_{xx} + \sum_{(\gamma-\delta)(A)=0} \frac{1}{\gamma(A)} R^{-\alpha}{}_{-\beta-\gamma\delta} p^\beta p^\gamma q^\delta$$
$$- \sum_{\alpha=\beta-\gamma\in\theta^+}\left(\frac{1}{\beta(A)} q^\gamma p^\beta{}_x + \frac{1}{\gamma(A)} q^\gamma{}_x p^\beta\right) T_{-\beta}{}^{-\alpha}{}_\gamma$$
$$+ \sum_{\substack{\alpha=\beta+\gamma\\ \beta,\gamma\in\theta^+}} \frac{1}{\gamma(A)} p^\beta p^\gamma{}_x T_{-\beta}{}^{-\alpha}{}_{-\gamma}. \tag{3.32b}$$

*Remark.* It follows from $N_{-\beta,-\gamma} = -N_{\beta,\gamma}$ that $T_{-\beta}{}^{-\alpha}{}_\gamma = -T_\beta{}^\alpha{}_{-\gamma}$ and $T_{-\beta}{}^{-\alpha}{}_{-\gamma} = -T^\alpha_{\beta\,\gamma}$.

These equations are more complicated than (3.16), reducing to the latter when the torsion $T \equiv 0$. Notice that the torsion terms are not cubically nonlinear, but quadratic with derivatives.

When $A$ is regular (distinct eigenvalues) $\mathscr{k} = \mathscr{h}$, the Cartan subalgebra of $\mathscr{g}$. As the eigenvalues of $A$ coallesce, $\mathscr{k}$ grows larger and the homogeneous space smaller. When $A$ reaches its "most degenerate" state, the homogeneous space is symmetric. This coallescing of eigenvalues gives rise to a sequence of reductions:

$$K_1 \subset K_2 \subset G \Rightarrow G/K_2 \subset G/K_1 . \tag{3.33}$$

We do not discuss the examples of these equations in detail. The reduction procedure is best visualised in terms of:

$$\frac{\mathrm{SU}(n)}{\mathrm{S}(\mathrm{U}(n_1)\times \ldots \times \mathrm{U}(n_m))}, \quad \sum_{j=1}^{m} n_j = n .$$

We give a simple example:

*3-Wave Hierarchy*

$$\frac{\mathrm{SU}(3)}{\mathrm{S}(\mathrm{U}(1)\times \mathrm{U}(1)\times \mathrm{U}(1))}$$

The linear problem is an equation in SU(3):

$$\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}_x = \begin{pmatrix} ia_1\lambda & q_1 & q_2 \\ -q_1^* & ia_2\lambda & q_3 \\ -q_2^* & -q_3^* & ia_3\lambda \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} . \tag{3.34}$$

The second order flow is:

$$\begin{aligned} iq_{1t} &= \frac{q_{1xx}}{a_1-a_2} + q_1\left(\frac{2q_1q_1^*}{a_1-a_2} + \frac{q_2q_2^*}{a_1-a_3} - \frac{q_3q_3^*}{a_2-a_3}\right) - \frac{q_{2x}q_3^*}{a_1-a_3} - \frac{q_2q_{3x}^*}{a_2-a_3}, \\ iq_{2t} &= \frac{q_{2xx}}{a_1-a_3} + q_2\left(\frac{q_1q_1^*}{a_1-a_2} + \frac{2q_2q_2^*}{a_1-a_3} + \frac{q_3q_3^*}{a_2-a_3}\right) - \frac{q_{1x}q_3}{a_1-a_2} - \frac{q_1q_{3x}}{a_2-a_3}, \\ iq_{3t} &= \frac{q_{3xx}}{a_2-a_3} + q_3\left(-\frac{q_1q_1^*}{a_1-a_2} + \frac{q_2q_2^*}{a_1-a_3} + \frac{2q_3q_3^*}{a_2-a_3}\right) + \frac{q_1^*q_{2x}}{a_1-a_3} + \frac{q_{1x}^*q_2}{a_1-a_2}, \end{aligned} \tag{3.35}$$

together with the complex conjugates.

The reduction corresponding to:

$$\frac{\mathrm{SU}(3)}{\mathrm{S}(\mathrm{U}(1)\times \mathrm{U}(2))} \subset \frac{\mathrm{SU}(3)}{\mathrm{S}(\mathrm{U}(1)\times \mathrm{U}(1)\times \mathrm{U}(1))} \tag{3.36}$$

is purely algebraic in terms of (3.34). We merely set $a_2 = a_3 = -\frac{1}{2}a_1$ and $q_3 \equiv 0$. In terms of the differential equations (3.35) the limiting procedure must be taken into account.

