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On the Structure of Space–Time Caustics†

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Abstract. Caustics formed by timelike and null geodesics in a space–time M are investigated. Care is taken to distinguish the conjugate points in the tangent space (T-conjugate points) from conjugate points in the manifold (M-conjugate points). It is shown that most nonspacelike conjugate points are regular, i.e. with all neighbouring conjugate points having the same degree of degeneracy. The regular timelike T-conjugate locus is shown to be a smooth 3-dimensional submanifold of the tangent space. Analogously, the regular null T-conjugate locus is shown to be a smooth 2-dimensional submanifold of the light cone in the tangent space. The smoothness properties of the null caustic are used to show that if an observer sees focusing in all directions, then there will necessarily be a cusp in the caustic. If, in addition, all the null conjugate points have maximal degree of degeneracy (as in the closed Friedmann–Robertson–Walker universes), then the space-time is closed.

1. Introduction

Gravitational focusing plays an important role in general relativity both observationally through the discovery of the gravitational lens effect (Walsh et al. 1979 [1]) and theoretically in the proofs of the singularity theorems (Hawking and Ellis [2]). A non-uniform gravitational field gives rise to tidal forces which tend to have a converging effect on a bundle of light rays. It is the attractive nature of the gravitational force which causes a bundle of rays to converge and focus rather than diverge. An observational effect of focusing is that objects are magnified as in a lens. In general, the magnification depends on the transverse direction. This means that images will be distorted; for example, a circular galaxy may appear elliptical.

The points where geodesics refocus are called conjugate points [2, Chapt. 4]. The geometric locus of such points, the conjugate locus, is said to be a caustic. The proofs of the singularity theorems depend on the existence of a pair of conjugate points

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along some geodesic. However, there is evidence that information on the global structure of space-times can also be gained from the global structure of caustics. In Rosquist [3], [4], properties of the conjugate locus were used to obtain restrictions on the topology of the universe under certain conditions. In some cosmological models (the rotating space–time homogeneous Class III models [6] with the well known Gödel model [2] as a special case), the conjugate locus of null geodesics is related to the causal structure through the existence of a closed null curve in the caustic In this paper we investigate general properties of caustics formed by nonspacelike geodesics in a space–time.

We deal with pseudo-Riemannian manifolds which are Hausdorff, paracompact, C^∞ and has a C^∞ non-degenerate metric. We adopt the usage that a *Lorentzian manifold* or *space-time* has metric signature $(-\,+\ldots+)$ while a *Riemannian manifold* is characterized by a positive definite metric. The conjugate locus of a Riemannian manifold has been extensively investigated in the past. A high point was reached in 1965 when Warner [5] showed that the regular conjugate locus (characterized by constant degree of degeneracy) in the tangent space is dense in the conjugate locus and that it is a smooth submanifold with the induced topology in the tangent space. Warner also gave canonical coordinates for the exponential map near regular conjugate points. Our goal is to obtain Lorentzian analogues of Warner's theorems (excluding canonical coordinates). As a result, two new global theorems for space-times will emerge.

The outline of the paper is as follows. The fundamental properties of conjugate points are reviewed in Sect. 2. The timelike conjugate locus in the tangent space is discussed in Sect. 3 and the null conjugate locus in the tangent space is examined in Sect. 4. Finally in Sect. 5, we treat the manifold conjugate locus, the caustic.

All manifolds will be n-dimensional unless otherwise specified. We follow the notation of Hawking and Ellis [2] as closely as possible. In particular, the differential of a map f will be denoted by f_*. We always parametrize geodesics by an affine parameter. Unlike [2], however, the tangent vector field along a geodesic γ will be denoted by γ'. We write covariant differentiation of a vector field V along γ as V'. We also deviate from [2] in our notation of scalar products which are written as $\langle\ ,\ \rangle$. Throughout the paper we concentrate our attention on geodesics emanating from a fixed point p in a manifold M and denote the exponential map at p by exp. The tangent space at p is denoted by T_p or $T_p(M)$ when we wish to emphasize the manifold. We will often consider tangent spaces of T_p considered as a manifold in its own right. Thus if K is an element of T_p, then the tangent space at K is written as $T_K(T_p)$.

2. Conjugate Points and Jacobi Fields

In this section we review the basic properties of conjugate points and Jacobi fields. Our arguments will apply both to the Riemannian and Lorentzian cases unless otherwise stated. Let $\gamma:[a,b]\to M$ be a geodesic with $\gamma(a)=p$ and $\gamma(b)=q$. Then $\gamma(v)=\exp((v-a)W)$, where $W=\gamma'(a)$. In the literature, a conjugate point could be either an element of the tangent space or a point in the manifold. Since we want to differentiate between these two aspects we say that $V=(b-a)W\in T_p$ is *T-conjugate* if exp is singular[1] at V. Further, if V is T-conjugate, then $q = \exp(V)$ is said to be *M-conjugate to p along γ*. Conjugate points are related to solutions of the geodesic deviation equation or Jacobi equation

$$Z'' + \mathbf{R}(Z,K)K = 0, \tag{2.1}$$

where $K = \gamma'$ and $\mathbf{R}$ denotes the Riemann tensor. Such solutions are called *Jacobi fields*. Equation (2.1) is a second order linear differential system. Therefore, the space of all (smooth) Jacobi fields along γ is $2n$-dimensional. For future purposes we define $\chi_t(\gamma)$ to be the space of continuous piecewise smooth vector fields along γ which vanish at p and $\gamma(t)$. In particular, if $t = a$ then a vector field is in $\chi_a(\gamma)$ if it vanishes at p. Let $J_a(\gamma)$ denote the space of smooth Jacobi fields in $\chi_a(\gamma)$. Then $\dim J_a(\gamma) = n$. Also, let $J_b(\gamma)$ be the space of smooth Jacobi fields in $\chi_b(\gamma)$. The *T-conjugate locus*, denoted by $T(p)$, is the set of all T-conjugate points in T_p and the *M-conjugate locus* is the set of all points in M which are M-conjugate to p along some geodesic. The T-conjugate locus, being the set of singular points of exp, is a closed set. Sometimes when there is no risk of confusion we will drop the prefixes T- and M- for conjugate points.

