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The Unitary Irreducible Representations of $\overline{\mathrm{SO}}_0(4,2)$

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Abstract. An exhaustive classification of all irreducible Harish-Chandra $\mathfrak{so}(4,2)$-modules, integrable to unitarizable projective representations of the conformal group, is established by infinitesimal methods: the classification is based

1) on the reduction upon the maximal compact subalgebra, associated with a lattice of points in $\mathbb{R}^3$, and

2) on a set of additional parameters upon which the eigenvalues of central elements of the enveloping algebra depend polynomially.

0. Introduction

The conformal group of Minkowski space has recurrently been a quite important tool in mathematical physics. Its unitary irreducible representations, true or projective ones, have been a favorite subject of research of many scientists [1]; but, in spite of the various methods used, there is no exhaustive list of them, at least to the author's knowledge [8].

The object of this paper is to give an exhaustive classification of the unitary dual of the universal covering of the connected component of the conformal group, $\bar{G}=\overline{\mathrm{SO}}_0(4,2)$; more precisely the problem studied is the (equivalent) Lie algebraic transcription of that statement: determine, up to infinitesimal equivalence, all Schur-irreducible representations π of $\mathfrak{g}=\mathfrak{so}(4,2)$, on a pre-Hilbert space of $\mathfrak{k}$-finite vectors ($\mathfrak{k}=\mathfrak{so}(4)\oplus\mathfrak{so}(2)$), such that $i\pi(X)$ is essentially self-adjoint for every X in $\mathfrak{g}$.

In this paper we shall only sketch the proof for the following reason: after solving the above problem for $\mathfrak{so}(3,2)$ [2], we developed a formalism for solving the $\mathfrak{so}(p,2)$ case, establishing necessary and sufficient conditions for unitarity [3]. Whereas this formalism would be too long to expose here in full detail, the final explicit formulas can be given rather concisely in the physically interesting case $p=4$.

The principle of the method used is the following: Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of a real semisimple Lie-algebra, and consider a $\mathfrak{g}$-module $\mathscr{E}$ having a pseudohermitian form for which $(M\varphi|\psi)+(\varphi|M\psi)=0$ for every φ, ψ in $\mathscr{E}$ and every M in $\mathfrak{g}$. If moreover one imposes upon $\mathscr{E}$ to be an irreducible Harish-Chandra module, i.e. a space of $\mathfrak{k}$-finite vectors, positive-definiteness of the pseudohermitian form is equivalent to unitarizability of the corresponding group representation (integrability is granted).

Bargmann [4] and Naïmark [5] have solved the problem by producing explicitly the generalized matrix elements for $\mathfrak{g}=\mathfrak{so}(2,1)$ and $\mathfrak{g}=\mathfrak{so}(3,1)$, respectively. This is the starting point of our method, together with the following observation: If $\mathfrak{k}$ is quite big, it is not easy to compute generalized matrix elements for every $\mathfrak{k}$-module occurring in $\mathscr{E}$ and every element of $\mathfrak{k}$; but that is not necessary, at least for the positive-definiteness problem, because if φ belongs to a given $\mathfrak{k}$-type, then $(\varphi|\varphi)$ and $(M\varphi|M\varphi)$ have the same sign for every M in $\mathfrak{k}$.

This leaves us with generalized matrix elements of the form $(\varphi|X\psi)$ with X in $\mathfrak{p}$. This involves the reduction of the tensor product $\mathfrak{p}\otimes\mathscr{E}^{\chi}$, $\mathfrak{p}$ being considered as a $\mathfrak{k}$-module, and $\mathscr{E}^{\chi}$ being any particular $\mathfrak{k}$-isotypic component occurring in the $\mathfrak{k}$ reduction of $\mathscr{E}$. By standard techniques one can obtain "basis-independent squared generalized matrix elements," generalizing the multiple-j-symbols obtained when $\mathfrak{k}=\mathfrak{so}(3)$: we call them "squared shift operators." Once some of their properties are established, one can reduce the positive-definiteness research to the determination of the sign of some completely factorized polynomials [such as $(m-j)(m+j+1)$, j fixed, m variable] at least in the case of $\mathfrak{g}=\mathfrak{so}(p,2)$.

All this is exposed in Sects. 1 and 2, where we give the initial formulas and sketch the procedure. The final formulas on which the positive-definiteness test is realized are given in Sect. 3. Section 4 contains the classification and Sect. 5 concluding remarks.

All known unitary irreducible projective representations of $\overline{\mathrm{SO}}_0(4,2)$ appear in Tables 1–3 (Sect. 4), in particular those for which the $\mathfrak{so}(2)$ eigenvalue (the energy operator for physicists) is bounded on one side, classified by Mack [6]. The keys of the classification are the $\mathfrak{k}$-reduction of $\mathscr{E}$, represented by a three dimensional lattice of points, and the eigenvalues of the center of the enveloping algebra, given by what we call the "characteristic polynomial" of $\mathfrak{g}$, introduced in Sect. 2, and used to identify unitarizable irreducible $\mathfrak{g}$-modules as factors of some indecomposable Harish-Chandra $\mathfrak{g}$-module.

1. General Features

1.a. Notations

Throughout this paper, let the sets of indices $\alpha, \beta, \ldots$ run over $\{1,2\}$; $i, j, \ldots$ over $\{3,4,5,6\}$; $I, J, \ldots$ over $\{1,2,3,4,5,6\}$. Define the metric tensor g_{IJ}, by $g_{IJ}=0$ if $I \neq J$, $g_{\alpha\beta}=1$ if $\alpha=\beta$, $g_{ij}=-1$ if $i=j$. For every set of indices used we shall adopt the Feynman notation convention for repeated indices, that is, for every indexed expression E_{IJ}, one defines $E_{II}=\sum_{I=I'}\sum_{J=J'} E_{I'J'}g_{IJ}$. In particular $g_{\alpha\beta}=2$, $g_{ii}=4$, $g_{II}=6$, and $E_{II}=E_{\alpha\alpha}+E_{ii}$. We introduce also the completely skew-symmetric tensors on 2, 4, and 6 indices, such that $\varepsilon_{\alpha\beta ijkl}=\varepsilon_{\alpha\beta}\alpha_{ijkl}$, $\varepsilon_{12}=\varepsilon_{3456}=1$.

1.b. The Lie Algebra Generators

Let $M_{IJ}=-M_{JI}$ be basis elements of the Lie algebra $\mathfrak{g}$, obeying the commutation relations:

$$[M_{IJ}, M_{I'J'}]=g_{JI'}M_{IJ'}-g_{JJ'}M_{II'}-g_{IJ'}M_{JJ'}+g_{IJ'}M_{JI'}. \tag{1.1}$$

The seven-dimensional maximal compact subalgebra $\mathfrak{k}=\mathfrak{so}(4)\oplus\mathfrak{so}(2)$ is spanned by elements $M_{\alpha\beta}$, M_{ij}; its supplementary subspace $\mathfrak{p}$, in a Cartan decomposition $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is spanned by $M_{\alpha i}$.

1.c. Remarkable Polynomials in the Enveloping Algebra $\mathscr{U}(\mathfrak{g})$ of $\mathfrak{g}$

Define

$$8F_{IJ}=\varepsilon_{IJKLK'L'}M_{KL}M_{K'L'}, \tag{1.2}$$

$$2\mathscr{C}_2=M_{IJ}M_{IJ}\,;\ 3\mathscr{C}_3=M_{IJ}F_{IJ}\,;\ 2\mathscr{C}_4=F_{IJ}F_{IJ}. \tag{1.3}$$

The elements of the center of $\mathscr{U}(g)$ are polynomials in $\mathscr{C}_2$, $\mathscr{C}_3$, $\mathscr{C}_4$. We shall introduce the following t-dependent element of the center, called the *characteristic polynomial of* $\mathfrak{g}$:

$$\mathscr{C}(t)=t^2(t^2-1)(t^2-4)+t^2(t^2-1)\mathscr{C}_2+t^2\mathscr{C}_4+[\mathscr{C}_3]^2. \tag{1.4}$$

The introduction of the characteristic polynomial is motivated by its close relation to the weights of finite-dimensional representations of $\mathfrak{g}$: consider a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which is also a Cartan subalgebra of $\mathfrak{k}$, and a basis of $\mathfrak{h}$, the elements of which are mutually "disjoint" such as $\{M_{12}, M_{34}, M_{56}\}$.