## 4. Hamiltonian Structure and $r$-Matrix for NLS Equations

In Sect. 3 we discussed NLS equations associated with various Hermitian symmetric spaces. We also considered a more general class of equation associated with reductive Lie algebras. We discussed various reductions of the above systems. An interesting feature is that the Hamiltonian structures of the above systems survive reduction. In this section we discuss this aspect in detail. We find the $r$-matrix approach most convenient for our purposes. We first consider the general framework of the $r$-matrix, and then discuss some associated generalised ferromagnets. Finally, we return to our NLS equations.

*The r-Matrix*. One aim of the inverse scattering method is to realise a transformation from the "field" variables to new ones, in terms of which the nonlinear evolution equation (NLEE) takes a simpler form. Such variables are defined by entries of the monodromy matrix $T_l(\lambda)$:

$$\frac{d}{dx} T_l(x,\lambda) = L(x,\lambda) T_l(x,\lambda), \qquad T_l(-l,\lambda) = 1), \qquad T_l(\lambda) \equiv T_l(l,\lambda). \tag{4.1}$$

It was noticed in [7, 8] that the classical $r$-matrix permits us to write down the Poisson brackets of the entries of $T_l(\lambda)$ in compact form:

$$\{T_l(\lambda) \underset{,}{\otimes} T_l(v)\} = [r(\lambda - v), T_l(\lambda) \otimes T_l(v)], \tag{4.2}$$

where the left hand side is an $n^2 \times n^2$ matrix of Poisson brackets $\{T_{ab}(\lambda), T_{cd}(v)\}$ of different entries of $n \times n$ matrices $T_l(\lambda)$ and $T_l(v)$. It is important that we can calculate $r(\lambda)$ from the equation:

$$\{ L(x,\lambda) \underset{,}{\otimes} L(y,v)\} = [r(\lambda - v), L(x,\lambda) \otimes I + I \otimes L(x,v)] \delta(x-y), \tag{4.3}$$

where we use only the matrix $L(x,y)$ and the Poisson brackets of its coefficient functions. We would like to underline also that with a given $r$-matrix it is always possible to construct a corresponding linear problem, the Hamiltonian structure of its coefficient functions and, as a result, integrable NLEE's.

To prove these statements we recall some results from [7, 8]. At first we obtain an equation for the $r$-matrix only. If we consider the Jacobi identity for the Poisson brackets of entries of three monodromy matrices $T(\lambda)$, $T(v)$, and $T(\mu)$, then using (4.2) we see that a sufficient condition for the Jacobi identity to be valid is the following equation for the $r$-matrix:

$$[r_{12}(\lambda - v), r_{13}(\lambda - \mu)] + [r_{12}(\lambda - v), r_{23}(v - \mu)] + [r_{13}(\lambda - \mu), r_{23}(v - \mu)] = 0. \tag{4.4}$$

This is the classical Yang-Baxter equation. It is written in the tensor product of three spaces $V_1 \otimes V_2 \otimes V_3$, and indices of the $r$-matrix show in which two of the three spaces this matrix is nontrivial. Equation (4.4), just as the Lax (integrability) equations, depends only upon the structure constants of the corresponding algebra $\mathscr{g}$, so is independent of representation. We can write down its solutions as linear combinations of the tensor products of basic elements of the algebra $\mathscr{g}$:

$$r(\lambda) = \sum_{\alpha,\beta} \omega_{\alpha\beta}(\lambda) e_\alpha{}^{(1)} e_\beta{}^{(2)}. \tag{4.5}$$

In this expression $\omega_{\alpha\beta}(\lambda)$ are complex-valued functions of $\lambda$, $e_\alpha{}^{(1)}$ and $e_\beta{}^{(2)}$ are independent generators of two copies of the same algebra $\mathscr{g}$, satisfying the commutation relations

$$[e_\alpha{}^{(a)}, e_\beta{}^{(b)}] = \delta_{ab} C_\alpha{}^\gamma{}_\beta e^a{}_\gamma, \qquad a, b = 1, 2. \tag{4.6}$$

The $r$-matrix corresponding to two dimensional Toda lattices was calculated in [8, 13]. Here we are interested in a limiting case of this, which has the form:

$$r(\lambda) = \frac{1}{\lambda}\left(\sum_{i=1}^{l} h_i \otimes h_i + \sum_{\alpha \in \Phi} e_\alpha \otimes e_{-\alpha}\right), \qquad l = \operatorname{rank} \mathscr{g}. \tag{4.7}$$

This $r$-matrix corresponds to the simple Lie algebra $\mathfrak{g}$ and the notation is that used in Sect. 2.