The *conjugate order* or *multiplicity* or *degree of degeneracy* of a T-conjugate point V. denoted by $\mathrm{ord}(V)$, is the dimension of the null space of exp at V. It can be shown that the order is equal to the dimension of the space of Jacobi fields along γ which vanish at both p and $\exp(V)$ (Cheeger and Ebin [7, Corollary 1.18]). Therefore, Jacobi fields can be used to study conjugate points. Since $\dim J_a(\gamma) = n$, the conjugate order is at most n. Let $K = \gamma'$. Then $(v - a)K$ is a Jacobi field in $J_a(\gamma)$ which does not give rise to conjugate points. Hence the maximal conjugate order is $n - 1$. This limit cannot be further reduced in general. For example, on the n-sphere, the antipode is a conjugate point of order $n - 1$. If $\mathrm{ord}(V) > 1$, then we say that the T-conjugate locus is *degenerate* at V.

A *ray* in T_p is a line $r:[0,\infty) \to T_p$ with $r(u) = uV$ for $u \in [0,\infty)$, where $V \in T_p$. If r is a ray, then $u \to \exp(r(u))$ is a geodesic. There is always a first T-conjugate point along any given ray in T_p. This follows from the fact that $T(p)$ is a closed set. If J is a Jacobi field and $K = \gamma'$, then $\langle J,K \rangle'' = \langle J'',K \rangle = -\langle R(J,K)K,K \rangle = 0$, where a prime denotes covariant differentiation along γ. Hence there are constants c and d such that $\langle J,K \rangle = cv + d$. Therefore, if J is smooth and $\langle J,K \rangle$ vanishes at two points, then $\langle J,K \rangle \equiv 0$. We have proved the following lemma:

Lemma 2.1 (Cheeger and Ebin [7], Proposition 1.12). *Let $\gamma:[a,b] \to M$ be a geodesic and J a smooth Jacobi field in $J_b(\gamma)$. Then $\langle J,\gamma' \rangle \equiv \langle J',\gamma' \rangle \equiv 0$.*

Define $J_a^{\perp}(\gamma) \equiv \{J \in J_a(\gamma) : \langle J,\gamma' \rangle \equiv 0\}$. By Lemma 2.1, only Jacobi fields in $J_a^{\perp}(\gamma)$ (i.e. only Jacobi fields perpendicular to γ) can give rise to conjugate points. Another useful lemma is:

Lemma 2.2 (Cheeger and Ebin [7], Proposition 1.13). *If q is not conjugate to p along γ, then a Jacobi field along γ is uniquely determined by its values at p and q.*

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1 If N and M are n-dimensional manifolds, then a map $f: N \to M$ is said to be singular at a point q in N if the linear map f_* is singular at q, that is if $f_*: T_q(N) \to T_{f(q)}(M)$ is not an isomorphism

Let $c \in (a,b)$ be a number such that $\gamma(c)$ is not conjugate to p along γ. Then, by Lemma 2.2, any Jacobi field in $J_a^\perp(\gamma)$ is uniquely determined by its value at $\gamma(c)$. Therefore, there is a one-one correspondence between $J_a^\perp(\gamma)$ and the orthogonal complement of $\gamma'(c)$ in $T_{\gamma(c)}$. But the orthogonal complement of a non-zero vector K is always $(n-1)$-dimensional. This applies both to Riemannian and Lorentzian space. The argument shows that $\dim J_a^\perp(\gamma) = n-1$, so that we again have the result that the order of a conjugate point is at most $n-1$. If K is a null vector, then K is contained in its own orthogonal complement. Thus for a null geodesic $\gamma(v)$, $J_a^\perp(\gamma)$ contains the Jacobi field $(v-a)K$, where $K=\gamma'$. But $(v-a)K$ is not in $J_b(\gamma)$. Hence the maximal order of a conjugate point along a null geodesic reduces by one to $n-2$.

We shall need the analogue of Lemma 2.2 for Jacobi classes along a null geodesic (see Appendix 2 for the definition of Jacobi classes).

Lemma 2.3 (Beem and Ehrlich [13, Lemma 8]). *If $\gamma:[a,b]\to M$ is a null geodesic, and $\gamma(b)$ is not conjugate to $\gamma(a)$, then a Jacobi class along γ is uniquely determined by its values at $\gamma(a)$ and $\gamma(b)$.*

3. The Timelike T-Conjugate Locus

Following Warner [5] we say that a timelike T-conjugate point $V \in T_p$ is *regular* if there is a neighbourhood B of V such that every ray in T_p contains at most one T-conjugate point in B. The set of all regular T-conjugate points will be denoted by $R(p)$. A T-conjugate point which is not regular is said to be *singular*. Roughly, singular conjugate points occur at intersections in the conjugate locus.

