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On $\widetilde{GL}(4, \mathbb{R})$-Covariant Extensions of the Dirac Equation*

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Abstract. Infinite component generalizations of both massless and massive Dirac equations are constructed which are covariant with respect to the double covering of the general linear group in four dimensions. These generalized Dirac equations can be made covariant with respect to the full diffeomorphism group of the spacetime manifold by replacing ordinary derivatives by covariant derivatives in the usual way.

1. Introduction

When ψ is a Dirac field transforming according to the complex four-dimensional spinor representation $a \to S(a)$ of the covering group $SL(2, \mathbb{C})$ of the connected (restricted) Lorentz group $SO_0(3, 1)$, then by

$$(D(a)\psi)(x) := S(a)\psi(\theta(a)^{-1}x)$$

one defines a representation of $SL(2, \mathbb{C})$ in the space of Dirac fields; here θ is the real vector representation of $SL(2, \mathbb{C})$. The Dirac operator $i\gamma^\nu\partial_\nu + m$ commutes with the representation D [we are used to saying that the Dirac equation $(i\gamma^\nu\partial_\nu + m)\psi = 0$ is covariant with respect to Lorentz transformations].

A natural question to ask in a general relativistic contex is whether the representation S of $SL(2, \mathbb{C})$ could be extended to the group $G = \widetilde{GL}(4, \mathbb{R})$ [universal covering of the general linear group $GL(4, \mathbb{R})$] in such a way that the Dirac equation would be G-covariant. What would then be the generalized Dirac matrices? Assuming that the extension is carried out, the Dirac equation will be covariant with respect to an arbitrary diffeomorphism J in $(\mathbb{R}^4, g)$ (g is a Lorentz metric). One has only to replace ∂_ν by the space-time covariant derivative ∇_ν,

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determined by the metric g and by the representation S, and define

$$(D(J)\psi)(x) := S\left(\frac{\partial J}{\partial x}(J^{-1}x)\right)\psi(J^{-1}x).$$

It is easy to check that D is a "two-valued representation" of the diffeomorphism group $\mathrm{Diff}\,\mathbb{R}^4$ in the space of Dirac fields. This is not a true representation since S is not a representation of $\mathrm{GL}(4,\mathbb{R})$ but of $\overline{\mathrm{GL}(4,\mathbb{R})}$. To make things more precise we can define $G(\mathbb{R}^4)$ as the group of all pairs (ω, J), where $J \in \mathrm{Diff}\,\mathbb{R}^4$ and $\omega : \mathbb{R}^4 \to G$ is a function such that $\theta(\omega(x)) = \frac{\partial J}{\partial x}(J^{-1}x)$ for all $x \in \mathbb{R}^4$; the composition law is $(\omega, J)\cdot(\omega', J') = (\omega'', J'')$, where $\omega''(x) = \omega(x)\omega'(J^{-1}x)$ and $J'' = J \circ J'$. One can define a proper representation of $G(\mathbb{R}^4)$ by

$$(D(\omega, J)\psi)(x) := S(\omega(x))\psi(J^{-1}x).$$

The discussion above can be generalized to the case of an arbitrary space-time manifold (M, g) which admits a principal G bundle $\tilde{F}$ as the double covering of the frame bundle F of M. If $\varphi : \tilde{F} \to F$ is the covering map then $\varphi(p\cdot a) = \varphi(p)\cdot\theta(a)$ for all $p \in \tilde{F}$ and $a \in G$. Let V be a linear space equipped with a spinor representation S of G. The associated vector bundle $\tilde{F}\times_G V$ consists of all equivalence classes $[(p, v)]$, where $(p, v) \in \tilde{F}\times V$ and $(p, v) \sim (p', v')$, if $p' = p\cdot a$, $v' = S(a^{-1})v$ for some $a \in G$, [1, Sect. I.5]. Fix a basis $\{e_1, e_2, \ldots\}$ in V. Given a local cross-section $x \to p(x)$ of $\tilde{F}$ one can define a local basis of $\tilde{F}\times_G V$ by $\hat{e}_k(x) := [(p(x), e_k)]$. In particular, if $x_1, \ldots, x_4$ are local coordinates on M then the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_4}$ form a local cross-section of F and one can choose (one of two) a local cross-section $p(x)$ of $\tilde{F}$ such that $\varphi(p(x)) = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_4}\right)$, and thus in this way to any coordinate system on M there is associated a basis $\{\hat{e}_k\}$ of the spinor bundle $\tilde{F}\times_G V$. Since the kernel of $\theta : G \to \mathrm{GL}(4,\mathbb{R})$ is $\mathbb{Z}_2$ and in a spinor representation it is represented (by definition) by the operators 1 and -1, the second choice for $p(x)$ leads to the basis $\{-\hat{e}_k\}$. In a coordinate transformation $y = y(x)$ the basis $\{\hat{e}_k\}$ is transformed to $\{S(\omega)\hat{e}_k\}$, where the function $x \to \omega(x) \in G$ satisfies $\theta(\omega) = \frac{\partial y}{\partial x}$. Thus working with a principal G-bundle enables us to define spinors and their transformation properties in a holonomic way, [6]: to a given set of local coordinates is always associated a basis (unique up to the sign) in the spinor space, in a natural way. This is in contrast to the case of ordinary $\mathrm{SL}(2,\mathbb{C})$ spinors, where one has to *choose* a section of the principal $\mathrm{SL}(2,\mathbb{C})$ bundle and the choice of the section is not related in any natural way to the choice of local coordinates on M (this corresponds to the fact that in general there is no natural choice for a set of local orthonormal vector fields on M, related to a given coordinate system).

A first obstacle to constructing G-covariant spinor field equations is that there are no finite dimensional spinor representations of G; this is due to the fact that the three fundamental representations of the simple subalgebra $\mathfrak{sl}(4,\mathbb{R}) \subset \mathfrak{gl}(4,\mathbb{R})$ consists of the vector, covector and the natural representation in the space $\mathbb{R}^4 \wedge \mathbb{R}^4$

and these are all integrable to true representations of the group $\mathrm{SL}(4,\mathbb{R})$. On the other hand there are lot of infinite dimensional spinor representations of G. In fact, "half" of the principal series belongs to this class.

If we wish our G-covariant field equation to be in some sense an extension of the usual Dirac equation, we run into a second difficulty: no irreducible continuous representation of G contains a Dirac spinor representation of the subgroup $\mathrm{SL}(2,\mathbb{C})$ (or any other finite dimensional spinor representation). We shall not prove this here; it is really a simple argument based on properties of the spectrum of a non-compact generator of $\mathrm{SL}(2,\mathbb{C})$. There are at least two ways around this obstacle. The first alternative is to replace the Lorentz metric by an euclidean metric and to consider $\mathrm{SU}(2)\times\mathrm{SU}(2)$ spinors instead of $\mathrm{SL}(2,\mathbb{C})$ spinors [$\mathrm{SU}(2)\times\mathrm{SU}(2)$ is the universal covering group of the rotation group $\mathrm{SO}(4)$]. We shall follow this route in Sects. 2 and 3. The second alternative is to work with non-integrable representations of the Lie algebra $\mathfrak{gl}(4,\mathbb{R})$; we shall consider this in Sect. 4.

In Sect. 2 we describe certain representations $T_\varepsilon^{(v)}$ ($\varepsilon=\pm i, v\in\mathbb{C}^4$) of G in a Hilbert space H_ε. The space H_ε is a subspace of the space $L^2(\bar{K})$ of square integrable functions on $\bar{K}=\mathrm{SU}(2)\times\mathrm{SU}(2)$, but it turns out to be more conveniently realized as a space of homogeneous functions of degree one of a Dirac spinor $z=(z_1,\ldots,z_4)$ and its complex conjugate $\bar{z}$. The vector z transforms according to the representation $D^{(1/2,0)}\oplus D^{(0,1/2)}$ of $\mathrm{SU}(2)\times\mathrm{SU}(2)$. The representation $T_\varepsilon^{(v)}$ contains the irreducible $\bar{K}$-components $D^{(1/2,0)}$ and $D^{(0,1/2)}$ with multiplicity one. In general, the multiplicity of $D^{(j_1,j_2)}$ in $T_\varepsilon^{(v)}$ is equal to $(1/2)\dim D^{(j_1,j_2)}$ if j_1+j_2 is a half-integer and is zero otherwise. The Lie algebra $\mathfrak{gl}(4,\mathbb{R})$ is realized as an algebra of first order differential operators on z and $\bar{z}$. Simple and explicit formulas are given for a set of generators of $\mathfrak{gl}(4,\mathbb{R})$.