*Magnets.* Using this $r$-matrix we can construct the linear problem corresponding to a magnet NLEE which is a Lie algebra $\mathfrak{g}$ generalisation of the SU(2) continuous Heisenberg Ferromagnet [9].

Let us substitute in place of $h_i^{(2)}$, $e_\alpha^{(2)}$ in (4.7) the functions $S_i(x)$, $S_\alpha(x)$. Then we obtain the matrix $S$ and a linear problem:

$$\begin{aligned} &S(x)=\sum_{i=1}^{l} h_i S_i(x)+\sum_{\alpha\in\Phi} S_\alpha e_\alpha\,, \qquad l=\operatorname{rank}\mathfrak{g}\,,\\ &\frac{d}{dx}f=\frac{1}{\lambda}S(x)f. \end{aligned} \tag{4.8}$$

Let us define Poisson brackets of the functions $S_a$ $(a=i,\alpha)$ using the structure constants of the Lie algebra $\mathfrak{g}$

$$\{S_a(x), S_b(y)\}=C_a{}^c{}_b S_c(x)\delta(x-y)\,. \tag{4.9}$$

As a result the linear problem (4.8) will satisfy (4.3) with $r$-matrix (4.7) and $L(\lambda)=\frac{1}{\lambda}S(x)$. To prove this, it is enough to notice that (4.3) in this case is nothing but (4.4) in which the last commutator is replaced by the Poisson bracket.

The NLEE corresponding to the second order flow of (4.8) is the following:

$$\partial_t S=[S, S_{xx}]\,. \tag{4.10}$$

Hence, the quantities $\mathscr{S}_m=\operatorname{tr} S^{m+1}$, $m=1, \ldots, \operatorname{rank}\mathfrak{g}$ are independent of $t$. Moreover, the $\mathscr{S}_m$ are mutually in involution and commute with all other variables $S_a(x)$:

$$\{\mathscr{S}_m, S_a\}=\{\mathscr{S}_m, \mathscr{S}_n\}=0\,. \tag{4.11}$$

These quantities are global invariants with respect to isospectral deformations of (4.8). Fixing their values we define an orbit on which the Poisson bracket (4.9) is nondegenerate. For the general set of $\{\mathscr{S}_m\}_1^{\operatorname{rank}\mathfrak{g}}$, the functional dimension of the orbit is $\dim\mathfrak{g}-\operatorname{rank}\mathfrak{g}$. We would like to consider some reductions where the Casimirs $\mathscr{S}_m$ are not all independent. However, in terms of the matrix $S(x)$ such reductions are very complicated. It is much simpler to consider the eigenvalues of the matrix $S(x)$ and to relate reductions of $S(x)$ to the coallescing of eigenvalues. We consider the case when all $\mathscr{S}_n$ are independent of $x$.

We thus write $S(x)$ in the form $S(x)=g^{-1}(x)Ag(x)$, where $A$ is a constant diagonal matrix. This relation does not define $g(x)$ uniquely, since $g$ can be multiplied on the left by any element of the centraliser of $A$. We can thus choose $g(x)$ so that:

$$g_x g^{-1}=\sum_{\alpha\in\theta^+}(q^\alpha e_\alpha+p^\alpha e_{-\alpha})\in \operatorname{Range}(\operatorname{ad}A)\,, \tag{4.12}$$

where we have used the notation of Sect. 2 and 3. Once again we have a reductive homogeneous space $G/K$, where $K=\{B\in G: BAB^{-1}=A\}$. Thus, in the notation of (2.3), $g_x g^{-1}\in \mathfrak{m}\cdot\theta^+$ is the set of positive roots which are nonzero when evaluated on $A$. Defining $\phi = gf$ and using (4.8) we obtain the equation $\left(\lambda \to \frac{1}{\lambda}\right)$:

$$\frac{\partial}{\partial x}\phi(x,\lambda) = (\lambda A + Q(x))\phi(x,\lambda), \tag{4.13}$$

where $Q(x) = \sum_{\alpha\in\theta^+} (q^\alpha e_\alpha + p^\alpha e_{-\alpha})$. This is exactly the equation discussed in Sect. 3. The above is just the gauge transformation between the generalised ferromagnets of this section and the generalised NLS equations of Sect. 3.