We wish to prove space-time analogues of the following two theorems:

Theorem 3.1 (Warner [5]). *Let M be a Riemannian manifold. Then the regular conjugate locus of a point p, $R(p)$, is a $(n-1)$-dimensional submanifold with the relative topology. Further, $R(p)$ is an open dense subset of $T(p)$ and for all $V \in R(p)$, $T_V(T_p) = T_V(R(p)) \oplus T_V(r)$ where $T_V(r)$ is the tangent space of the ray $v \to r(v) = vV$ at V.*

It follows from the last statement of Theorem 3.1 that $R(p)$ cannot be parallel to a ray in T_p.

Let $N(V) \in T_V(T_p)$ be the null space of exp at V. If $V \in R(p)$ we let $T(V)$ be that subspace of $N(V)$ which is tangential to $R(p)$ at V, i.e. $T(V) = N(V) \cap T_V(R(p))$. If $\mathrm{ord}(V) = k$, then $T(V)$ has dimension k or $k-1$.

Theorem 3.2 (Warner [5]). *Let M be a Riemannian manifold. If $V \in R(p)$ and* $\mathrm{ord}(V) = k > 1$, *then* $\dim\ T(V) = k$.

That is, if V is a regular T-conjugate point of order > 1, then the null space of $\exp_*$ is tangential to the conjugate locus at V.

Warner also gave canonical coordinates for exp near all regular conjugate points which have neighbourhoods in which the null space is everywhere tangential to the conjugate locus. We make no attempt to construct canonical coordinates in this paper. However, it should be remarked that Warner's construction is independent of the metric and therefore applies to the timelike case. Special problems arise in the null case because the null vectors do not form an open set in the tangent space.

Warner's theorems do not depend on the Riemannian structure as such but only on the following properties of the exponential map:

(R1) $\exp: T_p \to M$ is C^∞ and $\exp_*(r'(v)) \neq 0$ for all v, for every ray $r(v)$ in T_p (here $r'(v) \in T_{r(v)}(r)$ denotes the ray tangent).

(R2) The *regularity property*[2] of conjugate points: For any $V \in T_p, T_{\exp(V)}$ is spanned by $\exp_*(T_V(T_p))$ and $(\partial/\partial r) \exp_*(N(V))$, where $\partial/\partial r$ is a radial derivative. Explicitly, put $F = \exp$ and let $(x^1, \ldots, x^n)$ be coordinates on a neighbourhood of V such that $r = x^n$ is a radial coordinate. Then $\exp_*$ can be represented by the matrix $(\partial F^i/\partial x^j)$, where the F^i are components of exp in some coordinates at $\exp(V)$. Also, $(\partial/\partial r) \exp_*$ has the matrix representation $(\partial^2 F^i/\partial r \partial x^j)$.

(R3) The *continuity property* of conjugate points. Roughly, this means that conjugate points depend continuously on the initial direction of the geodesic. (An exact statement is given in Proposition 3.4.)

When formulating Warner's theorems for Lorentzian manifolds, the causal character of the geodesics is crucial. Indeed, as will be seen later on, (R2) and (R3) cannot be proved for spacelike geodesics with the methods used for nonspacelike geodesics. It could well be that (R2) and (R3) do not hold in the spacelike case (cf. discussion of how conjugate points vary with initial conditions in Penrose [14, p. 64]). Furthermore, since the maximal order of null conjugate points is one unit less than the maximal order of timelike conjugate points and since the nonspacelike rays do not form an open set in T_p we have to treat the null and timelike cases separately.

We will show that (R1)–(R3) are valid in the nonspacelike Lorentzian case. The only exception concerns the first part of (R1), namely the domain of definition of exp. In the timelike case one must consider the restriction of exp to future (or past) directed timelike vectors in T_p. However, this does not in any way affect Warner's argument. In the null case, exp must be restricted to N_p^+ (or N_p^-), where N_p^+ (respectively N_p^-) is the set of future (respectively past) directed null vectors in T_p. Warner's argument still goes through, however, provided that one works with neighbourhoods in N_p^+ instead of in the full tangent space T_p and that one makes necessary dimensional changes. For example, the dimension of the regular null conjugate locus is $n-2$.

Therefore to carry out our program of proving theorems analogous to 3.1 and 3.2 for space-times it suffices to establish the validity of the second part of (R1) together with (R2) and (R3) in the nonspacelike case.

Starting with (R1), we note that in the Riemannian case, the second part of (R1) follows from the fact that the tangent vector of a geodesic has constant length. The same argument applies to timelike and spacelike geodesics, so that property (R1) is valid in those cases too. Now consider a null geodesic. In this case the length of the tangent vector is always zero. Hence the length cannot be used to determine whether the tangent vector vanishes or not. Instead (R1) follows from the observation that if a tangent vector γ' is zero at one point then there would be an infinite proportionality factor beween the affine parameter v and another affine parameter u with $d/du \neq 0$ at

2 This term is introduced here by the present author

the same point. To see that, suppose that $\gamma(v)$ is a geodesic and that $\gamma'(a)=0$. Let K be a non-zero null vector at $\gamma(a)$ such that the geodesic $u \to \exp(uK)$ is the same geodesic as $\gamma(v)$ and $\exp(0K)=\gamma(a)$. Then $u = Av$ where A is a constant. But then also $K=(d/du)\,(a)=A^{-1}(d/dv)\,(a)=A^{-1}\gamma'(a)=0$ which is a contradiction. Hence (R1) applies to null geodesics as well. Finally we note that the second part of property (R1) can also be expressed by the statement that the null space of $\exp_*$ is never parallel to a ray in T_p.