Let π be a finite-dimensional i.r. of $\mathfrak{g}$, and denote again by π the i.r. of the complexified Lie algebra $\mathfrak{so}(6,\mathbb{C})$. Take an extremal vector of π, and let ij_1, ij_2, ij_3 be the eigenvalues of the basis elements. Suppose that $j_3\geqq j_2\geqq |j_1|$ (reordering them if needed), then one has

$$\pi(\mathscr{C}(j_3+2))=\pi(\mathscr{C}(j_2+1))=\pi(\mathscr{C}(j_1))=0,$$

that is, *the roots of $\mathscr{C}$, for every simple finite-dimensional representation of* $\mathfrak{g}$, *are fixed linear functions on the coordinates of its dominant weight.* This is not specific to $\mathfrak{so}(4,2)$: such characteristic polynomials can be defined in much more general situations.

It turns out, as will be exposed later, that constraints on the range of the $\mathfrak{k}$-reduction of a $\mathfrak{g}$-module are still related to the roots of $\mathscr{C}$ even in the infinite-dimensional case.

One can prove that there are $6\times 6=36$ t-depending elements $\mathscr{T}_{IJ}(t)$ in $\mathscr{U}(\mathfrak{g})$, such that the following identity holds:

$$\mathscr{T}_{IJ}(t)\cdot((t-2)g_{JK}+M_{JK})=((t-2)g_{IJ}+M_{IJ})\,\mathscr{T}_{JK}(t)=g_{IK}\mathscr{C}(t). \tag{1.5}$$

The $\mathscr{T}$'s are polynomials of degree 5 in t, their coefficients are linear combinations of g_{IJ}, M_{IJ}, F_{IJ}, $M_{IK}M_{KJ}$, $M_{IK}F_{KJ}$, and $F_{IK}F_{KJ}$. We shall call them *transition polynomials* of $\mathfrak{g}$.

Similar elements can be defined inside $\mathscr{U}(\mathfrak{so}(4))$: the characteristic polynomial of so(4) will be

$$C(t)=t^2(t^2-1)+t^2C_2+[C_2']^2, \tag{1.6}$$

where $C_2=\frac{1}{2}M_{ij}M_{ij}$, $C_2'=F_{12}$; there are again $4\times 4=16$ elements $T_{ij}(t)$ in $\mathscr{U}(\mathfrak{so}(4))$, linear in g_{ij}, M_{ij}, $\varepsilon_{ijkl}M_{kl}$, $M_{ik}M_{kj}$, and polynomial in t, such that:

$$T_{ij}^{(t)}((t-1)g_{jk}+M_{jk})=((t-1)g_{ij}+M_{ij})\,T_{jk}(t)=g_{ik}C(t). \tag{1.7}$$

The interest of the characteristic and transition polynomials is the following: one proves that if $\mathscr{F}$ is an arbitrary irreducible finite-dimensional $\mathfrak{g}$-module [respectively $\mathfrak{so}(4)$-module] and if $\mathscr{X}$ is the canonical six- (respectively four) dimensional $\mathfrak{g}$ [respectively $\mathfrak{so}(4)$]-module, with basis $\{x_I\}$ (respectively $\{x_i\}$), then, the reduction of the tensor product $\mathscr{X}\otimes\mathscr{F}=\bigoplus_\chi \mathscr{F}^\chi$ involves elements of the form $x_I\otimes\mathscr{T}_{IJ}(t)\cdot f$ [respectively $x_i\otimes T_{ij}(t)\cdot f$], $f\in\mathscr{F}$, for values of t such that $\mathscr{C}(t)=0$ [respectively $C(t)=0$] on $\mathscr{F}$.

1.d. $\mathfrak{g}$-*Modules*

We study $\mathfrak{g}$-modules $\mathscr{E}$ possessing the following properties:

1) $\mathscr{E}$ is an algebraic direct sum of $\mathfrak{k}$-isotypic components $\mathscr{E}=\bigoplus_\chi \mathscr{E}^\chi$, where each $\mathscr{E}^\chi$ is a finite multiple $\mathscr{M}^\chi\otimes\mathscr{F}^\chi$ of χ, $\mathscr{M}^\chi$ being a finite-dimensional vector space on which $\mathfrak{k}$ acts trivially.

2) $\mathscr{E}$ is pseudohermitian, that is, there is a sesquilinear form on $\mathscr{E}$ such that $(M\varphi|\varphi)+(\varphi|M\varphi)=0$ for every $M\in\mathfrak{g}$ and every $\varphi\in\mathscr{E}$.

3) $\mathscr{E}$ is irreducible.

We want to classify, within infinitesimal equivalence, such $\mathfrak{g}$-modules for which the sesquilinear form is positive definite. Our classification will be based on two criteria.

The first criterion is the range of χ; $\mathscr{E}$ will be assimilated to a set $\mathscr{L}$ of points in $\mathbb{R}^3$, such that χ will belong to $\mathscr{L}$ iff $\mathscr{E}^\chi\neq\{0\}$ belongs to the $\mathfrak{k}$-reduction of $\mathscr{E}$.

The second criterion will be the representation of the commutant of $\mathfrak{k}$, $\bar{\mathscr{B}}(\mathfrak{k})\subset\mathscr{U}(\mathfrak{g})$, and more precisely of $\mathscr{B}(\mathfrak{k})=\bar{\mathscr{B}}(\mathfrak{k})/\mathscr{U}(\mathfrak{k})$, on a suitably chosen subspace of $\mathscr{E}$, namely on points $(v;h,l)$ for which h takes its minimal value.

We point out that the two criteria are not always independent: it may happen that the determination of the range determines also the action of the commutant, or vice-versa; but this is not necessary.

1.e. $\mathfrak{k}$-*Modules*

Finite-dimensional irreducible (f.d.i.) representations of $\mathfrak{k}$ will be labelled by a triplet $\chi=(v;h,l)$ of (a priori complex) numbers, such that, if f belongs to the representation space $\mathscr{F}^\chi$, one has

$$(M_{\alpha\beta}-iv\varepsilon_{\alpha\beta})\cdot f=C(\pm h)\cdot f=C(\pm l)\cdot f=0. \tag{1.8}$$

Finite-dimensionality implies that both $h\pm l$ are integers and that $h^2\neq l^2$. Conventionally, we shall choose $h^2>l^2$, so that $2h\in\mathbb{N}+2$. When h is fixed, the parameter l can take all the $2h-1$ values $\{-h+1, -h+2, \ldots, h-1\}$. With this convention, χ and χ' are equivalent iff $\chi = \chi'$; the parameter ν will be taken real, so that χ is integrable to a unitary irreducible projective representation of $SO(4) \times SO(2)$, or, equivalently, to a u.i.r. of its universal covering $SU(2) \times SU(2) \times \mathbb{R}$.

Remark 1. The correspondence between our parametrization and the "usual" parametrization of u.i.r.'s of $SU(2) \times SU(2)$ by $D(j_1, j_2)$ is

$$h = j_1 + j_2 + 1, \qquad l = j_1 - j_2, \tag{1.9}$$

and the dimension of $\mathscr{F}^{(\nu; h, l)}$ is $h^2 - l^2 = (2j_1 + 1)\cdot(2j_2 + 1)$.

Remark 2. We shall often identify the representation χ, the space $\mathscr{F}^\chi$ and the representative point of χ in $\mathbb{R}^3$ in an abusive but convenient geometrical language: for instance, speaking about the multiplicity of a given k-module which is an isotypic multiple of $\mathscr{F}^\chi$, we shall say "the multiplicity of χ."

2. The Positive-Definiteness Problem

It is clear that the pseudohermitian form (p.h.f.) will be positive-definite on $\mathscr{E}$ iff it is so on every isotypic component $\mathscr{E}^\chi$, since $\mathscr{E}^\chi$ and $\mathscr{E}^{\chi'}$ are orthogonal for $\chi \neq \chi'$. Here $\mathscr{E}^\chi$ is conserved by $\mathfrak{k}$ but not (with the exception of the trivial module) by $\mathfrak{p}$.