In Sect. 3 we shall first construct a massless G-covariant field equation $\sum_{k=1}^{4} X^k \frac{\partial}{\partial x_k}\psi=0$ such that the field ψ transforms according to the representation $T_\varepsilon^{(v)}$ of G in H_ε ($v\in\mathbb{C}^4$ arbitrary). The vector operator $X=\{X_k\}$ can be defined simply by $X_k\psi=\lambda^{-1/2}R_{k4}\psi$, where the functions R_{k1} ($1\leqq k, 1\leqq 4$) are the matrix elements of $\bar{K}$ in the vector representation (written as functions of z) and $\lambda^{-1/2}$ is a normalizing factor. It turns out that in order for the corresponding massive equation $\left(\sum X^k\frac{\partial}{\partial x_k}+\Lambda\right)\psi=0$ to be G-covariant, the "mass" Λ has to be defined as an intertwining operator between the representations $T^{(v')}$ and $T^{(v)}$, where $T^{(v)}:=T_i^{(v)}\oplus T_{-i}^{(v)}$ and

$$(v_1',v_2',v_3',v_4')=(v_1,v_2,v_3,v_4+1).$$

This in turn is possible only for special values of v, which are determined. We shall show that the massive field equations can be obtained from a Lagrangean by a variational principle.

In Sect. 4 we shall discuss the reduction of the representations $T_\varepsilon^{(v)}$ with respect to the subgroup $\mathrm{SL}(2,\mathbb{C})$. In particular, in the unitary case $v-\varrho$ purely imaginary [ϱ is half the sum of positive roots in $\mathfrak{sl}(4,\mathbb{R})$] the representation $T_\varepsilon^{(v)}\downarrow\mathrm{SL}(2,\mathbb{C})$ is equivalent to twice the regular representation of $\mathrm{SL}(2,\mathbb{C})$; in the non-unitary case the equivalence is only infinitesimal. Via the "Weyl trick" one can construct from $T_\varepsilon^{(\nu)}$ new representations $V_\varepsilon^{(\nu)}$ of the Lie algebra $\mathfrak{gl}(4,\mathbb{R})$ [essentially ix_4 is replaced by time t; the generators of $\mathfrak{so}(3,1)$ will be represented by complex linear combinations of the generators of $\mathfrak{so}(4)$ in the old representation]. The operators X_k will be replaced by $\tilde{X}_k$'s which transform like a vector when commuting with the generators of $\mathfrak{sl}(4,\mathbb{R})$ in the representation $V_\varepsilon^{(\nu)}$. We shall show that the submatrix of $\tilde{X}_k$ corresponding to the four dimensional Dirac subspace in H_ε is just the ordinary Dirac matrix γ_k. Our field equations are therefore in a definite sense extensions of the usual Dirac equation.

The group $\mathrm{GL}(4,\mathbb{R})$ has been suggested as a gauge group for theories of gravitation, see [2] and references therein. The use of $\mathrm{GL}(4,\mathbb{R})$ and its semidirect product with the translation group $\mathbb{R}^4$ as a unification of gravity and strong interactions has been proposed in [7]; in this connection there has been some activity on the spinor representations of G, especially on the multiplicity free representations (representations of G in which each representation of the subgroup $\bar{K}$ occurs with multiplicity 0 or 1) [7, 8].

2. On a Class of Induced Representations of $\overline{\mathrm{GL}(4,\mathbb{R})}$

Let $\mathrm{GL}_+(4,\mathbb{R})$ be the group of real 4×4 matrices with positive determinant and let G denote its simply connected double covering group. In the Iwasawa decomposition $\mathrm{GL}_+(4,\mathbb{R})=KAN$ the group N consists of all upper triangular matrices with diagonal elements $=1$, A is the group of all positive diagonal matrices in $\mathrm{GL}_+(4,\mathbb{R})$ and $K=\mathrm{SO}(4)$. Similarly $G=\bar{K}AN$, where $\bar{K}\cong\mathrm{SU}(2)\times\mathrm{SU}(2)$ is the double covering of K. Let $\theta: G\to\mathrm{GL}_+(4,\mathbb{R})$ be the covering homomorphism. We shall identify $\mathbb{R}^4$ with the space of quaternions through $x\to\sum_{k=1}^{4} x_k q_k$, where the unit quaternions q_k are defined by

$$q_1=\begin{bmatrix}0 & i\\ i & 0\end{bmatrix},\quad q_2=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},\quad q_3=\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix},\quad q_4=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

in the space of complex 2×2-matrices. By this identification the group $\bar{K}$ acts in $\mathbb{R}^4$ as

$$\theta(g)x=axb^*\,;\qquad g=(a,b)\in\mathrm{SU}(2)\times\mathrm{SU}(2)\,. \tag{2.1}$$

Thus the kernel of the homomorphism θ consists of $(1,1)$ and $(-1,-1)$. Since $G/AN\cong\bar{K}\cong S^3\times S^3$ (as C^∞-manifolds; S^3 is the real 3-dimensional sphere) G acts as a Lie transformation group on the manifold $S^3\times S^3$. The differential of this action (from the left) defines a set of first order differential operators on $\bar{K}$ which gives a realization of the Lie algebra $\mathfrak{g}=\mathfrak{gl}(4,\mathbb{R})$. Explicit but complicated formulas for the differential operators have been given in [3] in terms of the spherical coordinates on S^3. Here we shall use a different set of coordinates which gives simple formulas and is more convenient when discussing the reducibility and equivalence questions of representations of G in the function spaces on $\bar{K}$.

The elements of $\mathrm{SU}(2)\cong S^3$ can be parametrized by unit vectors $z=(z_1,z_2)\in\mathbb{C}^2$, $|z_1|^2+|z_2|=1$. To the vector z one can associate the element

$$\begin{bmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix}\in\mathrm{SU}(2).$$

We shall extend in an obvious way the action of G on $S^3\times S^3\subset\mathbb{C}^2\times\mathbb{C}^2\cong\mathbb{C}^4$ to the whole space $\mathbb{C}^4$ by demanding the action to commute with the dilatations $(z,z')\to(\lambda z,\lambda',z')$, where $z,z'\in\mathbb{C}^2$ and λ,λ' are real positive numbers. In this way the Lie algebra $\mathfrak{g}$ will be realized as an algebra of first order differential operators in four complex variables z_j $(1\leqq j\leqq 4)$ and their complex conjugates $\bar{z}_k$. For the generators of the subalgebra $\mathfrak{k}$ (the Lie algebra of K and $\bar{K}$), one gets easily the following standard expressions $(j=1,2,3)$:

$$\begin{aligned} L_j&=[z_1z_2]q_j\begin{bmatrix}\dfrac{\partial}{\partial z_1}\\ \dfrac{\partial}{\partial z_2}\end{bmatrix}+[\bar{z}_1\bar{z}_2]\bar{q}_j\begin{bmatrix}\dfrac{\partial}{\partial \bar{z}_1}\\ \dfrac{\partial}{\partial \bar{z}_2}\end{bmatrix},\\ M_j&=[z_3z_4]q_j\begin{bmatrix}\dfrac{\partial}{\partial z_3}\\ \dfrac{\partial}{\partial z_4}\end{bmatrix}+[\bar{z}_3\bar{z}_4]\bar{q}_j\begin{bmatrix}\dfrac{\partial}{\partial \bar{z}_3}\\ \dfrac{\partial}{\partial \bar{z}_4}\end{bmatrix}.\end{aligned}\tag{2.2}$$

The commutators $[M_j,L_k]$ vanish and

$$[L_1,L_2]=2L_3,\qquad [M_1,M_2]=2M_3,\tag{2.3}$$

and the other non-vanishing commutators are obtained by a cyclic permutation of the indices (123).