Now we turn to property (R2). We prove the regularity property for the Riemannian and the Lorentzian nonspacelike case together to indicate where the differences arise. Warner showed that (R2) is equivalent to a property of Jacobi fields which is expressed by the following proposition:

Proposition 3.3 (Regularity Property). *Let $\gamma:[a,b]\to M$ be a geodesic in a Riemannian manifold or a nonspacelike geodesic in a Lorentzian manifold. If $\gamma(a)=p$, then T_q, the tangent space at $q=\gamma(b)$, is spanned by the values at q of the Jacobi fields in $J_a(\gamma)$ together with the values at q of the derivatives of the elements in $J_b(\gamma)$.*

Before proving the proposition recall that $J_b(\gamma)$ is the space of Jacobi fields which vanish at both $p=\gamma(a)$ and $q=\gamma(b)$. Also $k=\dim J_b(\gamma)$ is the order of q as an M-conjugate point to p along γ. Further we define

$$A_q=\{V\in T_q : V=J'(b) \quad \text{for some} \quad J\in J_b(\gamma)\},$$
$$B_q=\{V\in T_q : V=J(b) \quad \text{for some} \quad J\in J_a(\gamma)\}.$$

Then B_q is the subspace of T_q which is spanned by the values at q of the Jacobi fields in $J_a(\gamma)$ and A_q is the subspace of T_q which is spanned by the values at q of the derivatives of the Jacobi fields in $J_b(\gamma)$. Proposition 3.3 states that $A_q\oplus B_q=T_q$. Note also that the proposition is trivial if q is not conjugate to p along γ.

Proof of Proposition 3.3. Let $Z_i\in J_b(\gamma)(i=1,2,\ldots,k)$ be a basis for $J_b(\gamma)$ and choose $W_i\in J_a(\gamma)(i=1,2,\ldots,n-k)$ such that $\{Z_i\}$ and $\{W_i\}$ together form a basis for $J_a(\gamma)$. Then the W_i are not in $J_b(\gamma)$.

We assert that the $Z_i'(b)$ are linearly independent. For suppose that there exist numbers $a_i(i=1,2,\ldots,k)$ such that $\sum a_i Z_i'(b)=0$. Put $Z=\sum a_i Z_i$. Then $Z(b)=Z'(b)=0$, whence it follows that Z and hence all the a_i are identically zero. Thus $\dim A_q=k$ and $\{Z_i'(b)\}$ is a basis for A_q.

Further, $\{W_i(b)\}$ constitutes a basis for B_q. For suppose that there exist $b_i(i=1,2,\ldots,n-k)$ such that $\sum b_i W_i(b)=0$. Put $W=\sum b_i W_i$. Then $W\in J_b(\gamma)$ implying that W and hence all the b_i are identically zero. Hence $\dim B_q=n-k$ and $\dim A_q+\dim B_q=n$.

It remains to establish that $A_q\cap B_q=\{0\}$. For that purpose we show that A_q and B_q are orthogonal complements. First we observe that the scalar $\langle Z_i', W_j\rangle - \langle Z_i, W_j'\rangle$ is a constant along γ (see Cheeger and Ebin [7, p. 25]). But $Z_i(a)=W_j(a)=0$ implies that the constant is zero. Since $Z_i(b)=0$, we obtain $\langle Z_i'(b), W_j(b)\rangle=\langle Z_i(b), W_j'(b)\rangle=0$, showing that A_q and B_q are indeed orthogonal.

In a space with positive definite metric orthogonal complements span the entire space. Hence the proposition now follows for Riemannian manifolds. In the

Lorentzian case, on the other hand, orthogonal complements do not necessarily span T_q.

We proceed by assuming that A_q and B_q have a common non-zero vector V and derive a contradiction. Since V is orthogonal to itself it must be a null vector. There exist numbers a_i such that $V = \sum a_i Z_i'(b) \in B_q$. Put $Z = \sum a_i Z_i$. We will show that $Z = 0$. By the definition of Z we have $Z'(b) = V$. Further $Z \in J_b(\gamma)$ and by Lemma 2.1, $\langle Z', K \rangle = 0$, where $K = \gamma'$. Thus, at q we have

(1) $\langle Z', Z' \rangle = 0$,

(2) $\langle Z', K \rangle = 0$.

If K is timelike, the relations (1) and (2) together imply that $Z'(b) = 0$ which in conjunction with $Z(b) = 0$ implies that $Z = 0$. Now suppose γ is a null geodesic. For simplicity we work in four dimensions. The generalisation to n dimensions ($n \geqq 3$) is obvious. Choose a frame $E_a (a = 0, 1, 2, 3)$obtained by parallel transport along γ as in Hawking and Ellis [2, p. 86] where $E_0 = K$, $E_1 = L$ is a null vector with $\langle K, L \rangle = -1$ and E_2 and E_3 are unit spacelike vectors, orthogonal to each other and to K and L. Then if J is any Jacobi field in $J_b(\gamma)$, $J^1 = -J_0 = -\langle E_0, J \rangle$ is a constant. Since $Z(a) = 0$ we obtain $Z^1 = 0$. The length of a vector in the parallel frame is given by $\langle V, V \rangle = -2V^0V^1 + (V^2)^2 + (V^3)^2$. Then since $Z^1 = 0$, $Z'^1 = -\langle E_0, Z' \rangle$ and Z and Z' are null vectors it follows that $Z^2 = Z^3 = Z'^2 = Z'^3 = 0$ at q. Hence $Z = Z' = 0$ at q. When $Z^1 = 0$ the Jacobi equation in the parallel frame becomes (see [2]).