Vectors of the form $X\varphi$, with $X \in \mathfrak{p}$, and $\varphi \in \mathscr{E}^\chi$, span a $\mathfrak{k}$-module isomorphic to $\mathfrak{p} \otimes \mathscr{E}^\chi$, $\mathfrak{p}$ being considered as a k-module under the adjoint representation. The reduction of this tensor product into isotypic components is known; one has $\mathfrak{p} \otimes \mathscr{E}^\chi \subset \bigoplus_{\Delta\chi} \mathscr{E}^{\chi + \Delta\chi}$, where $\Delta\chi$ can be one of the triplets $(\pm\,; \varepsilon, 0)$ and $(\pm\,; 0, \varepsilon)$ with $\varepsilon^2 = 1$. There are in general eight points in the reduction, but some of them may vanish if $\chi + \Delta\chi$ does not correspond to a representation: if $h - 1 = \varepsilon l$, then $\mathscr{E}^{\chi + \Delta\chi} = \{0\}$ for $\Delta\chi = (\pm\,; -1, 0)$; or $(\pm\,; 0, \varepsilon)$.

The irreducibility requirement implies the following consequence on the range $\mathscr{L}$ of χ:

Proposition 1. *The set $\mathscr{L}$ is a lattice in $\mathbb{R}^3$, such that if two points $(\nu; h, l)$ and $(\nu'; h', l')$ are in $\mathscr{L}$, then $\nu - \nu' + h - h' + l - l' \in 2\mathbb{Z}$, $h - h' \in \mathbb{Z}$ and $l - l' \in \mathbb{Z}$, and that if $(\nu; h, l)$ is in $\mathscr{L}$, then $2h \in 2 + \mathbb{N}$ and $h - |l| \in \mathbb{N}$.*

With the help of the transition polynomials $T_{ij}(t)$ one can define, for every $X \in \mathfrak{p}$, and given χ, a *shift operator* $X^{\Delta\chi}$, such that $X^{\Delta\chi}\mathscr{E}^\chi \subset \mathscr{E}^{\chi + \Delta\chi}$ and that $X = \bigoplus_{\Delta\chi} X^{\Delta\chi}$. This splitting of X into components is χ-depending; we shall write $X^{\Delta\chi} = 0$ when $\chi + \Delta\chi$ does not correspond to a representation.

Define the X-depending squared shift operator (s.s.o.):

$$\Xi_X^{\Delta\chi} = -X^{-\Delta\chi} X^{\Delta\chi}.$$

The existence of the p.h.f. implies that, for $\varphi \in \mathscr{E}^\chi$, $(X^{\Delta\chi}\varphi | X^{\Delta\chi}\varphi) = (\varphi | \Xi_X^{\Delta\chi}\varphi)$ so that the squared shift operators establish a relation between the p.h.f. of $\mathscr{E}^\chi$ and that of $\mathscr{E}^{\chi + \Delta\chi}$. It is clear that

Theorem 0. *The pseudohermitian form on $\mathscr{E}$ is positive definite iff $(\varphi | \Xi_X^{\Delta\chi}\varphi) \geqq 0$ for every $\varphi \in \mathscr{E}^\chi$, every χ, every X, every $\Delta\chi$.*

This is however too general (it is true for quite a lot of Lie algebras) and there are too many "every"'s. We shall use the algebraic expression of the s.s.o.'s to obtain explicit inequalities concerning $\mathscr{L}$ on one hand (using the fact that if $\Xi_X^{\Delta\chi}\varphi=0$, then $X^{\Delta\chi}\varphi$ must be zero for a positive-definite lattice), and eigenvalues of s.s.o.'s on the other hand.

First one shows that there is no loss of generality if one sums over the 8 basis elements of $\mathfrak{p}$, obtaining thus squared shift operators independent of the choice of X in $\mathfrak{p}$; we shall denote them by $\Xi^{\Delta\chi}$; their algebraic expression involves elements of the form $M_{\alpha i}M_{\beta j}T_{ji}(t)(g_{\alpha\beta}\pm i\varepsilon_{\alpha\beta})$, for χ-depending values of t.

Next one shows that the eight (one for each choice of $\Delta\chi$) s.s.o.'s so obtained span the commutant $\bar{\mathscr{B}}(\mathfrak{k})$ of $\mathfrak{k}$, or more precisely the representation on every $\mathscr{M}^{\chi}$ (recall that $\mathscr{E}^{\chi}=\mathscr{M}^{\chi}\otimes\mathscr{F}^{\chi}$) of the quotient $\mathscr{B}(\mathfrak{k})=\bar{\mathscr{B}}(\mathfrak{k})/\mathscr{U}(\mathfrak{k})\cap\bar{\mathscr{B}}(\mathfrak{k})$.

Then, one establishes linear relations among the s.s.o.'s using the commutation relations (the simplest one being $M_{\alpha i}M_{\beta i}-M_{\beta i}M_{\alpha i}=-4M_{\alpha\beta}=-4iv\varepsilon_{\alpha\beta}$). It turns out that (say, for $v\geqq 0$), $\Xi^{(+;+,0)}$ is a linear combination $a_i\Xi^{\Delta\chi i}+b$, with $a_i, b\geqq 0$, of s.s.o.'s for which the parameter h does not increase.

This reduces the investigation to spaces of the form $\mathscr{K}^{\chi}=\mathscr{E}^{\chi}\cap\operatorname{Ker}\Xi^{(-;-,0)}$.

Further, one examines the positivity on $\mathscr{K}^{\chi}$, using the roots of the characteristic polynomial $\mathscr{C}(t)=0$. It turns out that the real roots of $\mathscr{C}(t)$ are related to the coordinates of χ for which a change of sign on $\Xi^{\Delta\chi}$ occurs. If this change of sign takes place after the corresponding Ξ vanishes, the full lattice $\mathscr{L}$ may contain an irreducible sublattice with positive-definiteness at every point; otherwise it is rejected. This procedure determines as well the multiplicity of every point in the lattice.

Finally, one is able to study $\mathscr{B}(\mathfrak{k})$ on a set of multiplicity-free $\mathfrak{k}$-modules, for which the algebraic expression of each $\Xi^{\Delta\chi}$ (identified by its unique eigenvalue) is known explicitly in terms of the coordinates of χ and of the roots of the characteristic polynomial.

The above procedure leads to the exhaustive classification of u.i.r.'s. Indeed, it is standard knowledge that we can restrict ourselves to the study of irreducible Harish-Chandra modules, the difficulty of such an approach lying precisely in determining which ones are unitarizable. This is solved by checking positivity of s.s.o.'s eigenvalues on k-modules of multiplicity one which lie on the boundary of the lattice: here we have outlined the proof that there is no need to check positivity elsewhere.

In the next section we shall give the algebraic expression of these eigenvalues on boundary k-modules.

3. Boundary Subspaces of a $\mathfrak{g}$-Module

3.a. The Ground Floor

Since the parameter h is a positive number by convention, we shall denote by H its lowest value. The subspace $\mathscr{E}_H$ of the (irreducible pseudo-hermitian), $\mathfrak{g}$-module $\mathscr{E}$ determined by $\mathscr{E}_H=\bigoplus_{v,l}\mathscr{E}^{(v;H,l)}$, summing over all possible values of v, l, will be called the *ground floor*, or the *ground-sublattice* of $\mathscr{E}$. Once the action of the stabilizer S_H of $\mathscr{E}_H$ on $\mathscr{E}_H$ is known, the $\mathfrak{g}$-module $\mathscr{E}$ is completely determined by a classical construction: see, e.g., Dixmier [7, Chap. 9.1, especially 9.1.5. and 9.1.6]. When $H \neq 1$, S_H is generated by shift operators $X^{(\varepsilon; 0, \varepsilon')}$, while when $H=1$ [trivial so(4)-module] S_H is generated by shift operators of the form $X^{(\varepsilon; -1, 0)}Y^{(\varepsilon'; 1, 0)}$: for $\varepsilon+\varepsilon'=0$ these operators stabilize each $\mathscr{E}^\chi$, while for $\varepsilon=\varepsilon'$ they translate v by 2ε. We shall write $\Omega^{\pm 2}=M_{\alpha i}M_{\beta i}M_{\beta' j}M_{\alpha' j}(g_{\alpha'\alpha} \pm i\varepsilon_{\alpha'\alpha})(g_{\beta'\beta} \pm i\varepsilon_{\beta'\beta})$, the corresponding squared shift operator.