We shall need also the corresponding differential operators L_j^r and M_j^r defined by the right action of $\bar{K}$ on itself; they turn out to be

$$\begin{aligned} L_j^r&=[\bar{z}_2z_1]\bar{q}_j\begin{bmatrix}\dfrac{\partial}{\partial \bar{z}_2}\\ \dfrac{\partial}{\partial z_1}\end{bmatrix}+[z_2\bar{z}_1]q_j\begin{bmatrix}\dfrac{\partial}{\partial z_2}\\ \dfrac{\partial}{\partial \bar{z}_1}\end{bmatrix},\\ M_j^r&=[\bar{z}_4z_3]\bar{q}_j\begin{bmatrix}\dfrac{\partial}{\partial \bar{z}_4}\\ \dfrac{\partial}{\partial z_3}\end{bmatrix}+[z_4\bar{z}_3]q_j\begin{bmatrix}\dfrac{\partial}{\partial z_4}\\ \dfrac{\partial}{\partial \bar{z}_3}\end{bmatrix}.\end{aligned}\tag{2.4}$$

These operators commute with L_k and M_k. We denote by e_{ij} the 4×4-matrix with 1 in the (i,j) place and all other matrix elements zero. The subalgebra

$$\mathfrak{k}\cong\mathfrak{so}(4)\subset\mathfrak{gl}(4,\mathbb{R})$$

is spanned by the elements $l_{ij}=e_{ij}-e_{ji}=-l_{ji}$ and the ad$\mathfrak{k}$ invariant complement of $\mathfrak{k}$ is spanned by $a_{ij}=e_{ij}+e_{ji}=a_{ji}$. In the left realization of $\bar{K}$ we have the correspondence

$$l_{ij} \to L_{ij} := (1/2)(L_k + M_k),\ (ijk) \text{ a cyclic permutation of } (123),$$
$$l_{i4} \to L_{i4} := (1/2)(L_i - M_i), \quad i = 1, 2, 3.$$

The operators L^r_{ij} are defined similarly. The commutation relations are

$$[L_{ij}, L_{kl}] = \delta_{jk} L_{il} - \delta_{ik} L_{jl} + \delta_{il} L_{jk} - \delta_{jl} L_{ik}, \tag{2.5}$$

and similarly for L^r_{ij}'s.

By a tedious but straightforward computation from the defining relation

$$(A_{ij}\psi)(z) := \frac{d}{dt} \psi(e^{-ta_{ij}} \cdot z)|_{t=0},$$

where ψ is a differentiable function of the variable $z \in \mathbb{C}^4$ and the dot means the action of G extended from $\bar{K} \subset \mathbb{C}^4$, one arrives at

$$A_{ij} = \frac{1}{\lambda} \sum_{k<l} f^{kl}_{ij} L^r_{kl}, \tag{2.6}$$

where

$$\begin{aligned} \lambda &= (|z_1|^2 + |z_2|^2)(|z_3|^2 + |z_4|^2), \\ f^{kl}_{ij} &= R_{ik} R_{jl} + R_{il} R_{jk}, \end{aligned} \tag{2.7}$$

and

$$R_{ij} = R_{ij}(z) := (1/2) \operatorname{tr} q^*_i \begin{bmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix} q_j \begin{bmatrix} \bar{z}_3 & \bar{z}_4 \\ -z_4 & z_3 \end{bmatrix}. \tag{2.8}$$

Thus the functions R_{ij} divided by the normalization coefficient $\sqrt{\lambda}$ are just the matrix elements of $SU(2) \times SU(2)$ in its vector representation. Because of the factor $1/\lambda$ in (2.6), the coefficient of each L^r_{kl} is essentially a function on $\bar{K} \subset \mathbb{C}^4$ only [they are invariant with respect to the dilatations $(z_1, z_2, z_3, z_4) \to (\lambda z_1, \lambda z_2, \lambda' z_3, \lambda' z_4)$]. Instead of reproducing the computations leading to (2.6) we shall give a simple proof that the A_{ij}'s satisfy correct commutation relations. This means that the following relations should hold also for the A_{ij}'s and L_{ij}'s:

$$[l_{ij}, a_{kl}] = \delta_{jk} a_{il} - \delta_{ik} a_{jl} + \delta_{jl} a_{ki} - \delta_{il} a_{kj}, \tag{2.9}$$

$$[a_{ij}, a_{kl}] = \delta_{jk} l_{il} + \delta_{ik} l_{jl} + \delta_{il} l_{jk} + \delta_{jl} l_{ik}. \tag{2.10}$$

The case of (2.9) is easy: Since λ is $\bar{K}$-invariant we have $L_{ij} \cdot \lambda = 0$. Since for a fixed j the functions R_{ij} transform like a vector with respect to the left action of $\bar{K}$, the functions f^{kl}_{ij} (for fixed k and l) behave like a symmetric second rank tensor upon commutation with the L's; the rest follows from the fact that $[L, L^r] = 0$.

We shall now attack the case (2.10). Using the fact that the functions f^{kl}_{ij}, now for fixed i and j, transform like a symmetric tensor with respect to the right action of $\bar{K}$ and using the relations (2.5) for the L^r_{kl}'s, one gets first

$$[A_{ij}, A_{kl}] = \lambda^{-2} \sum_{\substack{\alpha, \beta, \gamma = 1 \\ \alpha < \beta}}^{4} (f^{\alpha\gamma}_{ij} f^{\gamma\beta}_{kl} - f^{\alpha\gamma}_{kl} f^{\gamma\beta}_{ij}) L^r_{\alpha\beta}. \tag{2.11}$$

From the definition (2.8) follows that

$$\sum_{\alpha} R_{i\alpha} R_{j\alpha} = \lambda \cdot \delta_{ij}. \tag{2.12}$$

Inserting (2.7) in (2.11) and using (2.12) we get

$$\begin{aligned}[A_{ij}, A_{kl}] = \frac{1}{\lambda} \sum_{\alpha<\beta} [&\delta_{jk}(R_{i\alpha}R_{l\beta} - R_{i\beta}R_{l\alpha}) \\ &+ \delta_{il}(R_{j\alpha}R_{k\beta} - R_{j\beta}R_{k\alpha}) + \delta_{ik}(R_{j\alpha}R_{l\beta} - R_{j\beta}R_{l\alpha}) \\ &+ \delta_{jl}(R_{i\alpha}R_{k\beta} - R_{i\beta}R_{k\alpha})] L^r_{\alpha\beta}.\end{aligned} \tag{2.13}$$

Denoting by $L(X)$ and $L^r(X)$ the left and right realizations of an arbitrary element $X \in \mathfrak{k}$ one has

$$L(X) = L^r(R^{-1}(z) X R(z)), \tag{2.14}$$

and in particular

$$L_{ij} = L(l_{ij}) = \frac{1}{\lambda} \sum_{\alpha<\beta} (R_{i\alpha} R_{j\beta} - R_{i\beta} R_{j\alpha}) L^r_{\alpha\beta}. \tag{2.15}$$

Combining (2.15) with (2.13) one gets the desired commutation relations (2.10) for the operators A_{ij}.

