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The Itô–Clifford Integral III. The Markov Property of Solutions to Stochastic Differential Equations

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Abstract. It is shown that the solution to the Itô–Clifford stochastic differential equation $dX_t = F(X_t,t)d\Psi_t + d\Psi_t G(X_t,t) + H(X_t,t)dt$, where F, G, H are suitable Lipschitz functions and Ψ_t is the fermion martingale, satisfies a Markov property.

Introduction

The concept of a Markov process plays a central rôle in the theory of diffusion processes, and it is well-known that solutions of stochastic differential equations with respect to Brownian motion are Markov processes.

We consider here the fermion analogue of Brownian motion. The Itô–Clifford stochastic integral (the fermion analogue of the usual Itô-integral with respect to Brownian motion) was constructed in [2], and in [3] the existence and uniqueness of solutions to Itô–Clifford stochastic differential equations was established. We shall show that the solutions to certain of these equations enjoy a Markov property. We consider this to be a step towards a theory of quantum diffusion processes.

Markov properties of fermions and quantum Markov processes have been considered by many authors [1, 4, 5, 9–11, 14]. In [7], quantum diffusions for stochastic evolutions governed by quantum Brownian motion are discussed.

1. Notation and Definitions

We shall briefly outline some of the structure of the Clifford gage space that we shall need. For further details we refer to [2, 3, 6, 8, 12–14].

Let $\mathscr{C}$ denote the weakly-closed Clifford operator algebra over $L^2(\mathbb{R}_+)$; thus $\mathscr{C}$ is the W^*-algebra on the fermion Fock space $\wedge(L^2(\mathbb{R}_+))$ over $L^2(\mathbb{R}_+)$ generated by the fermion fields $\Psi(f) = C(f) + A(\bar{f})$, $f \in L^2(\mathbb{R}_+)$, where $C(\,\cdot\,)$, $A(\,\cdot\,)$ are the Fock fermion creation and annihilation operators, respectively.

For $t \in \mathbb{R}_+$, $\mathscr{C}_t$ denotes the W^*-subalgebra of $\mathscr{C}$ generated by the fields $\Psi(f)$ with $f = 0$ on (t, ∞). For $1 \leqq p < \infty$, $L^p(\mathscr{C})$ is the completion of $\mathscr{C}$ with respect to the norm $\|x\|_p = m(|x|^p)^{1/p}$, where $m(\cdot) = (\Omega, \cdot\,\Omega)$, Ω being the Fock vacuum. Here m is a central state on $\mathscr{C}$. If $\mathscr{B}$ is any W^*-subalgebra of $\mathscr{C}$, $L^p(\mathscr{B})$ is the corresponding completion of $\mathscr{B}$, and is a closed subspace of $L^p(\mathscr{C})$. Segal and Kunze have shown [8, 12] that $L^p(\mathscr{C})$ consists of certain operators on $\wedge(L^2(\mathbb{R}_+))$ affiliated to $\mathscr{C}$. Indeed, any $X \in L^p(\mathscr{C})$ can be written as $X = Y + iZ$, where Y and Z are self-adjoint operators on $\wedge(L^2(\mathbb{R}_+))$ affiliated to $\mathscr{C}$. It is therefore meaningful to talk about the algebra generated by elements of $L^2(\mathscr{C})$—it is that W^*-subalgebra generated by the spectral projections of their real and imaginary parts.

We recall the following definitions [3].

Definition 1.1. A map $X:\mathbb{R}_+ \to L^2(\mathscr{C})$ is said to be adapted if $X_t \in L^2(\mathscr{C}_t)$ for each $t \in \mathbb{R}_+$. A map $F: L^2(\mathscr{C}) \times \mathbb{R}_+ \to L^2(\mathscr{C})$ is said to be adapted if, for any $t \in \mathbb{R}_+$ and $X \in L^2(\mathscr{C}_t)$, we have $F(X,t) \in L^2(\mathscr{C}_t)$.

Definition 1.2. A map $F: L^2(\mathscr{C}) \times \mathbb{R}_+ \to L^2(\mathscr{C})$ is said to satisfy a locally uniform Lipschitz condition if for each $T \geqq 0$ there is a constant $K > 0$ such that

$$\|F(X,s) - F(Y,s)\|_2 \leqq K\|X - Y\|_2$$

for all $0 \leqq s \leqq T$ and all $X, Y \in L^2(\mathscr{C})$.

The following result is a special case of [3, Theorem 2.1].