It turns out that every point of the ground lattice has multiplicity one, and that the algebraic expression of the s.s.o.'s is quite similar to analogous expressions in $\mathscr{U}(\mathfrak{so}(2,2))$ concerning the s.s.o.'s with respect to an $\mathfrak{so}(2)\otimes\mathfrak{so}(2)$ diagonal basis. The knowledge of the s.s.o.'s on the ground floor determines completely the action of S_H [by means of the same construction as that passing from S_H to $\mathscr{U}(\mathfrak{g})$]. One has

Proposition 2. *The action of the stabilizer S_H on the ground lattice is completely determined by two parameters, labelled x_+^2, x_-^2, and by position of the ground lattice in the strip $|l| \leqq H-1$ of the (v, l) plane; the extent of the ground lattice may be limited by particular values of x_+^2, x_-^2. Both $x_\pm^2$ must be real for $H \neq 1$; for $H=1$ they may be either real or complex conjugate numbers, and the permutation $x_+^2 \leftrightarrow x_-^2$ yields equivalent representations. The roots of the characteristic polynomial are $\pm(H-1)$, $\pm\frac{1}{2}(x_+ + x_-)$, $\pm\frac{1}{2}(x_+ - x_-)$.*

The algebraic expressions of the squared shift operators on $\mathscr{E}^{(v; H, l)}$ are:

$$4\Xi^{(\varepsilon; 0, \varepsilon)}=(H-\varepsilon l)^{-1}\cdot(H-\varepsilon l-1)((v+l+\varepsilon)^2-x_+^2), \tag{3.1}$$

$$4\Xi^{(\varepsilon; 0, -\varepsilon)}=(H+\varepsilon l)^{-1}\cdot(H+\varepsilon l-1)((v-l+\varepsilon)^2-x_-^2), \tag{3.2}$$

$$4\Xi^{(\varepsilon; 1, 0)}=(H^2-l^2)^{-1}[2H((H+\varepsilon v+1)^2+H^2-l^2)-(H+l)x_+^2-(H-l)x_-^2], \tag{3.3}$$

for $H>1$, and

$$4\Xi^{(\varepsilon; +, 0)}=(v+\varepsilon)^2+(v+3\varepsilon)^2-x_+^2-x_-^2=2(v+2\varepsilon)^2+2-x_+^2-x_-^2, \tag{3.4}$$

$$\Omega^{2\varepsilon}=((v+\varepsilon)^2-x_+^2)((v+\varepsilon)^2-x_-^2), \tag{3.5}$$

for $H-1=0=l$.

3.b. Boundaries Involving l

The planes $h \mp l=1$ are "natural" boundaries of the lattice $\mathscr{L}$; however $\mathscr{E}^{(v; h, \varepsilon(h-1))}$ may vanish for special values of the parameters $x_\pm^2$. If this is not the case, there are nonvanishing $\mathscr{U}(\mathfrak{k})$-linear combinations of elements F_{ij} [see (1.2)], defining shift operators from $\mathscr{E}^{(v; h, \varepsilon(h-1))}$ to $\mathscr{E}^{(v; h\pm 1\cdot\varepsilon(h\pm 1-1))}$. The multiplicity of such $\mathfrak{k}$-modules is always 1 (if it is not 0), and the squared shift operators are equal to $\mathscr{C}(h)$ (for $h \to h+1$) or to $\mathscr{C}(h-1)$ (for $h \to h-1$). This feature gives a geometrical significance to the result $0=\mathscr{C}(H-1)$ appearing in Theorem 1.

The parameter l may also be limited by "vertical" planes $\varepsilon l \leqq \varepsilon L$. In that case $L+\varepsilon$ is a root of $\mathscr{C}$; it can be equal to $\pm(H-1)$ or to a combination of x_+, x_-. The algebraic expressions of the commutant are given by formulas (3.1) and (3.2) modified by exchanging h^2 and l^2, $(H-1)^2$ and $(L+\varepsilon)^2$ and subsequently permuting roots of $\mathscr{C}(t)$. Many situations may occur: l may be bounded by one or two vertical walls, and even be constant; comparatively few of them occur in positive-definite lattices, so we shall not list them separately, except for those for which $l=L$ is constant.

If $L=0$, then $\mathscr{C}(0)=\mathscr{C}(1)=0=\mathscr{C}(a)$, the third couple of roots $\pm a$ being arbitrary if $H=1$, and equal to $H-1$ in other cases. One has on $(v;h,0)$:

$$4\Xi^{(\varepsilon;\varepsilon',0)}=h^{-1}\cdot(h+\varepsilon')((\varepsilon'h+\varepsilon v+1)^2-a^2), \tag{3.6}$$

and for $h=H>1$, one must have $v=0$ when $h=H$.

The only other possibility for constant l, is $\pm L=H-1$. The characteristic polynomial has the form $\mathscr{C}(t)=(t^2-(H-2)^2)(t^2-(H-1)^2)(t^2-H^2)$. The lattice is one-dimensional: for every point one has $l^2=(H-1)^2$, $v^2=h^2$, and there are four such lattices for every H (two if $H=1$) corresponding to choices of sign for l and v. The only nonzero s.s.o.'s on $(\pm h,h,L)$ are:

$$\Xi^{(\pm\varepsilon,\varepsilon,0)}=(\varepsilon h+H)(\varepsilon h-H+2). \tag{3.7}$$

These lattices correspond to the well-known "*ladder representations*" of $\mathfrak{g}$.

3.c. *Oblique Walls*

Whenever the lattice $\mathscr{L}$ contains a point $\chi_0=(v_0;h_0,l_0)$ on which both $\Xi^{(-\varepsilon;-\varepsilon',0)}$ and $\Xi^{(-\varepsilon,0,-\varepsilon'')}$ are zero, then these two s.s.o.'s vanish as well all over the plane $\varepsilon v+\varepsilon' h+\varepsilon'' l=\varepsilon v_0+\varepsilon' h_0+\varepsilon'' l_0$, so that all points of the lattice obey $\varepsilon(v-v_0)+\varepsilon'(h-h_0)+\varepsilon''(l-l_0)\geqq 0$. There is a choice among that roots $\pm a$, $\pm b$, $\pm c$ of the characteristic polynomial such that, if $(v;h,l)$ is on the wall:

$$\varepsilon v+\varepsilon' h+\varepsilon'' l=a+b+c+2. \tag{3.8}$$

All points on the wall are of multiplicity one, and the expressions of the s.s.o.'s stabilizing the boundary are:

$$\begin{aligned}
&\Xi^{(-\varepsilon;\varepsilon',0)}=(\varepsilon' h-\varepsilon'' l)^{-1}\cdot\Phi(\varepsilon' h);\\
&\Xi^{(\varepsilon;-\varepsilon',0)}=(h^2-l^2)^{-1}\cdot(\varepsilon' h+\varepsilon'' l-1)\cdot\Phi(\varepsilon' h-1),\\
&\Xi^{(-\varepsilon;0,\varepsilon'')}=(\varepsilon' l-\varepsilon' h)^{-1}\cdot\Phi(\varepsilon'' l);\\
&\Xi^{(\varepsilon;0,-\varepsilon'')}=(l^2-h^2)^{-1}\cdot(\varepsilon' h+\varepsilon'' l-1)\cdot\Phi(\varepsilon' l-1),
\end{aligned} \tag{3.9}$$

with $\Phi(x)=(x-a)(x-b)(x-c)$.

Considering the intersection of the oblique wall with the ground floor, one finds that there is always a second oblique wall, $\varepsilon v+v' h-\varepsilon'' l\geqq a+b-c+2$, meeting the first one at the edge $\varepsilon'' l=c$, $\varepsilon v+\varepsilon' h=a+b+2$. If $\varepsilon'=1$, this edge may be unlimited towards increasing h's, so that the parameter v is bounded on one side, by a bound depending on h, but not absolutely-bounded; while, if $\varepsilon'=-1$, v is absolutely bounded on one side. Such representations are the so-called "weight representations" of $\mathfrak{g}$.