Next we shall construct certain reducible representations of G closely related to the principal series. Let V be the linear space consisting of all linear combinations of functions of the type $\lambda^{-(1/4)(n-1)} p(z, \bar{z})$, where p is a homogeneous polynomial of degree n, $n = 0, 1, 2, \ldots$. Elements of V are therefore homogeneous of degree one. Clearly V is invariant under the action of the differential operators L_{ij} and A_{ij}. Let S be the linear operator in V defined by $(Sh)(z, \bar{z}) := h(iz, -i\bar{z})$ and let $V_\varepsilon \subset V$ be the eigenspace of S corresponding to the eigenvalue ε ($\varepsilon = \pm i, \pm 1$). It is easily seen from the definitions that the operators L_{ij} and

$$A^{(v)}_{ij} := A_{ij} + \frac{1}{\lambda} \sum_{k=1}^{4} f^{kk}_{ij} v_k \tag{2.16}$$

commute with S for any $v = (v_1, v_2, v_3, v_4) \in \mathbb{C}^4$. By (2.16) we have in fact defined an induced representation of G in the space V_ε. The inducing subgroup is PAN, where $P \subset \mathrm{SU}(2) \times \mathrm{SU}(2)$ consists of the powers of $t = (q_3, q_3)$ (thus P contains exactly four elements). The subgroups P and A of G commute and one can define a 1-dimensional representation $U^{(v)}_\varepsilon$ of PAN by setting $U^{(v)}_\varepsilon(g) = 1$ for $g \in N$,

$$U^{(v)}_\varepsilon(t^n) = (-\varepsilon)^n, \quad \text{and} \quad U^{(v)}_\varepsilon(g) = g_1^{v_1} g_2^{v_2} g_3^{v_3} g_4^{v_4}$$

for $g = \mathrm{diag}(g_1, \ldots, g_4) \in A$. Let $T^{(v)}_\varepsilon$ denote the representation of G induced by $U^{(v)}_\varepsilon$. Let dk be the normalized Haar measure on $\bar{K}$. Then the carrier space $H^{(v)}_\varepsilon$ of the representation $T^{(v)}_\varepsilon$ is the space of square integrable (with respect to dk) complex-valued functions f on $\bar{K}$ such that

$$f(kx^{-1}) = U^{(v)}_\varepsilon(x) f(k) \quad \forall\, x \in P. \tag{2.17}$$

Since $\bar{K}$ is compact the space of functions V_ε (restricting the domain of functions from $\mathbb{C}^4$ to $S^3 \times S^3$) is clearly dense in $H_\varepsilon^{(v)}$: it is enough to note that $f(kt)=(Sf)(t)$. Computing the differential $dT_\varepsilon^{(v)}$ at a_{ij} gives just the expression (2.16). The operator $dT_\varepsilon^{(v)}(l_{ij})$ is of course equal to L_{ij}. The representations $T_\varepsilon^{(v)}$ are not quite principal series representations since the latter are induced by the subgroup MAN, where M is the centralizer of A in $\bar{K}$, [4]. The group P is a proper subgroup of M; the latter consists of 16 elements $(\pm q_j, \pm q_j)$, all signs, $j=1,2,3,4$.

Let us next consider the reduction of the representations $T_{\pm i}^{(v)}$ with respect to the maximal compact subgroup $\bar{K}$. The irreducible representations of $\bar{K}=\mathrm{SU}(2)\times\mathrm{SU}(2)$ are characterized by a pair of numbers (l_1, l_2) (highest weight) such that the dimension of the corresponding (equivalence class of) representation(s) $D^{(l_1,l_2)}$ is $(2l_1+1)(2l_2+1)$, where l_1, l_2 can obtain the values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. According to [5, Theorem 3.5], the multiplicity of $D^{(l_1,l_2)}$ in $T_\varepsilon^{(v)}$ is equal to the multiplicity of the restriction $U_\varepsilon^{(v)}|_{\bar{K}\cap PAN} = U_\varepsilon^{(v)}|_P$ in $D^{(l_1,l_2)}$; this does not depend on $v\in\mathbb{C}^4$ and we denote the multiplicity by $\chi(l_1,l_2)$. In a $(2l+1)$-dimensional irreducible representation of SU(2) one can define a basis $v(m)$, $m=-l, -l+1, \ldots, +l$, such that the action of the group element $\mathrm{diag}(e^{i\gamma}, e^{-i\gamma})$ is given by a multiplication of $v(m)$ by $e^{2im\gamma}$; thus in the tensor product basis $v(m_1)\otimes v(m_2)$ for $\mathrm{SU}(2)\times\mathrm{SU}(2)$ the element (q_3, q_3) is represented by the multiplication operator $e^{i\pi(m_1+m_2)}$. The number of pairs (m_1, m_2) for which $e^{i\pi(m_1+m_2)}=-\varepsilon$ is equal to $\frac{1}{2}(2l_1+1)(2l_2+1)$ (when $\varepsilon=\pm i$) if l_1+l_2 is a half-integer and it is zero otherwise; it follows that $\chi(l_1,l_2)=\frac{1}{2}\dim D^{(l_1,l_2)}$ when $l_1+l_2\in\mathbb{N}+\frac{1}{2}$ and $\chi(l_1,l_2)=0$ otherwise. In particular, the sum of two lowest dimensional representations $D^{(l_1,l_2)}$ in $T_{\pm i}^{(v)}$ is the "Dirac representation" $D^{(1/2,0)}\oplus D^{(0,1/2)}$, with multiplicity one. The carrier space of this subrepresentation is spanned by the monomials $z_1, \ldots, z_4$ in the case $\varepsilon=+i$ and by $\bar{z}_1, \ldots, \bar{z}_4$ when $\varepsilon=-i$. Because of the restriction $l_1+l_2\in\mathbb{N}+\frac{1}{2}$, $T_{\pm i}^{(v)}$ contains no proper 1-valued representations of SO(4). Similarly $T_{\pm 1}^{(v)}$ contains only proper representations of SO(4).

3. $\overline{\mathrm{GL}(4,\mathbb{R})}$-Covariant Field Equations

Let T and T' be two continuous representations of G in Hilbert spaces H and H', respectively. A set $X=\{X_k\}_{k=1}^4$ of four linear operators $X_k:=H\to H'$ is called a vector operator if

$$T'(g)X_k T(g^{-1}) = \sum_{j=1}^{4} \theta(g)_{jk} X_j \tag{3.1}$$

for all $g\in G$ and $k=1,2,3,4$ [remember that $\theta(g)$ is a real 4×4-matrix]. Let $C^\infty(\mathbb{R}^4, H)$ be the linear space of all C^∞-functions $\psi:\mathbb{R}^4\to H$. We can define a representation D of G in $C^\infty(\mathbb{R}^4, H)$ [and similarly D' in $C^\infty(\mathbb{R}^4, H')$] by

$$(D(g)\psi)(x) := T(g)\psi(\theta(g)^{-1}x). \tag{3.2}$$

We shall consider the differential operator

$$\mathscr{D} := \sum_{k=1}^{4} X^k \frac{\partial}{\partial x_k} : C^\infty(\mathbb{R}^4, H) \to C^\infty(\mathbb{R}^4, H'), \tag{3.3}$$

where $X^k = \sum_l h^{kl} X_l$ and $h = (h^{kl})$ is a space-time metric. It follows easily from (3.1) that

$$D'(g)\mathscr{D}D(g^{-1}) = \tilde{\mathscr{D}} \quad \forall g \in G, \tag{3.4}$$

where $\tilde{\mathscr{D}}$ is again given by (3.3) but using the transformed metric $\tilde{h} = \theta(g)h\theta(g)^t$. If ψ is a solution of $\mathscr{D}\psi = 0$ then $\tilde{\mathscr{D}}D(g)\psi = D'(g)\mathscr{D}\psi = 0$ so that $D(g)\psi$ is a solution of the same differential equation with respect to the metric $\tilde{h}$; therefore we shall call a differential operator which satisfies a condition of the type (3.4) a G-covariant differential operator. The field equation $\mathscr{D}\psi = 0$ could be thought to describe a massless particle. The corresponding massive field equation would be

$$(\mathscr{D} + \Lambda)\psi = 0, \tag{3.5}$$

where $\Lambda : H \to H'$ is a linear operator. If the time coordinate is x_4 and X_4 is invertible then the spectrum of $X_4^{-1}\Lambda$ would give the different masses associated to the field ψ. Assuming that

$$T'(g)\Lambda T(g^{-1}) = \Lambda \quad \forall g \in G, \tag{3.6}$$

the field $D(g)\psi$ is a solution of (3.5) whenever ψ is a solution for all $g \in G$. An operator Λ satisfying (3.6) is usually called an intertwining operator for the pair (T, T'). In particular, if Λ is invertible, then from (3.6) follows $T'(g) = \Lambda T(g)\Lambda^{-1}$ so that T and T' are equivalent representations.