Theorem 1.3. *Let $F,G,H: L^2(\mathscr{C}) \times [t_0,\infty) \to L^2(\mathscr{C})$ be adapted, continuous and satisfy a locally uniform Lipschitz condition on $[t_0,\infty)$, and let $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+)$. Then, for any $Z \in L^2(\mathscr{C}_{t_0})$, there is a unique continuous adapted L^2-process $(X_t)_{t \geqq t_0}$ satisfying the stochastic differential equation*

$$dX_t = F(X_t,t)d\Psi_t(u) + d\Psi_t(u)G(X_t,t) + H(X_t,t)dt \tag{1.1}$$

on $[t_0,\infty)$ with $X_{t_0} = Z$, and where $\Psi_t(u) = \Psi(u\chi_{[0,t]})$.

Of course, this means that X_t satisfies the stochastic integral equation

$$X_t = Z + \int_{t_0}^{t} F(X_s,s)d\Psi_s(u) + \int_{t_0}^{t} d\Psi_s(u)G(X_s,s) + \int_{t_0}^{t} H(X_s,s)ds \tag{1.2}$$

where $\int \cdot\, d\Psi_s(u)$ is the Itô–Clifford stochastic integral constructed in [2].

2. The Markov Property

We shall see that X_t enjoys a Markov property. To formulate this, we need some more notation. For any interval $I \subseteq [t_0,\infty)$, let $\mathscr{A}_I$ denote the W^*-algebra generated by $\mathbb{1}$ and the solution X_t of Eq. (1.2) for $t \in I$. We shall write $\mathscr{A}_s$ for $\mathscr{A}_{[s,s]}$. Since the solution (X_t) is adapted, i.e. $X_t \in L^2(\mathscr{C}_t)$ for $t \geqq t_0$, it follows that $\mathscr{A}_I$ is a W^*-subalgebra of $\mathscr{C}_t$ whenever $I \subseteq [t_0,t]$.

Let β be the parity operator [2, 3]. Then $\beta:\mathscr{C} \to \mathscr{C}$ is the spatial automorphism implemented by the self-adjoint unitary operator $\Gamma(-\mathbb{1})$ on $\wedge(L^2(\mathbb{R}_+))$. ($\Gamma(-\mathbb{1})$ acts on an n-particle vector as $(-1)^n$ [13].) Since $\mathscr{A}_I \subseteq \mathscr{C}$, it follows that $\beta(\mathscr{A}_I) \subseteq \mathscr{C}$. Let $\tilde{\mathscr{A}}_I = \mathscr{A}_I \vee \beta(\mathscr{A}_I)$, the W^*-subalgebra of $\mathscr{C}$ generated by $\mathscr{A}_I$ and $\beta(\mathscr{A}_I)$. Then $\beta(\tilde{\mathscr{A}}_I) = \tilde{\mathscr{A}}_I$, and, for any $s \geqq t_0$, $\tilde{\mathscr{A}}_s \subseteq \mathscr{C}_s$.

Finally, we shall need to consider the algebra generated by field differences. For fixed $u \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}_+)$, as in Theorem 1.3, and for $s \geqq t_0$, let $\mathscr{F}_s$ denote the W^*-subalgebra of $\mathscr{C}$ generated by the field differences $\{\Psi_\tau(u) - \Psi_s(u) : \tau \geqq s\}$. Here $\tilde{\mathscr{A}}_s \vee \mathscr{F}_s$ is the W^*-subalgebra of $\mathscr{C}$ generated by $\tilde{\mathscr{A}}_s$ and $\mathscr{F}_s$. Thus, $\tilde{\mathscr{A}}_s \vee \mathscr{F}_s$ is generated by X_s, $\beta(X_s)$ and the fields $\{\Psi(u\chi_{[\tau,s]}) : \tau \geqq s\}$. We note that $\beta(\tilde{\mathscr{A}}_s \vee \mathscr{F}_s) = \tilde{\mathscr{A}}_s \vee \mathscr{F}_s$.

Lemma 2.1. *Let $F,G,H : L^2(\mathscr{C}) \times [t_0,\infty) \to L^2(\mathscr{C})$ satisfy the requirements of Theorem 1.3, and, in addition, suppose that for any $X \in L^2(\mathscr{C})$ and $\tau \in [t_0,\infty)$, $F(X,\tau)$ is affiliated to the unital W^*-subalgebra of $\mathscr{C}$ generated by X and $\beta(X)$; similarly for G and H. Then $X_t \in L^2(\tilde{\mathscr{A}}_s \vee \mathscr{F}_s)$ for all $t_0 \leqq s \leqq t$, where $(X_t)_{t \geqq t_0}$ is the solution to Eq. (1.2).*

Remark. The condition that F (and G, H) be such that $F(X,\tau)$ is affiliated to the algebra generated by X and $\beta(X)$ will be satisfied if $F(X,\tau)$ is given by (possibly τ-dependent) functions of the real and imaginary parts of X via the functional calculus (i.e. the spectral theorem), and also if X is self-adjoint and F (and G, H) are as in Sect. 4 of [3].