4. Classification of Positive-Definite $\mathfrak{g}$-Modules

4.a. Indecomposable Harish-Chandra Modules

Suppose now that $\mathscr{E}$ is a dense analytic subspace of $\mathfrak{k}$-finite vectors of a Banach space $\bar{\mathscr{E}}$ on which a representation of $\bar{G}$ is defined, such that the center of $\mathscr{U}(\mathfrak{g})$ as well as the center $\mathscr{Z}$ of $\bar{G}$, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}$, are represented by a priori complex-valued scalars. If $\mathscr{E}$ contains a k-submodule $\mathscr{E}^\lambda$ of multiplicity one, its invariant $\mathfrak{g}$-submodule $\mathscr{U}(\mathfrak{g})\mathscr{E}^\lambda$ is indecomposable, and all irreducible unitarizable $\mathfrak{g}$-modules are factors of such a module. Identifying the Casimirs $\mathscr{C}_2, i\mathscr{C}_3, \mathscr{C}_4$ with their eigenvalues, let $(\mathscr{C}_2, i\mathscr{C}_3, \mathscr{C}_4; \mu, \gamma)$ denote such a module, the spanning element of $\mathscr{Z}$ being represented by:

$$\exp(2\pi M_{56}) = (-1)^{2\gamma}; \qquad \exp\pi(M_{12}+M_{34}+M_{56}) = e^{i\pi(\mu+\gamma)}, \tag{4.1}$$

the range of γ, μ being $\{0, \frac{1}{2}\}$ for γ and $\mathbb{C}/2\mathbb{Z}$ for μ; they are related to the coordinates of the representative point $\chi = (v; h, l)$ of a $\mathfrak{k}$-isotypic component by

$$h \equiv l \equiv \gamma \,(\mathrm{mod}\,\mathbb{Z}); \qquad v+h+l \equiv \mu+\gamma+1\,(\mathrm{mod}\,2\mathbb{Z}). \tag{4.2}$$

For finite covering representations μ must be a rational number; in particular, representations of SU(2, 2) must obey μ integer, those of $\mathrm{SO}_0(4,2)$ μ integer and $\gamma = 0$, and those of the centerless group Ad SU(2, 2) $\mu = \gamma = 0$.

Pseudohermitian structure for each irreducible factor of the module $\mathscr{E} = (\mathscr{C}_2, i\mathscr{C}_3, \mathscr{C}_4; \mu, \gamma)$ restricts the range of the five parameters from the set $\mathscr{M}^{\mathbb{C}} = \mathbb{C}^3 \times \mathbb{C}/2\mathbb{Z} \times \{0, \frac{1}{2}\}$ to $\mathscr{M}^{\mathbb{R}} = \mathbb{R}^3 \times \mathbb{R}/2\mathbb{Z} \times \{0, \frac{1}{2}\}$; but we shall not restrict it for the moment; a point in $\mathscr{M}^{\mathbb{C}}$ will denote the biggest possible indecomposable $\mathfrak{g}$-module available (if any), with the convention that we identify $\mathfrak{g}$-modules which contain the same irreducible factors and differ only by permutations among invariant submodules and quotient modules.

The results of the preceding sections can be transposed to this frame (except for specific considerations about pseudohermitian forms) with minor changes; in particular the s.s.o.'s formulas of Sect. 3 still hold. It follows that not every point of $\mathscr{M}^{\mathbb{C}}$ defines a $\mathfrak{g}$-module, but only those for which $\mathscr{C}(H-1)=0$ with $H-\gamma \in \mathbb{Z}$, H being the ground floor of the lattice. The surface in $\mathbb{C}^3$ (Casimir's space) defined by $\mathscr{C}(t)=0$ is a parabolic cylinder (or a double plane if $t=0$). When t^2 varies, $\mathscr{C}(t)=0$ describes a cubic bundle of cylinders, any two of which intersect along a parabola (or a straight line if $i\mathscr{C}_3 = 0$ is one of them); three distinct surfaces meet at two points $(\mathscr{C}_2, \pm i\mathscr{C}_3, \mathscr{C}_4)$ or one if $i\mathscr{C}_3 = 0$. For each value of γ one must retain a discrete subfamily of quadrics, i.e. a Zariski closed subset of $\mathscr{M}^{\mathbb{C}}$. Using all information about walls, one has:

Proposition 3. *A point $m = (\mathscr{C}_2, i\mathscr{C}_3, \mathscr{C}_4; \mu, \gamma)$ of $\mathscr{M}^{\mathbb{C}}$ defines a $\mathfrak{g}$-module iff there is $H \in 1+\gamma+\mathbb{N}$ such that $\mathscr{C}(H-1)=0$; when this happens we shall design by m_H the biggest fully reducible $\mathfrak{g}$-module such that every factor of m_H is a factor of m and has ground floor $h = H' \geqq H$. If m_H contains factors with ground floor $H' > H$, then $\mathscr{C}(H'-1)=0$, and the lattice of the factor $m_H/m_{H'}$ is bounded by the walls $|l| \leqq H'-1$. To every m correspond at most three ground floors.*

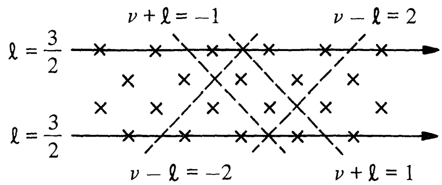

Fig. 1. Multiple splitting for $H=5/2$

We shall denote by $m[H]$ the quotient of m_{H-1} by the biggest $m_{H'}$ strictly included in m_H, if any, so that $m[H]$ contains only factors with ground floor H.

4.b. Reduction of $m[H]$

To find the irreducible factors of $m[H]$ one may restrict the study to the ground floor, using Eqs. (3.1)–(3.4). The plane $h=H$ may be divided by at most two lines parallel to the $v+\varepsilon l=0$ direction, for each (or for both) choice(s) of $\varepsilon=\pm 1$, yielding thus 1×2, 2×2, 1×3, 2×3 or 3×3 regions, some of which may have no common points with the strip $|l|\leqq H-1$. The situation depends on the number of roots of $\mathscr{C}$ which are congruent to H modulo $\mathbb{Z}$. One has:

Proposition 4. *Let $\pm a$, $\pm b$ be the remaining roots of $\mathscr{C}$, besides $\pm(H-1)$, such that $i\mathscr{C}_3=(H-1)\cdot ab$. Then:*

1) *If neither $H-a$, $H-b$ is in $\mathbb{Z}$, then $m[H]$ is irreducible, expect if there is a choice of ε, $\varepsilon'\in\{-1,+1\}$ such that*

$$\varepsilon'(a+\varepsilon b)\equiv\mu-H+\varepsilon\gamma\,(\mathrm{mod}\,2\mathbb{Z}). \tag{4.3}$$

When this happens (for, say, $\varepsilon'=1$), the ground sublattice of $m[H]$, containing points which satisfy $v+\varepsilon l-(a+\varepsilon b)-1\in 2\mathbb{Z}$ and $|l|\leqq H-1$, splits into two sublattices each one containing the points satisfying $\pm(v+\varepsilon l-(a+\varepsilon b))\in 1+2\mathbb{N}$, when $a+\varepsilon b\notin\mathbb{Z}$ or $a+\varepsilon b=0$; it splits into three sublattices when $a+\varepsilon b\in\mathbb{Z}-\{0\}$, the extremal ones satisfying $\pm(v+\varepsilon l-|a+\varepsilon b|)\in 1+2\mathbb{N}$, the middle one $|a+\varepsilon b|-1\geqq|v+\varepsilon l|$; the $\mathfrak{g}$-module $m[H]$ splits accordingly to the direct sum of two or three factors.