$$\begin{aligned} (d^2/dv^2) Z^0 &= R_{10n0} Z^n, \\ (d^2/dv^2) Z^m &= -R_{m0n0} Z^n. \end{aligned} \qquad (m, n = 2, 3) \tag{3.1}$$

The 1-equation becomes an identity. Note that the 2- and 3-equations decouple from the rest. Hence the 2-vector Z^m obeys a homogeneous second order linear differential system. The initial conditions at q then imply $Z^2 = Z^3 = 0$ for all v. Thus $Z = Z^0 K$ and the equation for Z^0 gives $Z^0 = cv + d$ where c and d are constants. Since $Z \in J_b(\gamma)$ we must have $c = d = 0$. Hence $Z = 0$. Then $a_i = 0$ for all i implying $V = 0$ in contradiction to our assumption that V is non-zero. Consequently, the assumption must be wrong and A_q and B_q have no non-zero vector in common. [ ]

Next we discuss (R3). The continuity property is well known for Riemannian manifolds. However, no easily available proof seems to exist in the literature. Here, we give proofs for the timelike and null cases. Our proof of the timelike continuity property can easily be carried over to Riemannian manifolds. The precise statement in that case is:

Proposition 3.4 (Continuity Property of Conjugate Points) (Morse [9, Lemma 13.1, p. 235]). *Let M be Riemannian manifold. If $V \in T_p(M)$ is the j'th T-conjugate point along the ray $v \to vV$, then there is a convex neighbourhood D of V, such that the number of T-conjugate points (counted with multiplicities) in $r \cap D$, for each ray r which intersects D, is constant and equals* ord(V). *Moreover, the first T-conjugate point on each segment $r \cap D$ is the j'th T-conjugate point along r.*

Our convention for ordering conjugate points is that V is the j'th T-conjugate point along the ray $r: v \to vV$ in T_p if there are exactly $j-1$ T-conjugate points (counted with multiplicities) preceding V on r. To prove the continuity property one uses index theory (see Appendix for a brief sketch of timelike and null index theory). It follows from the Morse Index Theorem that conjugate points are isolated along a geodesic. There is no index theory for spacelike geodesics. The reason is that any spacelike geodesic can be approximated by curves with both longer and shorter length than that of the spacelike geodesic itself. That is, unlike the Riemannian and timelike cases, a spacelike geodesic is neither minimal nor maximal.

We use ideas from Patterson [10] in our approach to the proof of the continuity property. As the proof is rather long we break it up by first proving two lemmas. First, we make the following standard definition: Let $\gamma:[a,b]\to M$ be a geodesic with $\gamma(a)=p$ and $\gamma(b)=q$. Then a *k-parameter geodesic variation* of γ is a smooth mapping $h:B\times[a,b]\to M$, where B is a neighbourhood of $\mathbf{0}\in\mathbb{R}^k$, and $h(0,v)=\gamma(v)$ for all $v\in[a,b]$ and the curves $v\to h(\alpha,v)$ are geodesic for all $\alpha\in B$.

For convenience, we denote a k-parameter variation by γ_α, where $\gamma_\alpha(v)=h(\alpha,v)$. Following Patterson [10] we define a *normal sequence* $\{t_i\}_{i=0}^{k}$, $a=t_0<t_1<\ldots<t_{k-1}<t_k=t$, for $\gamma|[a,t]$ by the requirement that the intervals $[t_i,t_{i+1}]$ have no pairs of conjugate points.

Lemma 3.5. *Given a k-parameter geodesic variation γ_α of γ, there exist neighbourhoods D of $\mathbf{0}\in\mathbb{R}^k$ and U of $b\in\mathbb{R}$ and a sequence $\{t_i\}_{i=0}^{m-1}$ such that $a=t_0<t_1<\ldots<t_{m-1}<t_m=t$ is a normal sequence for $\gamma_\alpha|[a,t]$ for all $(\alpha,t)\in D\times U$.*

Proof. The compact set $\gamma([a,b])$ can be covered by a finite number of normal neighbourhoods N_j. We may choose a finite sequence $\{U_j\}_{j=0}^{s}$ of compact *overlapping* intervals ($U_{j+1}\cap U_j$ has non-zero length for all $j=1,\ldots,s-1$) such that the geodesic segments $\gamma(U_j)$ cover $\gamma([a,b+\delta])$ and $\gamma(U_j)\subset N_j$ for all j, and where $\delta>0$ is a number such that $\gamma([b-\delta,b+\delta])$ is contained in one of the N_j. Then, for any j, there is a neighbourhood D_j of $\mathbf{0}\in\mathbb{R}^k$ such that $\{\gamma_\alpha(t):t\in U_j,\alpha\in D_j\}\subset N_j$. Put $D=\cap D_j$ and let $d\in(0,\delta)$ be a number which is less than the minimum of the lengths of $U_j\cap U_k$ when $U_j\cap U_k$ is nonempty. Since the intervals U_j have an overlap (if any) which is at least d, the conditions $t\in[a,b]$ and $0<t'-t<d$ guarantee that, for all α, the set $\gamma_\alpha([t,t'])$ is contained in one of the N_j and hence that $\gamma_\alpha(t')$ is not conjugate to $\gamma_\alpha(t)$ along γ_α. Next choose a sequence $a=t_0<t_1<\ldots<t_{k-1}<b$ such that $|t_{i+1}-t_i|<d/2$ for $i=0,1,\ldots,k-2$ and $d/3<b-t_{k-1}<d/2$. Define $U=\{t:|t-b|<d/3\}$. It is now straightforward to check that D, U and $\{t_i\}_{i=0}^{k-1}$ satisfy the requirements of the lemma. [ ]