Proof of Lemma 1.2. Let $X_t^{(n)}, n = 0, 1, 2, \ldots$ be defined inductively as in [3] by $X_t^{(0)} = X_s$, $\forall t \geqq s$, and

$$X_t^{(n+1)} = X_s + \int_s^t F(X_\tau^{(n)},\tau)\,d\Psi_\tau(u) + \int_s^t d\Psi_\tau(u)\,G(X_\tau^{(n)},\tau) + \int_s^t H(X_\tau^{(n)},\tau)\,d\tau.$$

Then $X_t^{(0)} = X_s \in L^2(\tilde{\mathscr{A}}_s \vee \mathscr{F}_s)$, and from the definition of the Itô–Clifford integral, it follows by induction that $X_t^{(n)} \in L^2(\tilde{\mathscr{A}}_s \vee \mathscr{F}_s)$ for all $n \geqq 0$, and $t_0 \leqq s \leqq t$.

But it was shown in [3] that $X_t^{(n)} \to X_t$ in $L^2(\mathscr{C})$ as $n \to \infty$, and so we deduce that $X_t \in L^2(\tilde{\mathscr{A}}_s \vee \mathscr{F}_s)$, $\forall t_0 \leqq s \leqq t$. QED.

Theorem 2.2. *With the notation and conditions of Lemma 2.1, the process $(X_t)_{t \geqq t_0}$ is a Markov process in the following sense: for any $s \geqq t_0$ and $f \in L^1(\tilde{\mathscr{A}}_{[s,\infty)})$, we have*

$$m(f \mid \tilde{\mathscr{A}}_{[t_0,s]}) = m(f \mid \tilde{\mathscr{A}}_s), \tag{2.1}$$

where $m(\cdot \mid \mathscr{B})$ denotes the conditional expectation with respect to the subalgebra $\mathscr{B}$ of $\mathscr{C}$.

Proof. By Lemma 2.1, X_t and therefore $\beta(X_t)$ both belong to $L^2(\tilde{\mathscr{A}}_s \vee \mathscr{F}_s)$ for any $t \geqq s$, and so we see that $\tilde{\mathscr{A}}_{[s,\infty]} \subseteq \tilde{\mathscr{A}}_s \vee \mathscr{F}_s$.

Now, by continuity, it is sufficient to prove the theorem for $f \in \tilde{\mathscr{A}}_{[s,\infty)}$. But, by the previous remark $\tilde{\mathscr{A}}_{[s,\infty)} \subseteq \tilde{\mathscr{A}}_s \vee \mathscr{F}_s$, and so, by linearity and the σ-weak continuity of the conditional expectation map, we may assume that f is of the form

$$f = y_1 z_1 \ldots y_n z_n \tag{2.2}$$

for some $n \in \mathbb{N}$ and $y_i \in \tilde{\mathscr{A}}_s, z_i \in \mathscr{F}_s$, $1 \leqq i \leqq n$.

Furthermore, since $\beta(\tilde{\mathscr{A}}_s) = \tilde{\mathscr{A}}_s$, any element y in $\tilde{\mathscr{A}}_s$ can be written as $y = y_+ + y_-$, with $y_\pm = \frac{1}{2}(y \pm \beta(y)) \in \tilde{\mathscr{A}}_s$, and $\beta(y_\pm) = \pm y_\pm$. We may assume, therefore, that each y_i in Eq. (2.2) has definite parity (—it is here that we must use $\tilde{\mathscr{A}}_s$ rather than $\mathscr{A}_s$). Similarly, since $\beta(\mathscr{F}_s)=\mathscr{F}_s$, we may assume that each z_i in Eq. (2.2) has definite parity.

But $\tilde{\mathscr{A}}_s \subseteq \mathscr{C}_s$, and for any $y\in\mathscr{C}_s$, $z\in\mathscr{F}_s$, the canonical anticommutation relations imply that $yz = \pm zy$, where the minus sign occurs when both y and z are odd. Thus the proof of Eq. (2.1) is reduced to the case when f is of the form $f = yz$ with $y\in\tilde{\mathscr{A}}_s$ and $z\in\mathscr{F}_s$.