2) *If $H-b\in\mathbb{Z}$, $H-a\notin\mathbb{Z}$, then $m[H]$ is irreducible except if there is a choice of $\varepsilon'\in\{+1,-1\}$ such that $\varepsilon'(a+b)\equiv\mu-H+\gamma\,(\mathrm{mod}\,2\mathbb{Z})$; when this happens (for, say, $\varepsilon'=1$) the ground sublattice splits into four subsets (α,α') for all choices of $\alpha,\alpha'\in\{+1,-1\}$, such that all points of each satisfy both conditions $\alpha(v-a+l-b)\in 1+2\mathbb{N}$, and $\alpha'(v-a-l+b)\in 1+2\mathbb{N}$. There are as many irreducible factors of $m[H]$, as many nonempty subsets (α,α'), that is four if $|b|\leqq H-2$, three if $0\neq|b|\geqq H-1$ (the subset $(|b|/b,-|b|/b$ is empty) or two if $0=b=H-1$ (both $(+,-)$ and $(-,+)$ being empty).*

3) *If both a and $b\in\mathbb{Z}+\gamma$, then $m[H]$ is irreducible except if $a+b+H\equiv\mu+\gamma(\mathrm{mod}\,2\mathbb{Z})$. When this happens, the lines $v+l=a+b$, $v-l=a-b$, $v+l=-a-b$, $v-l=-a+b$ separate the plane $h=H$ into four (if $a=b=0$), six (if $a^2=b^2\neq 0$) or nine (if $a^2\neq b^2$) regions which divide the ground sublattice into as many subsets; there are as many irreducible factors in $m[H]$ as many nonempty subsets, that is:*

(with $\alpha^2 = \alpha'^2 = 1$*)*

$$\text{if}\quad a=b=0=H-1: \textit{two}: \{v \geqq 1\} \quad \textit{and} \quad \{v \leqq -1\},$$

$$\text{if}\quad a=b=0 \leqq H-2: \textit{four}: \{\alpha(v+l)>0, \alpha'(v-l)>0\},$$

$$\text{if}\quad \sup(0, H-2) < a = \varepsilon b: \textit{four}: \{\alpha(v+\varepsilon l) > 2a\} \quad \textit{and} \quad \{\alpha(v+\varepsilon l) < 2a, \alpha(v-\varepsilon l) > 0\},$$

$$\text{if}\quad 0 < a = \varepsilon b \leqq H-2: \textit{six}: \{\alpha(v+\varepsilon l) > 2a, \alpha'(v-\varepsilon l) > 0\}, \{|v+\varepsilon l| < 2a, \alpha(v-\varepsilon l) > 0\},$$

$$\text{if}\quad H-1=0=b<a: \textit{three}: \{\alpha v > a\} \quad \textit{and} \quad \{|v| < a\},$$

$$\begin{aligned} \text{if}\quad \sup(0, H-2) < \varepsilon b < a: \textit{five}: &\{\alpha(v+\varepsilon l) > a+\varepsilon b\}, \\ &\{a-\varepsilon b < \alpha(v-\varepsilon l), \alpha(v+\varepsilon l) < a+\varepsilon b\}, \\ &\{|v-\varepsilon l| < a-\varepsilon b\}, \end{aligned}$$

$$\begin{aligned} \text{if}\quad |b| < H-2 < a: \textit{seven}: &\{\alpha(v+l) > a+b, \alpha(v-l) > a-b\}, \\ &\{\alpha(v+\alpha' l) > a+\alpha' b, |v-\alpha' l| < a-\alpha' b\}, \\ &\{|v+l| < a+b, |v-l| < a-b\}, \end{aligned}$$

$$\begin{aligned} \text{if}\quad |b| < a \leqq H-2: \textit{nine}: &\{\alpha(v+l) > a+b, \alpha'(v-l) > a-b\}, \\ &\{\alpha(v+\alpha' l) > a+\alpha' b, |v-\alpha' l| < a-\alpha' b\}, \\ &\{|v+l| < a+b, |v-l| < a-b\}. \end{aligned}$$

In Fig. 1, we give a seven-region splitting, for $H-1=a=3/2$, $b=-\frac{1}{2}$.

One easily sees that the maximal number of factors of $m[H]$ occurs in Case 3 and it is nine. If one sums up for the three distinct determinations of the ground floor, one finds a maximal number of $5+7+9=21$ factors for a point in $\mathscr{M}^{\mathbb{C}}$. One also sees that roots of $\mathscr{C}$ may have many other lattice geometrical properties besides characterizing the ground floor. A geometrical property, which we have not encountered because it is not linked to the ground floor but to oblique walls, is the following:

Proposition 5. *If an irreducible Harish-Chandra module has a lower limit for* $\varepsilon \mathrm{Re}(v)$*, reached when* $v = v_0$*, then* $\varepsilon v_0 - 2$ *is a root of the characteristic polynomial.*

4.c. Positive Definite Irreducible Factors

We shall now give the classification of all unitarizable factors among those classified in Propositions 3 and 4. We shall group them into three big families using lattice geometrical criteria. Indeed, when the lattice $\mathscr{L}$ is known, the corresponding point m in $\mathscr{M}^{\mathbb{C}}$ has two degrees of liberty; if $\mathscr{L}$ has more boundaries than the ground floor and the $|l| \leqq h-1$ walls, m is reducible, and one or two of the remaining parameters are determined by the additional boundaries. So we may speak of an n-nongeometrical-parameters-depending g-module with $n=0, 1, 2$. Of course all unitarizable lattices will correspond to points in $\mathscr{M}^{\mathbb{R}}$.

Lattices of the first family correspond to irreducible m_H; we shall label them by $(\mu; a, b, ; H)$ such that $\pm$a, $\pm$b are roots of $\mathscr{C}$; any permutation of the roots which does not alter the quantities a^2+b^2, a^2b^2 and $(H-1)\cdot ab$ yields of course equivalent lattices. Positive-definiteness depends only on the parameters' range. All positive-definite such lattices are listed in Table 1.

Table 1. Positive-definite $\mathfrak{g}$-modules depending on 2 nongeometrical parameters.
Ground floor $h=H$; $\mu\equiv\nu+l+h+1+\gamma(2\mathbb{Z})$; $\mathscr{C}(\pm a)=\mathscr{C}(\pm b)=0$, $i\mathscr{C}_3=(H-1)ab$

1) $a^2<b^2\leqq 0$
2) $a^2<0=H-1<b^2<1$
3) $\varepsilon^2=1$, $a\neq\bar{a}=\varepsilon b$; $|a+\bar{a}|<1$; $\mu-H+\varepsilon\gamma\notin[-|a+\bar{a}|,|a+\bar{a}|]\,(\mathrm{mod}\,2\mathbb{Z})$
4) $0\leqq a^2, b^2$; $\gamma=0$; $|a|+|b|<1$; $\mu-H\notin[-|a|-|b|,|a|+|b|]\,(\mathrm{mod}\,2\mathbb{Z})$
5) $0\leqq b^2\leqq a^2<1/4$; $\gamma=1/2$; $a=|a|$; $\mu-H-\tfrac{1}{2}\in\,]a-b,1-a-b[\,(\mathrm{mod}\,2\mathbb{Z})$
6) $H-1=0\leqq b^2<a^2<1$; $\mu\in\,]1-|a|+|b|,1+|a|-|b|[\,(\mathrm{mod}\,2\mathbb{Z})$

Table 2. Positive-definite $\mathfrak{g}$-modules depending on 1 nongeometrical parameter.
Ground floor $h=H$; $\mathscr{C}(\pm a)=\mathscr{C}(\pm b)=0$, $i\mathscr{C}_3=(H-1)ab$

Type	Additional lattice boundaries	Geometrical parameters' range	a, b	Nongeometrical parameters' range
1	$\lvert\alpha l-\varepsilon(H-1)\rvert\leqq h+\alpha\nu-x-2$	$x\in\,]-1,\infty[$ $\alpha^2=1$, $\varepsilon^2(H-1)=H-1$	$(x+y)/2$ $\varepsilon(x-y)/2$	$y^2\in\,]-\infty,\mathrm{Inf}(2H-2-2m-x)^2[$ $m\in\mathbb{Z}$, $m\neq 0$
2	$\lvert\nu+\varepsilon l\rvert\leqq h-H$ $\lvert\nu-\varepsilon l\rvert\leqq h+H-2$	$\varepsilon^2=1$	$(y+1)/2$ $\varepsilon(-y+1)/2$	$y^2\in\,]-\infty,\mathrm{Inf}(2H-3-2m)^2[$ $m\in\mathbb{N}$
3	$l=0$	$h+\nu\equiv H+N\,(\mathrm{mod}\,2\mathbb{Z})$ $H=1$, $\lvert N\rvert\leqq 1$	a, 1	$a^2\in\,]-\infty,N^2[$