For any $v \in \mathbb{C}^4$ and $v' \in \mathbb{C}^4$ such that

$$v'_k = v_k, \quad k = 1, 2, 3, \quad v'_4 = v_4 + 1 \tag{3.7}$$

we can associate a vector operator X to the pair of representations $(T, T') = (T_\varepsilon^{(v)}, T_\varepsilon^{(v')})$. The carrier space of both of the representations is the same $H_\varepsilon = H_\varepsilon^{(v)} = H_\varepsilon^{(v')}$, defined in Sect. 2. We define

$$X_k f := (1/\sqrt{\lambda}) R_{k4} f, \quad f \in H_\varepsilon, \quad 1 \leqq k \leqq 4. \tag{3.8}$$

Since $\bar{K}$ is compact and each $\lambda^{-1/2} R_{k4}$ is a continuous function with maximum absolute value 1 on $\bar{K}$, the operators are bounded and $\|X_k\| = 1$ (with respect to the L^2-norm in H_ε). In order for (3.8) to make sense we have to show that the subspace $H_\varepsilon \subset L^2(\bar{K})$ is an invariant subspace for X. By (2.17) this is so if

$$(\lambda^{-1/2} R_{k4})(ax) = (\lambda^{-1/2} R_{k4})(a)$$

for all $a \in \bar{K}$ and $x \in P$. Since P is generated by $t = (q_3, q_3)$ it is enough to consider the case $x = t$. Looking at the definitions (2.7) and (2.8) of the functions λ and R_{k4} we notice that both functions are indeed invariant with respect to the right action by t.

Since we are dealing with differentiable representations of the Lie group G it is sufficient to prove the infinitesimal version of (3.1). Setting $g = g(s)$ in (3.1), where $g(s)$ is one of the 1-parameter subgroups of G generated by the vectors l_{ij} and a_{ij} in g, and taking the derivative with respect to s at $s = 0$ we see that (3.1) is equivalent with

$$[L_{ij}, X_k] = \delta_{jk} X_i - \delta_{ik} X_j, \tag{3.9}$$

$$A_{ij}^{(v')} X_k - X_k A_{ij}^{(v)} = \delta_{jk} X_i + \delta_{ik} X_j. \tag{3.10}$$

Now for the operator X_k defined by (3.8), Eq. (3.9) follows immediately from the fact that the functions $\lambda^{-1/2}R_{k4}$ transform like a vector under the left action of $\bar{K}$ (they are matrix elements of the vector representation). We shall verify (3.10) by a direct computation.

$$\begin{aligned}A_{ij}^{(\nu')}X_k - X_k A_{ij}^{(\nu)} &= \frac{1}{\lambda}\sum_{\alpha<\beta}[f_{ij}^{\alpha\beta}L_{\alpha\beta}^r, X_k] + \frac{1}{\lambda}f_{ij}^{44}X_k \\ &= \lambda^{-3/2}f_{ij}^{44}R_{k4} + \lambda^{-3/2}\sum_{\alpha<\beta} f_{ij}^{\alpha\beta}L_{\alpha\beta}^r R_{k4} \\ &= \lambda^{-3/2}f_{ij}^{44}R_{k4} + \lambda^{-3/2}\sum_{\alpha<4} f_{ij}^{\alpha4}R_{k\alpha} \\ &= \lambda^{-3/2}\sum_{\alpha=1}^{4} f_{ij}^{\alpha4}R_{k\alpha} = \lambda^{-3/2}\sum_{\alpha=1}^{4}(R_{i\alpha}R_{j4}+R_{i4}R_{j\alpha})R_{k\alpha} \\ &= \delta_{ik}\lambda^{-1/2}R_{j4} + \delta_{jk}\lambda^{-1/2}R_{i4} \\ &= \delta_{ik}X_j + \delta_{jk}X_i .\end{aligned}$$

On the third step we have used the fact that the R_{ij}'s transform like a vector under the right action of $\bar{K}$ (for any fixed i). The orthogonality properties (2.12) have been used on the sixth step.

In order to be able to construct massive G-covariant field equations we have to double the space. Here we are more interested on the spinorial case $\varepsilon = \pm i$ and we define

$$T^{(\nu)} := T_{+i}^{(\nu)} \oplus T_{-i}^{(\nu)}, \qquad H := H_{+i} \oplus H_{-i} . \tag{3.11}$$

Slightly modifying Lemma 8.10.8 and using Theorem 8.10.16 in [4] we can conclude that the restrictions of $T^{(\nu)}$ and $T^{(\nu')}$ to the subgroup $\overline{\mathrm{SL}(4,\mathbb{R})}$ allow an intertwining operator if

$$\nu' - \varrho \sim \sigma(\nu - \varrho) \quad \text{for some} \quad \sigma \in W, \tag{3.12}$$

and

$$(\nu-\varrho)_i > (\nu-\varrho)_j \quad \text{for} \quad 1 \leqq i < j \leqq 4 . \tag{3.13}$$

Here W is the Weyl group of $\overline{\mathrm{SL}(4,\mathbb{R})}$ associated to the Cartan subgroup $A_0 = A \cap \overline{\mathrm{SL}(4,\mathbb{R})}$, and ϱ is half the sum of the roots of A_0 in N. Explicitly, $\varrho = \frac{1}{2}(3,1,-1,-3)$ and W acts in $\mathbb{C}^4$ as a group of permutations of the coordinates. The equivalence $\mu \sim \mu'$ is defined by

$$\mu \sim \mu' \Leftrightarrow \mu'_j = \mu_j + c, \qquad 1 \leqq j \leqq 4,$$

for some $c \in \mathbb{C}$. By (2.16) $A_{ij}^{(\mu)} = A_{ij}^{(\mu')}$ if $i \neq j$ and

$$A_{ii}^{(\mu)} - A_{jj}^{(\mu)} = A_{ii}^{(\mu')} - A_{jj}^{(\mu')},$$

when $\mu \sim \mu'$ [a simple application of (2.7) and (2.12)]. Thus the restrictions of $T^{(\mu)}$ and $T^{(\mu')}$ to $\overline{\mathrm{SL}(4,\mathbb{R})}$ are equivalent when $\mu \sim \mu'$.

If now ν and ν' are related by (3.7) it is an easy exercise in linear algebra to show that (3.12) and (3.13) are satisfied if and only if

$$\nu - \varrho \sim \tfrac{1}{8}(3,1,-1,-3) = \tfrac{1}{4}\varrho .$$

In this case the vector $\nu' - \varrho$ is obtained from $\nu - \varrho$ acting by the permutation $\sigma_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ and then adding the vector $\frac{1}{4}(1,1,1,1)$.

Lemma 8.10.8 in [4] gives an explicit formula for the intertwining operator $\Lambda : H \to H$. Let M' be the normalizer of A in $\bar{K}$; then $M \subset M'$ is a normal subgroup and $W \cong M'/M$. Let m_o be a representative of the class $\sigma_o \in W$ in M' $\left(\text{for example, } m_o = \frac{1}{\sqrt{2}}(q_1 + q_3, q_2 + q_4) \in \mathrm{SU}(2) \times \mathrm{SU}(2)\right)$ and let $\tilde{f}$ denote the extension to G of a continuous function f on $\bar{K}$ given by

$$\tilde{f}(kan) := U^{(\nu)}(a) f(k)\,; \qquad k \in \bar{K}, \qquad a \in A, \qquad n \in N. \tag{3.14}$$