To proceed, we note that $\mathscr{C}_s$ and $\mathscr{F}_s$ are independent [13] and, therefore, for $g\in\tilde{\mathscr{A}}_s$, we have

$$\begin{aligned} m(gm(yz|\tilde{\mathscr{A}}_s)) &= m(gyz) \\ &= m(gy)m(z), \quad \text{since} \quad gy\in\mathscr{C}_s, \\ &= m(gym(z)). \end{aligned}$$

Hence $m(yz|\tilde{\mathscr{A}}_s) = ym(z)$.

Similarly, for any $h\in\tilde{\mathscr{A}}_{[t_0,s]}$,

$$\begin{aligned} m(hm(yz|\tilde{\mathscr{A}}_{[t_0,s]})) &= m(hyz) \\ &= m(hy)m(z), \quad \text{since} \quad hy\in\mathscr{C}_s, \\ &= m(hym(z)), \end{aligned}$$

and so $m(yz|\tilde{\mathscr{A}}_{[t_0,s]}) = ym(z)$.

We conclude that $m(yz|\tilde{\mathscr{A}}_{[t_0,s]}) = m(yz|\tilde{\mathscr{A}}_s)$ and the result follows.

If the $\mathscr{A}_s$ are invariant under β for all s, we have the stronger (and more desirable) result:

Corollary 2.3. *If $\beta(\mathscr{A}_s)=\mathscr{A}_s$ for all s, then*

$$m(f\,|\mathscr{A}_{[t_0,s]}) = m(f\,|\,\mathscr{A}_s)$$

for all $t_0 \leqq s$ and $f\in L^1(\mathscr{A}_{[s,\infty)})$.

Corollary 2.4. *$(\Psi_t(u))_{t\in\mathbb{R}_+}$ is a Markov process in the stronger sense of Corollary 2.3.*

Proof. For any I, $\beta(\mathscr{A}_I)=\mathscr{A}_I$ since $\beta(\Psi_t(u)) = -\Psi_t(u)$. Furthermore, $\Psi_t(u)$ satisfies the stochastic differential equation Eq. (1.1) on $[0,\infty)$ with $F(X,\tau)=\mathbb{1}$, $G=H=0$, and the initial condition $X_0=0$. QED.

Corollary 2.4 can also be seen immediately either by noticing that $\Psi_t(u) = \Psi_s(u)+(\Psi_t(u)-\Psi_s(u))$, for $s\leqq t$, and proceeding as in the proof of Theorem 2.2, or by noting that, for the case $X_t=\Psi_t(u)$, we have $\beta(\mathscr{A}_s)=\mathscr{A}_s$ for all $s\geqq 0$, and $m(\cdot|\mathscr{A}_{[0,s]}) = D^{-1}\Gamma(e_s)D$, $m(\cdot|\mathscr{A}_s)=D^{-1}\Gamma(q_s)D$, and $m(\cdot|\mathscr{A}_{[s,\infty)}) = D^{-1}\Gamma(p_s)D$, where $D: L^2(\mathscr{C})\to \wedge(L^2(\mathbb{R}_+))$ is the duality transform [2, 6, 13]. Here e_s is the orthogonal projection of $L^2(\mathbb{R}_+)$ onto the subspace spanned by elements of the form uv, where v has support in $[0, s]$, q_s that onto the subspace spanned by the single element $u\chi_{[0,s]}$, and p_s that onto the subspace spanned by elements of the form uv', where v' is constant on $[0, s]$.

Clearly $e_s p_s = e_s q_s$, and the result follows.

Remark. It is not clear, in general, when $\mathscr{A}_s = \tilde{\mathscr{A}}_s$. A sufficient condition is as follows. Let $v \in L^2(\mathbb{R}_+)$ with $\|v\|_2 = 1$ and supp $v \subseteq [s,\tau]$, $\tau > s$. Then, for any $h \in L^1(\mathscr{C}_s)$, $\beta(h) = \Psi(v) h \Psi(v)$. (Using the canonical anticommutation relations, this is easily seen when h is a monomial in the fields, and the general result follows by linearity and continuity.) It follows that $\beta(\mathscr{A}_s) \subseteq \mathscr{A}_s \vee \mathscr{F}_{[s,\tau]}$ for all $\tau > 0$, where $\mathscr{F}_{[s,\tau]}$ is the W^*-subalgebra of $\mathscr{C}$ generated by $\{\Psi_t(u) - \Psi_s(u) : s \leqq t \leqq \tau\}$. We see that $\beta(\mathscr{A}_s) = \mathscr{A}_s$, i.e. $\tilde{\mathscr{A}}_s = \mathscr{A}_s$, if $\mathscr{A}_s = \bigcap_{\tau > s} (\mathscr{A}_s \vee \mathscr{F}_{[s,\tau]})$.

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