Table 3. Positive-definite $\mathfrak{g}$-modules determined by their lattice.
Ground floor $h=H$; $\mathscr{C}(\pm a)=\mathscr{C}(\pm b)=0$; $i\mathscr{C}_3=(H-1)ab$

Type	Additional boundaries	Range of parameters	a, b
1	$\lvert h-H_c\rvert+\lvert\alpha l-\varepsilon H+\varepsilon\rvert\leqq\alpha\nu-N$ $\lvert l\rvert\leqq H_c-1$	$H_c\in H+1+\mathbb{N}$, $H_c<N-1$ $\alpha^2=1$, $\varepsilon^2(H-1)=H-1$	εH_c, $N-2$
2	$h-\alpha\varepsilon l-1\leqq\alpha\nu-N$ $\lvert l\rvert\leqq H-1$	$H<N$, $\alpha^2=1$, $\varepsilon^2(H-1)=H-1$	εH, $N-2$
3	$\lvert\alpha l-L\rvert\leqq h-H+\alpha\nu-N$	$\lvert L\rvert<N<H+1$, $\alpha^2=1$ $H-\lvert L\rvert\in 2+\mathbb{N}$	L, $N-1$
4	$\lvert\alpha l-L\rvert+h+1\leqq\alpha\nu$	$H-\lvert L\rvert\in 2+\mathbb{N}$, $\alpha^2=1$	L, H
5	$\lvert\alpha l-H+1\rvert+\lvert\alpha\nu-N\rvert\leqq h-L-1$ $L\leqq\alpha l$	$H-L\in 2+\mathbb{N}$ $\lvert N\rvert<L$, $\alpha^2=1$	$L-1$, N
6	$\lvert\varepsilon\nu-l\rvert\leqq h-H$ $L\leqq\alpha l\leqq h-\alpha\varepsilon\nu+H-2$	$H-L\in 2+\mathbb{N}$ $0<L$, $\alpha^2=\varepsilon^2=1$	$L-1$, εL
7	$\lvert\alpha\nu-N\rvert\leqq h-\alpha l-1$ $H-1\leqq\alpha l$	$\lvert N\rvert<H\neq 1$, $\alpha^2=1$	$H-2$, N
8	$\lvert\nu-N\rvert\leqq h-1$; $l=0$	$\lvert N\rvert<H=1$	1, N
9	$\lvert\nu\rvert\leqq h-H$; $l=0$	$H\in 2+\mathbb{N}$	1, 0
10	$h=\alpha\nu$; $l=\varepsilon(H-1)$	$\alpha^2=1$, $\varepsilon^2(H-1)=H-1$	$H-2$, εH
11	$h-1=l=\nu$	(trivial)	1, 2

Lattices of the second family depend on a unique nongeometrical parameter; in Table 2 we list all such lattices, as well as their relation to the roots a, b of the characteristic polynomial. The lattice boundaries contain extremal lattice points.

In Table 3 finally, figure all lattices which determine the point m of $\mathscr{M}^{\mathbb{R}}$ of which they are factors. The relation between the geometrical parameters of the lattice and the roots of $\mathscr{C}$ is indicated.

4.d. Relations Among the Different Types of $\mathfrak{g}$-Modules

The $\mathfrak{g}$-modules of Table 1 appear like analytic continuations of the first family which corresponds to the principal series of representations; the width of the analytic continuation depends in general continuously on the parameter μ. One should notice that for $H=1$ [trivial $\mathfrak{so}(4)$-module] there are two kinds of continuation which do not appear for $H \notin 1$. These are the types 1.2 and 1.6. The limiting case of 1.2 appears in Table 2 as 2.3 and contains $\mathfrak{g}$-modules with two-dimensional lattice ($l=0$). In Fig. 2 we illustrate types 1.4, 1.5, 1.6 (i.e. when both a and b are real) in an (x, y)-plane with $x=|a+b|$, $y=|a-b|$. (In case $H=1$, we can suppose $x \geqq y$ because of additional symmetry.)

The abscissa λ of the boundary corresponds to $\mathrm{Inf}|v \pm l-1|$ for H integer and to $\mathrm{Inf}\,|v+l-1|$ for H half integer, when v, l vary on the ground sublattice, μ being fixed. The board, $x=\lambda \mp 1$ corresponds to two factors of family 2.1 for which $(v+l)$ is bounded above or below by $1+x$ and $1-x$; when $x=\lambda=1$ one gets three factors, $v+l=0$ (type 2.2) and $\varepsilon(v+l) \geqq 2$ (type 2.1) on the ground floor. The point A corresponds, for noninteger λ to the splitting of Proposition 4, Case 2: all four components (three if $H=3/2$) are unitarizable, the components of the extremal vertices of each component being $\pm(v;l)=(\lambda, 1)$, $(\lambda+1, 0)$, $(\lambda-1, 0)$, $(\lambda, -1)$ in the integer nontrivial case and $(\lambda-1/2, 3/2)$, $(\lambda+1/2, 1/2)$, $(\lambda-3/2, 1/2)$, $\lambda-1/2, -1/2)$ in the half integer case. For sufficiently big H they are classified as 3.3 and 3.5; for $H=5/2$, the $|l|=3/2$ component shifts from 3.5 to 3.7; for $H=2$ both components with $|l|=1$ do so; for $H=3/2$ the singleton $\pm(\lambda-\frac{1}{2}, -\frac{1}{2})$ is in 3.7, the two other ones in 2.1. The point B explodes into seven factors: three with ground floor $h=1$, and ground floor restrictions $\varepsilon v \leqq -1-\lambda$, $\varepsilon v \geqq 3-\lambda$ (type 3.2) and $\varepsilon v=1-\lambda$ (type 3.8); six with ground floor $h=2$, among which two ground floor singletons $\{l=\pm 1, \varepsilon v=1-\lambda\}$ – (type 3.7) and two of the type 3.3 with extremal ground floor vertex $\{l=0, \varepsilon v=-\lambda\}$ and $\{l=0, \varepsilon v=2-\lambda\}$. When λ becomes 0, four more ground floor singletons appear, among which $H=2$, $v=l=0$

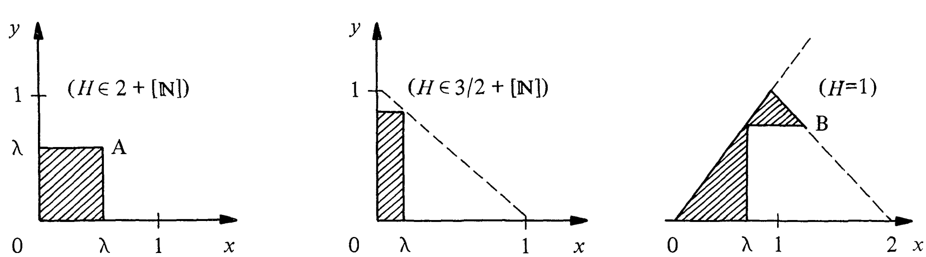

Fig. 2. Real-valued families of Table 1 (shaded)

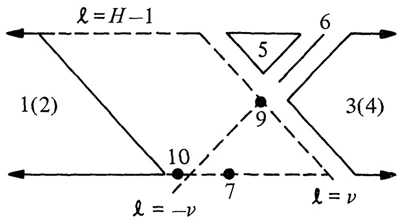

Fig. 3. Ground floor contours of IIIn

(type 3.9), the whole pattern being symmetric with respect to the $\nu \to -\nu$ involution: there are eleven components altogether.

Figure 3 gives an illustration of the ground floor boundaries of types occuring in Table 3. The position of the sublattice with respect to the main diagonals plays a great role for the positive-definiteness: 3.3 and 3.4 differ only in the positioning, 3.4 being maximally distant from the origin positive-definite lattice with this form: the abscissa of the summit is $\pm\nu = H+1$ and it is also the lowest bound of $\pm\nu$ for the whole lattice. The same is true for type 3.5: there are triangles with their summit in the unbounded regions $l \leqq \varepsilon\nu$, but they are not positive-definite. When the triangle reduces to one point we have type 3.7 ($\pm l = H-1$): this singleton may go farther than the diagonal $l = \pm\nu$, up to $\pm\nu = H$. At that point it decomposes into a lattice of type 3.6 and ground floor $H+1$ and a singleton 3.10, which is a so-called "ladder representation." Notice that types 3.1, 3.2, 3.4, 3.10, and 3.11 (the trivial one) are the so-called "weight representations," for which $\pm\nu$ has a lower bound [6].