Note that the above proof does not depend on the metric. Therefore we may use the lemma for variations of both timelike and null geodesics.

Lemma 3.6. *Let M be a Lorentzian manifold. If $V\in T_p(M)$ is timelike and V is not T-conjugate, then there is a neighbourhood of V in which the index is constant and equal to the index at V.*

Proof. Let γ be the geodesic defined by $v\to\gamma(v)=\exp(vV)$. Define a $(n-1)$-parameter geodesic variation γ_α of γ such that $\gamma_\alpha(0)=p$ for all α. Then all geodesics $v\to\gamma_\alpha(v)$ are timelike for α in a sufficiently small neighbourhood B of $\mathbf{0}\in\mathbb{R}^{n-1}$. A neighbourhood of V now corresponds to a neighbourhood $(\mathbf{0}, 1) \in B \times \mathbb{R}$. For any t (such that the geodesic is defined) and α we can define the index form $I = I_{\alpha,t}$ for $\gamma_\alpha | [0, t]$ by (A.1.1)

$$I(X, Y) = -\int_0^t (\langle X', Y' \rangle + \langle \mathbf{R}(X, K)Y, K \rangle) dv, \tag{3.2}$$

where $K = \gamma'_\alpha$ and X, $Y \in \chi_t(\gamma_\alpha)$. We wish to show that $I_{\alpha,t}$ depends continuously on (α, t) in a neighbourhood of $(\mathbf{0}, 1)$ in $\mathbb{R}^{n-1} \times U$ where U is a neighbourhood of $1 \in \mathbb{R}$. We do that by showing that $I_{\alpha,t}$ can be regarded as a family of forms, depending continuously on α and t, defined on a *fixed finite*-dimensional vector space. This is the same technique (slightly extended) which is used in Cheeger and Ebin [7] for the proof of the index theorem itself.

The first step is to decompose $\chi_t(\gamma)$ in two subspaces χ^1 and χ^2 such that χ^1 and χ^2 are orthogonal with respect to I, I is negative definite on χ^1 and the dimension of χ^2 is finite. As a consequence $\operatorname{ind}(I) = \operatorname{ind}(I | \chi^2) < \infty$. In the proof of the index theorem one shows that I depends continuously on the endpoint $\gamma(t)$ in a neighbourhood of $t = 1$ if q is not conjugate to p along γ. Here we must show that I depends continuously on the endpoint if variations of γ are also allowed.

By Lemma 3.5 there are neighbourhoods D of $\mathbf{0} \in \mathbb{R}^{n-1}$ and U of $1 \in \mathbb{R}$ and a sequence $\{t_i\}_{i=0}^{k-1}$ such that $0 = t_0 < t_1 < \ldots < t_k = t$ is a normal sequence for $\gamma_\alpha | [0, t]$ for all $(\alpha, t) \in D \times U$. Given (α, t) in $D \times U$ we define

$$\chi^1 = \{X \in \chi_t(\gamma_\alpha) : X(t_i) = 0 \text{ for all } i\}. \tag{3.3}$$

Then $I_{\alpha,t} | \chi^1$ is negative definite. For let I_i be the index form on $\gamma_\alpha | [t_i, t_{i+1}]$ so that $I_{\alpha,t} = \sum I_i$. By [8, Theorem 9.22] and Lemma 2.1, $I_i | \chi^1$ is negative definite. Hence so is $I_{\alpha,t} | \chi^1$.

Now let $\chi^2 = J_\alpha\{t_i\}$ be the subspace of $\chi_t(\gamma_\alpha)$ consisting of broken Jacobi fields which break only at the t_i. Then χ^1 and χ^2 are orthogonal with respect to $I_{\alpha,t}$. To show that, let $X \in \chi^1$ and $Y \in \chi^2$. Since Y is a broken Jacobi field it follows by (A.1.2) that

$$I_{\alpha,t}(X, Y) = \sum \Delta_i \langle Y', X \rangle. \tag{3.4}$$

But X vanishes at the jumps of Y'. Hence $I_{\alpha,t}(X, Y) = 0$ showing that χ^1 and χ^2 are indeed orthogonal. Together with the negative definiteness of χ^1 this shows that $I = I | \chi^2$.