Let $N_o := N \cap m_o N m_o^{-1}$ and $d\dot{n}$ a N-invariant measure on N/N_o. Then

$$(\Lambda f)(k) = \int_{N/N_o} \tilde{f}(knm_o) d\dot{n}. \tag{3.15}$$

The vector Λf is really in H because the subspace $H \subset L^2(\bar{K})$ is characterized by the eigenvalue -1 of the operator S^2 and

$$(S^2 f)(k) = f(kt^2), \qquad t^2 = (-1, -1) \in \bar{K},$$

so that the operators S^2 and Λ commute (the element t^2 commutes with AN). This could be expected of course from the fact that no intertwining operator can mix the spinorial representation in H with a vectorial representation characterized by $S^2 = +1$. We can make (3.15) a little bit more explicit by first computing

$$\theta(m_o) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \in K$$

and

$$\theta(m_o) \cdot \begin{bmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & x_{23} & x_{24} \\ 0 & 0 & 1 & x_{34} \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \theta(m_0)^{-1} = \begin{bmatrix} 1 & -x_{23} & x_{24} & 0 \\ 0 & 1 & -x_{34} & 0 \\ 0 & 0 & 1 & 0 \\ x_{12} & -x_{13} & x_{14} & 1 \end{bmatrix},$$

so that $m_o N m_o^{-1} \cap N$ consists of those elements $n = (x_{ij})$ for which $x_{14} = x_{13} = x_{12} = 0$. Setting

$$\bar{n}(x) = \begin{bmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we can write (3.15) as

$$(\Lambda f)(k)=\int_{\mathbb{R}^3}\tilde{f}(k\bar{n}(x)m_0)dx_{12}dx_{12}dx_{13}dx_{14}. \tag{3.16}$$

Of course, any scalar multiple of Λ is also an intertwining operator for $\overline{SL(4,\mathbb{R})}$.

Obviously the formula (3.6) is not valid for an element $g \neq 1$ in the centre of G: if $g=e^{\alpha}\cdot 1$ ($\alpha\in\mathbb{R}$) then $T^{(v)}(g)=\exp\left(\alpha\sum_{j=1}^{4}v_j\right)\cdot\mathbf{1}$ and $T^{(v')}(g)\Lambda T^{(v)}(g^{-1})=e^{\alpha}\Lambda$ for any operator Λ when v' and v are related by (3.7). Thus we have

$$T^{(v')}(g^{-1})\Lambda T^{(v)}(g)=(\det g)^{-1/4}\Lambda \tag{3.17}$$

for all $g\in G$. This generalization of (3.6) is perfectly acceptable on physical grounds since Λ is assumed to be related to the mass of the system and the mass is multiplied by $e^{-\alpha}$ in a dilatation $x\to e^{\alpha}x$ in $\mathbb{R}^4$.

To complete this section we examine the Lagrangian density associated to the field equation (3.5). Let A^* denote the hermitian conjugate of an operator A in H (with respect to the L^2-scalar product). Then $g\to T^{(v)}(g)^{*-1}$ is also a continuous representation of G in H; we denote it by $W^{(v)}$. Since $T^{(v)}|_{\bar{K}}$ is unitary, we have

$$W^{(v)}(k)=T^{(v)}(k)\quad\forall k\in\bar{K}.$$

On the other hand, by a partial integration from (2.6) and (2.16)

$$A_{ij}^{(v)*}=-A_{ij}+\frac{1}{\lambda}\sum_k f_{ij}^{kk}(v-2\varrho)_k$$

for any real vector v. Thus $W^{(v)}=T^{(-v+2\varrho)}$ for any real v. Denote by σ_1 the transposition $\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}$. Then for $v=\frac{5}{4}\varrho$ and $v'=v+(0,0,0,1)$ we have $-v'+\varrho\sim\sigma_1(v-\varrho)$. The permutation σ_1 is realized through $a\to m_1 a m_1^{-1}$ $(a\in A)$, where as the element $m_1\in M'$ we can take for example $m_1=\frac{1}{\sqrt{2}}(q_2+q_4,-q_2+q_4)$. We can now define an intertwining operator $\Omega:H\to H$ such that

$$W^{(v')}(g^{-1})\Omega T^{(v)}(g)=(\det g)^{-1/4}\Omega. \tag{3.18}$$

The operator Ω is given by (3.15) with m_o replaced by m_1. The analog of (3.16) is

$$(\Omega f)(k)=\int_{\mathbb{R}^3}\tilde{f}(k\hat{n}(x)m_1)dx_{12}dx_{23}dx_{13}, \tag{3.19}$$

where

$$\hat{n}(x)=\begin{bmatrix}1&x_{12}&x_{13}&0\\0&1&x_{23}&0\\0&0&1&0\\0&0&0&1\end{bmatrix}.$$

For a differentiable function $\psi:\mathbb{R}^4\to H$ we define the density

$$\mathscr{L}_{\psi}:=\langle\Omega\psi,\sum X^k\partial_k\psi+\Lambda\psi\rangle. \tag{3.20}$$

If $\varphi = (\sum X^k \partial_k + \Lambda)\psi$ then, by the covariance of the differential operator, φ transforms according to the representation $T^{(v')}$ of G when ψ is transformed by $T^{(v)}$. Thus for any $g \in G$ [denoting $x' = \theta(g)^{-1}x$],

$$\begin{aligned}\mathscr{L} \to \mathscr{L}^g(x) &= \langle \Omega T^{(v)}(g)\psi(x'), T^{(v')}(g)\varphi(x')\rangle \\ &= \langle T^{(v')}(g)^* \Omega T^{(v)}(g)\psi(x'), \varphi(x')\rangle \\ &= \langle W^{(v')^{-1}}(g)\Omega T^{(v)}(g)\psi(x'), \varphi(x')\rangle \\ &= \langle (\det g)^{-1/4}\Omega\psi(x'), \varphi(x')\rangle \\ &= (\det g)^{-1/4}\mathscr{L}(x').\end{aligned}$$

It follows in particular that the functional

$$\mathscr{L}(\psi) = \int \mathscr{L}_\psi(x) d^4x \tag{3.21}$$

is invariant with respect to all $g \in \overline{\mathrm{SL}(4,\mathbb{R})}$. The ordinary Dirac Lagrangian is

$$\bar{\psi}(i\gamma^\nu \partial_\nu + m)\psi = \langle \gamma_o \psi, (i\gamma^\nu \partial_\nu + m)\psi\rangle,$$

where now $\langle \cdot, \cdot \rangle$ is the euclidean inner product in $\mathbb{C}^4$; thus the operator Ω in (3.20) plays the role of γ_o. Analogously to the Dirac case the field equation (3.5) is obtained by a variational principle from (3.20) and (3.21) through an independent variation of ψ and $\Omega\psi$.

4. Reduction with Respect to the Lorentz Subgroup

We shall first ask the question which irreducible representations of the subgroup $\mathrm{SL}(2,\mathbb{C}) \subset G$ [the double covering of $\mathrm{SO}_o(3,1) \subset \mathrm{GL}_+(4,\mathbb{R})$] are contained in the representations $T_\varepsilon^{(v)}$. In this section we shall keep $v \in \mathbb{C}^4$ arbitrary.

We cannot apply the Mackey subgroup theorem [5, Theorem 3.5], since it is valid only for unitarily induced representations. Instead, we shall use a more direct orbit method which gives also a concrete decomposition in the space $L^2(\bar{K})$.

The group $J = \mathrm{SO}_o(3,1)$ consists by definition of those matrices $u \in \mathrm{SL}(4,\mathbb{R})$ which preserve the quadratic form $g(x) = x_1^2 + x_2^2 + x_3^2 - x_4^2$ and $u_{44} \geqq 1$ (the direction of time x_4 is preserved). Let the group J act on $K \cong \mathrm{GL}_+(4,\mathbb{R})/AN$ from the left. There are exactly six J-orbits in K,

$$K = J\cdot 1 \cup J\cdot k_1 \cup J\cdot k_2 \cup J\cdot(-1) \cup J\cdot(-k_1) \cup J\cdot(-k_2),$$

where

$$k_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad k_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

The orbit $J\cdot 1$ consists of those matrices $u\in \mathrm{SO}(4)$ for which the subspace $V(u)\subset \mathbb{R}^4$ spanned by the first three column vectors of the matrix u is space-like [i.e. $g(x)$ is positive definite on $V(u)$] and the fourth column vector is on the side of the positive x_4-axis from the 3-plane $V(u)$. The orbit $J\cdot(-1)$ is characterized in the same way except that the positive x_4-axis is replaced by the negative x_4-axis. The cases $J\cdot(\pm k_1)$ [respectively $J\cdot(\pm k_2)$] differ from the first two in the way that the restriction of $g(x)$ to $V(u)$ has signature $++-$ (respectively it is degenerate, signature $++0$). Since $J\cap AN=\{1\}$ and $J\cap k_1ANk_1^{-1}=\{1\}$, the orbits $J\cdot(\pm 1)$ and $J\cdot(\pm k_1)$ are diffeomorphic with the group J. On the other hand $J\cap k_2ANk_2^{-1}$ consists of the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & \beta & \alpha \end{bmatrix}; \qquad \alpha=\cosh\xi\,, \qquad \beta=\sinh\xi\,,$$

for $\xi\in\mathbb{R}$. Therefore the orbits $J\cdot(\pm k_2)$ are lower dimensional and they are of measure zero in K.