*NLS Equations.* After this transformation it is easy to see that the linear problem (4.13) satisfies Eq. (4.3) with the same $r$-matrix (4.7). If we define Poisson brackets of coefficient functions:

$$\{q^\alpha(x), p^\beta(y)\} = \alpha(A)\delta_{\alpha\beta}\delta(x-y). \tag{4.14}$$

Let us consider, for example, in the right hand side of Eq. (4.3) the term which does not contain coefficient functions $q^\alpha$, $p^\alpha$, with $L$ given by (4.13),

$$\begin{aligned}&\left[\frac{1}{\lambda - v}\left(\sum_{i=1}^{l} h_i\otimes h_i + \sum_{\alpha\in\Phi} e_\alpha\otimes e_{-\alpha}\right), \lambda\sum_j a_j h_j\otimes I + v\sum_j a_j I\otimes h_j\right]\\ &\quad = -\sum_{\alpha\in\Phi}\left(\sum_i a_i\alpha(h_i)\right)e_\alpha\otimes e_{-\alpha} = -\sum_{\alpha\in\Phi}\alpha(A)e_\alpha\otimes e_{-\alpha}.\end{aligned} \tag{4.15}$$

But this expression coincides with the left hand side of (4.3):

$$\begin{aligned}\{L(x,\lambda) \overset{\otimes}{,} L(y,\lambda)\} &= \left\{\sum_{\alpha\in\theta^+}(q^\alpha e_\alpha + p^\beta e_{-\alpha} \otimes \sum_{\beta\in\theta^+}(q^\beta e_\beta + p^\beta e_{-\beta})\right\}\\ &= \delta(x-y)\sum_{\alpha\in\theta^+}\alpha(A)\,(e_\alpha\otimes e_{-\alpha} - e_{-\alpha}\otimes e_\alpha).\end{aligned} \tag{4.16}$$

To prove the cancellation of other terms we have to use the special properties of the Cartan-Weyl basis (2.2); in particular $N_{\alpha,\beta} = N_{-\alpha-\beta,\alpha}$. This is a straightforward calculation.

We can see that Eq. (4.14)–(4.16) are consistent with respect to a degeneration of the element $A$ in $\mathscr{h}$. For regular $A$, all $\alpha(A) \neq 0$ (i.e. $C_g(A) = \mathscr{h}$) so that all $q^\alpha, p^\alpha \not\equiv 0$. For a degenerate element (some eigenvalues coincide) we have $\alpha(A) = 0$ for some roots. For Eqs. (4.15) and (4.16) to be consistent, we must have $q^\alpha \equiv p^\alpha \equiv 0$ for all such $\alpha$. As a result the subset $\theta^+$ of roots is smaller than $\Phi^+$ and we reduce our system to a smaller number of components.

We have thus shown that for the various reductions discussed in Sect. 3 the given Poisson bracket does not degenerate.

## 5. Conclusions

It is very natural to continue the research of this paper in several directions: first, there is the definition of action angle-variables and proof of complete integrability of these systems. The difficulty of this problem is evident; the systems are multicomponent but only possess "one series" of local integrals. For the vector NLS $\left(\text{the case of } \frac{SU(n+1)}{SU(n)}\right)$ a procedure has been proposed in [14]. Second, it would be interesting to quantise these equations in the framework of the quantum inverse scattering method [7, 8]. From these considerations we would be able to calculate the quantum $R$-matrix, which describes the commutation relations of the quantum scattering data. We underline that the $R$-matrix, contrary to its classical limit $r$, depends upon the representation.

There are also various questions of a more geometric and algebraic flavour. The NLS equations presented here were defined in the tangent space of the corresponding symmetric space, but no mention was made of the role of the actual symmetric space. In this paper reduction was achieved by taking a nested sequence of coset spaces (3.33). This approach is not limited to the systems of equations found in this paper. However, we cannot expect the Hamiltonian structure to be so well behaved in a more general context.

Finally, to each NLS equation presented here there corresponds a DNLS equation, isospectral to:

$$\phi_x=(\lambda^2 A+\lambda Q(x,t))\phi\,. \tag{5.1}$$

As usual, the Hamiltonian structure is not canonical in this case.

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