Since $\{t_i\}$ is a normal sequence for γ, it follows by Lemma 2.2 that a Jacobi field on $\gamma_\alpha | [t_i, t_{i+1}]$ is uniquely determined by its values at the end points. Hence for any value of α there is an isomorphism

$$J_\alpha\{t_i\} \approx T_{\gamma_\alpha(t_1)} \oplus \ldots \oplus T_{\gamma_\alpha(t_{k-1})}. \tag{3.5}$$

Let us choose a basis frame E_a along γ. In fact, it is possible to extend the frame to γ_α in a neighbourhood of γ such that E_a depends smoothly on $(\alpha, t) \in D \times [0, b_0]$, where D is a neighbourhood of $\mathbf{0} \in \mathbb{R}^{n-1}$ and $b_0 > 1$. We may now identify the spaces $\chi^2 = J_\alpha\{t_i\}$ for different values of α. Explicitly, we consider their elements as equal if the components in the E_a frame are equal. This is possible because of the isomorphism (3.5). We can now regard $I_{\alpha,t}$ as a family of forms on the fixed finite-dimensional space $\chi^2 = J_0\{t_i\}$. Also, for any interval $[t_i, t_{i+1}](i = 1, \ldots, k-2)$, the Jacobi equation can be regarded as a differentiable system on the fixed vector space spanned by the frame vectors E_a and depending on the $n-1$ parameters α. On $[t_{k-1}, t]$ it depends on the n parameters (α, t). Therefore, by the standard theory of ordinary differential equations, the derivatives $X'(t_i)$ depend continuously on (α, t) for fixed $X(t_i)(i = 1, \ldots, k-1)$. Since $I_{\alpha,t}|\chi^2$ is given by (3.4), it too depends continuously on (α, t).

Now suppose $q = \gamma(1)$ is not conjugate to p along γ. Then the null space of I is zero or, in other words, I is nondegenerate. Let χ^+ be a subspace of χ^2 such that I is positive definite on χ^+ and $\dim \chi^+ = \mathrm{ind}(I)$. Let χ^- be the orthogonal complement of χ^+ with respect to I. Then I is negative definite on χ^-. Since I depends continuously on (α, t) and χ^+ and χ^- are finite-dimensional, there is a neighbourhood C of $\mathbf{0} \in \mathbb{R}^{n-1}$ such that $I_{\alpha,t}$ is positive definite on χ^+ and negative definite on χ^- respectively for all $\alpha \in C$. Therefore the index is the same for all geodesics $\gamma_\alpha : [0, t] \to M$ with $(\alpha, t) \in C \times U$.[ ]

We are now in a position to prove the continuity property for timelike geodesics.

Proposition 3.7 (Timelike Continuity Property). *Let M be a Lorentzian manifold. If $V \in T_p(M)$ is a timelike vector which is the j'th T-conjugate point along the ray $v \to vV$, then there is a convex neighbourhood D of V such that the number of T-conjugate points (counted with multiplicities) on $r \cap D$ for each ray r which intersects D is constant and equals* $\mathrm{ord}(V)$. *Moreover, the first T-conjugate point on each segment $r \cap D$ is the j'th T-conjugate point along r.*

Proof. Let $V \in T_p$ be the j'th T-conjugate point along the ray $v \to vV$. Since conjugate points are isolated along a timelike ray, we may choose a number $v_1 > 1$ such that exp is non-singular at all points vV for which $v \in (1, v_1]$. If $\mathrm{ord}(V) = k$, then the index at $v_1 V$ is $i = j - 1 + k$. By Lemma 3.6 there is a neighbourhood B of $v_1 V$ such that the index is equal to i throughout B. In the same manner we can find a $v_2 < 1$ and a neighbourhood C of $v_2 V$ such that the index is equal to $j - 1$ throughout C. The final step is to choose a convex neighbourhood D of V consisting of ray segments which start in C and end in B. Then D has the required properties.[ ]

This completes our discussion of the proof that Warner's theorems (Theorems 3.1 and 3.2) hold in the timelike case. The timelike theorems can be stated in a way completely analogous to the Riemannian case. Let $R_T(p)$ be the regular timelike T-conjugate locus, i.e. the timelike part of $R(p)$.

Theorem 3.8. *Let M be a space-time. Then the regular timelike T-conjugate locus of a point p, $R_T(p)$, is a $(n-1)$-dimensional submanifold of T_p with the relative topology. Further, $R_T(p)$ is an open dense subset of the timelike T-conjugate locus and $T_V(T_p) = T_V(R_T(p)) \oplus T_V(r)$ at all points $V \in R_T(p)$ where $T_V(r)$ is the tangent space of the ray $v \to r(v) = vV$ at V.*

Theorem 3.9. *Let M be a space-time. If V is in the regular timelike T-conjugate locus and* $\mathrm{ord}(V) = k > 1$, *then* $\dim T(V) = k$.

4. The Null T-Conjugate Locus

Let N_p be the set of null vectors in T_p. A null T-conjugate point $K \in T_p$ is said to be *regular* if there is a neighbourhood B of K in N_p such that every ray in N_p contains at most one T-conjugate point in B. The null analogue of Lemma 3.6 is:

Lemma 4.1. *Let M be a Lorentzian manifold. If $K_0 \in T_p$ is a null vector which is not T-conjugate, then there is a neighbourhood of K_0 in N_p in which the index is constant and equal to the index at K_0.*

Proof. The general outline of the proof is the same as in the timelike case. Let γ be the null geodesic defined by $v \to \gamma(v) = \exp(vK_0)$. Define a $(n-2)$-parameter geodesic variation γ_α of γ such that $\gamma_\alpha(0) = p$ for all α and such that the geodesics $v \to \gamma_\alpha$ are all null. Then γ_α represents a part of the null cone at p. Define the index form $\bar{I}_{\alpha,t}$ for $\gamma_\alpha | [0, t]$ by (A.2.4)

$$\bar{I}(\bar{X}, \bar{Y}) = -\int_0^t (\langle \bar{X}', \bar{Y}' \rangle - \langle \bar{\mathbf{R}}(\bar{X}, K)K, \bar{Y} \rangle) dv, \tag{4.1}$$

where $K = \gamma'_\alpha$ and $\bar{X}, \bar{Y} \in \mathfrak{X}_t(\gamma_\alpha)$.