Let $\hat{k}_1, \hat{k}_2\in\bar{K}$ such that $\theta(\hat{k}_j)=k_j$ $(j=1,2)$. There are again exactly six $\mathrm{SL}(2,\mathbb{C})$-orbits in $\bar{K}$, each of them containing precisely one of the elements 1, $\hat{k}_1$, $\hat{k}_2$, t, $\hat{k}_1 t$, $\hat{k}_2 t$, where $t=(q_3,q_3)\in\bar{K}$ as earlier, $\theta(t)=-1\in K$. [Note that the kernel $\ker\theta\cong\mathbb{Z}_2$ is contained in $\mathrm{SL}(2,\mathbb{C})$.] Let us denote $Q(k)=\mathrm{SL}(2,\mathbb{C})\cdot k$, $k\in\bar{K}$. The right action of t on $\bar{K}$ defines a diffeomorphism [which commutes with the action of $\mathrm{SL}(2,\mathbb{C})$] between the orbits $Q(k)$ and $Q(kt)$. On the other hand $f(kt)=\varepsilon f(k)$ for a function $f\in H_\varepsilon$, therefore f is completely determined (in the L^2-sense) giving its restriction to the two orbits $Q(1)$ and $Q(\hat{k}_1)$. The orbits $Q(\hat{k}_2)$ and $Q(\hat{k}_2 t)$ don't count since they are of measure zero. Denoting by dq the $\mathrm{SL}(2,\mathbb{C})$ invariant measure on $Q(1)$ and $Q(\hat{k}_1)$ induced by the Haar measure of $\mathrm{SL}(2,\mathbb{C})$, let $\frac{dk}{dq}$ be the Radon-Nikodym derivative of dk with respect to dq. We define a unitary isomorphism

$$\begin{gathered}\tau: H_\varepsilon \to L^2(Q(1),dq)\oplus L^2(Q(\hat{k}_1),dq)\,,\\ \tau(f):=\sqrt{2}\left(f|_{Q(1)}\cdot\left(\frac{dk}{dq}\right)^{1/2},\ f|_{Q(\hat{k}_1)}\cdot\left(\frac{dk}{dq}\right)^{1/2}\right).\end{gathered} \tag{4.1}$$

The factor $\sqrt{2}$ is included since $Q(1)\cup Q(\hat{k}_1)$ is one half of the space $\bar{K}$ (in measure).

The subalgebra $\mathfrak{so}(3,1)\cong\mathfrak{sl}(2,\mathbb{C})$ in $\mathfrak{gl}(4,\mathbb{R})$ is spanned by the vectors l_{12}, l_{23}, l_{31} and a_{14}, a_{24}, a_{34}. By an abuse of notation we denote the corresponding right invariant differential operators on $\mathrm{SL}(2,\mathbb{C})$ [or on $Q(1)$ and $Q(\hat{k}_1)$] by L_{ij}, A_{i4}. The mapping τ pulls the restriction of $T_\varepsilon^{(\nu)}$ to $\mathrm{SL}(2,\mathbb{C})$ into a reducible representation of $\mathrm{SL}(2,\mathbb{C})$ in the subspaces $L^2(Q(1),dq)$ and $L^2(Q(\hat{k}_1,dq))$ of $\tau(H_\varepsilon)$. The representation of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is given by the operators $L_{ij}(i,j\leqq 3)$ and

$$\tilde{A}_{i4}=A_{i4}+\sum_{k=1}^{4}\tilde{f}_{i4}^{kk}(\nu-\varrho)_k\,, \tag{4.2}$$

where $\tilde{f}_{i4}^{kk}$ denotes the restriction of the function f_{i4}^{kk} to $Q(1)$, respectively $Q(\hat{k}_1)$. A function f on $Q(x)$ $(x=1,\hat{k}_1)$ can be considered as a function on $\mathrm{SL}(2,\mathbb{C})$ via the diffeomorphism $g\to g\cdot x$ $(g\in \mathrm{SL}(2,\mathbb{C}))$. The weight v is shifted by $-\varrho$ because of the factor $\left(\frac{dk}{dq}\right)^{1/2}$ in (4.1). To proceed we need

Lemma. *For each integer* $1\leqq k\leqq 4$ *there exists a* C^∞*-function* h_k *on* $\mathrm{SL}(2,\mathbb{C})$ *such that* $L_{ij}h_k=0$ $(1\leqq i,j\leqq 3)$ *and* $A_{i4}h_k=\tilde{f}_{i4}^{kk}$ $(1\leqq i\leqq 3)$.

Proof. By a simple computation from (2.6) and (2.7) we get

$$A_{i4}f_{j4}^{kk}=A_{j4}f_{i4}^{kk}. \tag{4.3}$$

If we set $Z_j=A_{j4}$ $(j=1,2,3)$, $(Z_4,Z_5,Z_6)=(L_{12},L_{23},L_{31})$ and $f_j=f_{j4}^{kk}$ $(j=1,2,3)$, $f_j=0$ for $j=4,5,6$ then we are looking for a function h such that $Z_jh=f_j$ $(1\leqq j\leqq 6)$. The Frobenius integrability conditions for this first order differential system are

$$Z_if_j-Z_jf_i=\sum_{k=1}^{6}C_{ij}^kf_k, \tag{4.4}$$

where C_{ij}^k's are the structure constants,

$$[Z_i,Z_j]=\sum C_{ij}^kZ_k.$$

Using (4.3) and the commutation relations (2.10) it is easily seen that (4.4) are satisfied. Thus the existence of a local solution h to $Z_jh=f_j$ is clear. Since $L_{ij}h=0$ $(1\leqq i,j\leqq 3)$, h is really a function on $\mathrm{SL}(2,\mathbb{C})/\mathrm{SU}(2)$. The homogeneous space is diffeomorphic to $\mathbb{R}_+\times\mathbb{R}^2$ and therefore we are effectively dealing with a Frobenius problem on $\mathbb{R}_+\times\mathbb{R}^2$. The vector fields Z_1, Z_2, Z_3 are linearly independent and thus the system $Z_jh=f_j$ will be equivalent to $\frac{\partial}{\partial y_j}h^{(0)}=f_j^{(0)}$ $(j=1,2,3)$, where y_1,y_2,y_3 are the Cartesian coordinates on $\mathbb{R}_+\times\mathbb{R}^2$, $f_j^{(0)}$ $(j=1,2,3)$ are some functions on $\mathbb{R}_+\times\mathbb{R}^2$ such that $\frac{\partial}{\partial y_j}f_l^{(0)}=\frac{\partial}{\partial y_l}f_j^{(0)}$. Since $\mathbb{R}_+\times\mathbb{R}^2$ is starshaped, the global existence of the solution $h^{(0)}$ follows from the Poincaré lemma. □

Let $H_\varepsilon^c=C^\infty(\bar{K})\cap\{f\in H_\varepsilon|Q(x)\cap\operatorname{supp}f$ compact for $x=1$, $x=\hat{k}_1\}$ ($\operatorname{supp}f$ denotes the support of f). By the differentiability of the $\mathrm{SL}(2,\mathbb{C})$ action on the orbits $Q(x)$, the dense subspace $H_\varepsilon^c\subset H_\varepsilon$ is $\mathrm{SL}(2,\mathbb{C})$-invariant.