It would be too long to expose in what way types 3 can be obtained as factors of type 2 for special values of y. Types 3, 5, 6 of Fig. 3 have been positioned to suggest the splitting of a $\mathfrak{g}$-module of type 2.1 for $y=1$. We shall just point out an irregularity in the supremum of y for type 2.1: while in general this supremum is a number between 0 and 1, determined by x modulo $2\mathbb{Z}$, when x is sufficiently near $2(H-1)$ the supremum is between 1 and 2, and it reaches $y=2$ when $x=2(H-1)$. This can be roughly explained as follows: when x is an integer and y reaches 0 or 1, then the three roots $H-1$, $\frac{1}{2}(x-y)$, $\frac{1}{2}(x+y)$ or $\mathscr{C}$ are congruent modulo $\mathbb{Z}$, so that the considered $\mathfrak{g}$-module reaches the intersection of two or three quadrics in $\mathscr{M}^{\mathbb{R}}$, and the indecomposable module splits into factors corresponding to distinct ground floors. But there is no such splitting when $x=2(H-1)$, $y=0$, $H-1$ being a triple root in that case.

5. Complements

5.a. Finite-Dimensional $\mathfrak{g}$*-Modules*

The method sketched in Sects. 2 and 3 gives also the finite-dimensional $\mathfrak{g}$-modules. They appear as factors in the decomposition of $m \in \mathscr{M}^{\mathbb{R}}$ when $\mu \in \mathbb{Z}(\operatorname{mod} 2\mathbb{Z})$ and when the roots of $\mathscr{C}$ are all $\mathbb{Z}$-congruent to 0 or $\frac{1}{2}$, provided there are no double roots. One gets then the $5+7+9=21$ component reduction of Proposition 4, and one of them is finite-dimensional.

For the trivial representation the roots of $\mathscr{C}$ are $(0,1,2)$, for SU(2, 2), they are $(5/2, 3/2, \pm 1/2)$, for SO(4, 2) $(0,1,3)$ and for the adjoint representation they are $(0,2,3)$; if a is the biggest root, then $a=1+\sup(h)=2+\sup(v)$ and this is to compare to the ground floor property for any $\mathfrak{g}$-module, which says that $1-\mathrm{Inf}(h)$ is also a root of $\mathscr{C}$.

5.b. *Relation with* $\mathfrak{so}(2,2)$*-Modules*

When the squared linear combinations of the roots $x_\pm^2$ are known, as well as μ and H, there is a unique $\mathfrak{g}$-module $m[H]$, irreducible or reducible to at most 9 factors. On the other hand, γ and $\mu-H\equiv v+l(\mathrm{mod}\,2\mathbb{Z})$ determine a two-dimensional lattice $\mathscr{L}$ on a (v,l) plane; if $x_\pm^2$ are fixed numbers, the quantities $B_{\varepsilon,\varepsilon'}=(\varepsilon v+\varepsilon' l+1)^2-x_{\varepsilon/\varepsilon}^2$ define a structure of $\mathfrak{so}(2,2)$-module on the space $\mathscr{C}=\bigoplus_{(v,l)\in\mathscr{L}}\mathscr{C}^{(v,l)}$, $\mathscr{C}^{(v,l)}$ being one-dimensional, spanned by the basis vector $\varphi^{(v,l)}$, such that if $X_{\varepsilon,\varepsilon'}$ is a nilpotent generator of a Cartan-Weyl basis of $\mathfrak{so}(2,2)$ one has $X_{\varepsilon,\varepsilon'}\varphi^{(v,l)}=\sqrt{B_{\varepsilon,\varepsilon'}}\varphi^{(v+\varepsilon,l+\varepsilon')}$. One has thus a correspondence from the set of indecomposable $\mathfrak{g}$-modules with ground floor H to that of so(2, 2)-modules obeying $l-\gamma\equiv 0(\mathrm{mod}\,\mathbb{Z})$. This correspondence is one-to-one for $H\neq 1$, and one-to-two for $H=1$, because x_+^2 and x_-^2 are interchangeable when $H=1$.

The reduction of $m[H]$ into irreducible factors given in Proposition 4 reflects this correspondence, and the constraint $|l|\leqq H-1$ masks it, because some factors are killed. Notice that positive-definiteness of so(2, 2)-modules is neither necessary (singletons 3.10, and 3.7 for $H-1\leqq|N|<H$) nor sufficient (types 3.3, 3.4 are limited to $|N|\leqq H+1$) for positive-definiteness of the associated $\mathfrak{g}$-module, though there is coïncidence over most cases. The characterization of this correspondence [which can be defined also when $\mathfrak{g}=\mathfrak{so}(2k,2)$, $2k>4$] in, say, group-theoretical terms, is, to our knowledge, an open question.

5.c. *Multiplicities*

The multiplicity of the k-type $\chi=(v;h,l)$ is always bounded by $h-H+1$ for an indecomposable module m_H, and this upper bound is reached for all points such that $|l|\leqq H-1$; the multiplicity is equal to $h-|l|$ when $H-1\leqq|l|\leqq h-1$. When m_H contains a factor $m_{H'}$, $H<H'$ (Proposition 3) the multiplicity stops increasing in $m[H]=m_H/m_{H'}$, its upper bound being $H'-H$. When there is an oblique wall, $h+\varepsilon v+\varepsilon' l\geqq a+b+(H-1)+2$, and no $H'>H$, the upper bound $m_\chi=\mathrm{Inf}(h-H+1,h-|l|)$ is reached for points satisfying $\varepsilon v+\varepsilon' l\geqq a+b+1+h-H$; the multiplicity decreases for points satisfying $h+\varepsilon v-\varepsilon' l\geqq a+b+2-(H-1)$ and $a+b+1+H-h\leqq\varepsilon v+\varepsilon' l<a+b+1+h-H$, its value being then $m_\chi=\mathrm{Inf}(\frac{1}{2}(h+\varepsilon v+\varepsilon' l-a-b-H-1)+1,h-|l|)$.

These three results, combined in all possible ways, give the multiplicity for all types of irreducible $\mathfrak{g}$-modules. The only positive-definite types which have unbounded multiplicity are types 1, types 2.1, and 3.3. Among the other ones, types 2.2, 2.3, 3.2, 3.4, and 3.6–3.11 have multiplicity equal to one for every χ. The starting point majoration leading to these results is a by-product of the positive definiteness research sketched in Sect. 2.

5.d. Concluding Remarks

We have determined here all unitarizable $\mathfrak{g}$-modules, but we have not constructed them. An effective realization may be done in two ways: either construct the Harish-Chandra modules as induced representations, expressing the infinitesimal generators by means of differential operators, and construct the positive Harish-Chandra kernels for each unitarizable factor; or proceed algebraically and study the representation of the commutant of $\mathscr{U}(k)$ on each subspace $\mathscr{E}^{\chi}$. The last approach is not too complicated for multiplicity-free representations, but there are a lot of calculations for the other cases. Our approach started in fact from that point of view: consider all generalized matrix elements (squared so as to remain in the commutant), then discard most of them by majorations. It results quite naturally that the $\mathfrak{g}$-modules are not presented in terms of functional spaces, and that strictly speaking harmonic analysis is only used to transcribe results into a group theoretical language and establish comparisons (with the exception of considerations about the center of $\bar{G}$).

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8. When this manuscript had already been typed, the author discovered that an article by A.W. Knapp and B. Speh, entitled: Irreducible unitary representations of SU(2, 2), had just been published in J. Funct. Anal. **45**, 41 (1982). There has been no interference, since their classification and the one given here are based on entirely different principles; a lot of work should be needed even to compare the results. Their results concern the 4×4 matrix group (not the universal covering): in this common domain a rapid survey hints equivalence of the two classifications

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