Theorem. *The representation of* $\mathrm{SL}(2,\mathbb{C})$ *on* H_ε^c *defined by* $T_\varepsilon^{(v)}$ *is infinitesimally equivalent to a direct sum of two regular representations on the space* $C_0^\infty(\mathrm{SL}(2,\mathbb{C}))$ *(*C^∞*-functions of compact support) by an unbounded operator* $F_\varepsilon^{(v)}$*; if* $v-\varrho$ *is purely imaginary the operator* $F_\varepsilon^{(v)}$ *is unitary.*

Proof. Let $h_1,\dots,h_4$ be as in the lemma and define

$$h^{(v)}:=\exp\left(-\sum_{k=1}^{4}(v-\varrho)_kh_k\right).$$

Now $h^{(v)-1}\tilde{A}_{i4}h^{(v)}=A_{i4}$ and $h^{(v)-1}L_{ij}h^{(v)}=L_{ij}$ for $1\leqq i,j\leqq 3$. We can set

$$F_\varepsilon^{(v)}(f):=h^{(v)}\cdot\tau(f),\quad f\in H_\varepsilon^c.$$

For each $f\in H_\varepsilon^c$ we have

$$F_\varepsilon^{(v)-1}A_{i4}F_\varepsilon^{(v)}(f)=A_{i4}^{(v)}f,$$
$$F_\varepsilon^{(v)-1}L_{ij}F_\varepsilon^{(v)}(f)=L_{ij}f,\quad 1\leqq i,j\leqq 3.$$

Clearly the multiplication by $h^{(v)}$, and thus also $F_\varepsilon^{(v)}$, is a unitary operator when $v-\varrho$ is purely imaginary. □

Next we wish to relate the field equations described by the representations $T_\varepsilon^{(v)}$ to the Dirac equation. As a first step we note that using the Weyl trick one can obtain a new (nonintegrable) representation $V_\varepsilon^{(v)}$ of the Lie algebra $\mathfrak{gl}(4,\mathbb{R})$ from $dT_\varepsilon^{(v)}$. The representation $V_\varepsilon^{(v)}$ has the property that it is composed from finite dimensional spinor representations of $\mathrm{SL}(2,\mathbb{C})$. Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of the real Lie algebra $\mathfrak{g}=\mathfrak{gl}(4,\mathbb{R})$. Define a $\mathbb{R}$-linear injection

$$\begin{aligned}&\eta:\mathfrak{g}\to\mathfrak{g}_{\mathbb{C}},\\ &\eta(l_{jk})=l_{jk},\qquad \eta(a_{jk})=a_{jk}\quad\text{for}\quad 1\leqq j,\quad k\leqq 3,\\ &\eta(l_{j4})=il_{j4},\qquad \eta(a_{j4})=ia_{j4}\quad\text{for}\quad 1\leqq j\leqq 3,\\ &\eta(a_{44})=a_{44}.\end{aligned}\tag{4.5}$$

If $g=\mathrm{diag}(1,1,1,-1)$ then the linear mapping $\tilde{\eta}:\mathfrak{g}\to\mathfrak{g}_{\mathbb{C}}$, $\tilde{\eta}(x):=\eta(gx)$, is a Lie algebra homomorphism and the image of $\mathfrak{so}(3,1)$ is contained in the complexification of $\mathfrak{so}(4)$. We define

$$V_\varepsilon^{(v)}(x):=dT_\varepsilon^{(v)}(\tilde{\eta}(x)),\tag{4.6}$$

where $dT_\varepsilon^{(v)}$ is extended from $\mathfrak{g}$ to $\mathfrak{g}_{\mathbb{C}}$ by $\mathbb{C}$-linearity. Then $V_\varepsilon^{(v)}$ is a representation of $\mathfrak{g}$ in $H_\varepsilon^{(v)}$, and there is a one-to-one correspondence between the finite dimensional irreducible subrepresentations of $\mathfrak{so}(4)$ contained in $T_\varepsilon^{(v)}$ and those of $\mathfrak{so}(3,1)$ contained in $V_\varepsilon^{(v)}$. In particular, the (reducible) Dirac representation of $\mathfrak{so}(3,1)$ occurs with multiplicity one in both of the cases $T_{\pm i}^{(v)}$.

Let us define the operators $\tilde{X}_k:H_\varepsilon\to H_\varepsilon$,

$$\begin{aligned}\tilde{X}_kf&=\frac{i}{\sqrt{\lambda}}R_{k4}f,\quad 1\leqq k\leqq 3\\ \tilde{X}_4f&=\frac{1}{\sqrt{\lambda}}R_{44}f\quad(f\in H_\varepsilon).\end{aligned}\tag{4.7}$$

Then $\tilde{X}=\{\tilde{X}_k\}_{k=1}^4$ is a vector operator with respect to the representations $V_\varepsilon^{(v)}$ and $V_\varepsilon^{(v')}$ of $\mathfrak{g}$ [again, v' is given by (3.7)]. Let π be the projection of $H_{\pm i}$ onto the four dimensional subspace $\mathscr{H}_{\pm i}$ which is the carrier space of the Dirac representation $D^{(1/2,0)}\oplus D^{(0,1/2)}$. We define the linear operators ($\varepsilon=\pm i$)

$$\gamma_k:\mathscr{H}_\varepsilon\to\mathscr{H}_\varepsilon,\quad \gamma_kf:=4\pi\tilde{X}_kf.\tag{4.8}$$

In the realization of H_ε by functions on $\mathbb{C}^4$, the space $\mathscr{H}_{\pm i}$ is spanned by the monomials $z_1,\dots,z_4$ and $\mathscr{H}_{-i}$ is spanned by $\bar{z}_1,\dots,\bar{z}_4$. We claim that with respect to these bases the operators γ_k are given by the matrices

$$\gamma_k = \left[\begin{array}{c|c} 0 & iq_k \\ \hline -iq_k & 0 \end{array}\right] (1 \leqq k \leqq 3), \qquad \gamma_4 = \left[\begin{array}{c|c} 0 & q_4 \\ \hline q_4 & 0 \end{array}\right] \tag{4.9}$$

in the case $\varepsilon = +i$, and by their complex conjugate matrices in the case $\varepsilon = -i$. This means that the projections of the operators $\tilde{X}_k$ onto the Dirac subspace are just the Dirac matrices. From (4.5), (4.7), and (4.8) follows that $\{\gamma_k\}$ transforms like a vector operator with respect to $\mathfrak{so}(3,1)$, and consequently it is sufficient to verify (4.9) only for one operator, say γ_4. For that purpose consider the functions $R_{44}z_j$ on the sphere $|z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2 = 1$. From (2.8) we have

$$\begin{aligned} R_{44}z_1 &= \tfrac{1}{2}(z_1\bar{z}_3 + z_3\bar{z}_1 + z_2\bar{z}_4 + z_4\bar{z}_2)z_1 \\ &= \tfrac{1}{4}z_3(|z_1|^2 + |z_2|^2) + \tfrac{1}{4}z_3(|z_1|^2 - |z_2|^2) \\ &\quad + \tfrac{1}{2}(z_1^2\bar{z}_3 + z_1 z_2 \bar{z}_4 + z_1 \bar{z}_2 z_4) . \end{aligned} \tag{4.10}$$

Using the explicit expressions (2.2) for the generators it is easily seen that the last two terms in (4.10) transform according to the representation $D^{(1,1/2)}$ and thus

$$\gamma_4 z_1 = 4\pi R_{44} z_1 = z_3(|z_1|^2 + |z_2|^2) = z_3 .$$

By a similar computation we get $\gamma_4 z_2 = z_4$, $\gamma_4 z_3 = z_1$, $\gamma_4 z_4 = z_2$ and so we are done with the case $\varepsilon = +i$. The case $\varepsilon = -i$ is treated analogously.

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