By Lemma 3.5 there are neighbourhoods D of $\mathbf{0} \in \mathbb{R}^{n-2}$ and U of $1 \in \mathbb{R}$ and a sequence $\{t_i\}_{i=0}^{k-1}$ such that $0 = t_0 < \ldots < t_{k-1} < t_k = t$ is a normal sequence for $\gamma_\alpha | [0, t]$ for all $(\alpha, t) \in D \times U$. Given (α, t) in $D \times U$ we define

$$\chi^1 = \{\bar{X} \in \mathfrak{X}_t(\gamma_\alpha) : \bar{X}(t_i) = [\gamma'_\alpha(t_i)] \text{ for all } i\}. \tag{4.2}$$

To prove that $\bar{I} | \chi^1$ is negative definite, let $\bar{I}_i$ be the index form on $\gamma_\alpha | [t_i, t_{i+1}]$ so that $\bar{I} = \sum \bar{I}_i$. By [8, Theorem 9.69], $\bar{I}_i$ is negative definite. Hence so is $\bar{I} | \chi^1$.

Now let $\chi^2 = J_\alpha\{t_i\}$ be the subspace of $\mathfrak{X}_t(\gamma_\alpha)$ consisting of broken Jacobi classes which break only at the t_i. To show that χ^1 and χ^2 are orthogonal with respect to $\bar{I}_{\alpha,t}$, let $\bar{X} \in \chi^1$ and $\bar{Y} \in \chi^2$. Since $\bar{Y}$ is a broken Jacobi class (A.2.6)

$$\bar{I}(\bar{X}, \bar{Y}) = \sum \Delta_i \langle\langle \bar{Y}', \bar{X} \rangle\rangle. \tag{4.3}$$

But $\bar{X}$ vanishes at the jumps of $\bar{Y}'$ and so $\bar{I}_{\alpha,t}(\bar{X}, \bar{Y}) = 0$. Hence $\bar{X}$ and $\bar{Y}$ are orthogonal with respect to $\bar{I}_{\alpha,t}$. This shows that $\bar{I} = \bar{I} | \chi^2$.

By Lemma 2.3, a Jacobi class along $\gamma_\alpha | [t_i, t_{i+1}]$ is uniquely determined by its values in $G(\gamma_\alpha(t_i))$ and $G(\gamma_\alpha(t_{i+1}))$. Hence for any value of α there is an isomorphism

$$J_\alpha\{t_i\} \approx G(\gamma_\alpha(t_1)) \oplus \ldots \oplus G(\gamma_\alpha(t_{k-1})). \tag{4.4}$$

As in the timelike case, let E_a be a basis frame along γ_α in a neighbourhood of γ. We now identify the space $J_\alpha\{t_i\}$ for different α using the isomorphism (4.4). The index form may then be regarded as a family of forms on the fixed finite-dimensional vector space $J_0\{t_i\}$. To show that $\bar{I}_{\alpha,t}$ depends continuously on (α, t) we must show that the $\bar{X}'(t_i) \in G(\gamma(t_i))$ depend continuously on (α, t) if $\bar{X} \in \chi^2$, i.e. if $\bar{X}(t_i)(i = 1, \ldots, k-1)$ are given.

Choose $V_i \in N(\gamma(t_i))$ with $\pi(V_i) = \bar{X}(t_i)$. Then for any α such that E_a is defined, there is a unique Jacobi field J along γ_α with $J(t_i) = V_i$. Further, J and hence the $J(t_i)$ depend continuously on (α, t). Put $\bar{X} = \pi(J)$. Then $\bar{X}$ is a Jacobi class with the prescribed values at the t_i. Also $\bar{X}'(t_i) = \pi(J'(t_i))$ by the definition of covariant differentiation of Jacobi classes. Finally, since π is a continuous map, the $\bar{X}'(t_i)$ depend continuously on (α, t). [ ]

We can now state the continuity property for null geodesics. The main difference is that everything now takes place inside the light cone N_p.

Proposition 4.2 (Null Continuity Property). *Let M be a Lorentzian manifold. If $K \in T_p(M)$ is a null vector which is the j'th T-conjugate point along the ray $v \to vK$, then there is a neighbourhood D of K in N_p such that the number of T-conjugate points (counted with multiplicities) in $r \cap D$ for each ray r which intersects D is constant and equals* ord (K). *Moreover, the first T-conjugate point on each segment $r \cap D$ is j'th T-conjugate point along r.*

The proof is the same as in the timelike case except that neighbourhoods should be taken in N_p instead of in the full tangent space T_p. We can now state the null analogues of Theorems 3.8 and 3.9. Let $R_N(p)$ be the regular null T-conjugate locus, i.e. the subset of $R(p)$ consisting of null vectors.

Theorem 4.3. *Let M be a space–time. Then the regular null T-conjugate locus of a point p, $R_N(p)$, is a $(n-2)$-dimensional submanifold of N_p with the relative topology. Further, $R_N(p)$ is an open dense subset of the null T-conjugate locus and $T_K(N_p) = T_K(R_N(p)) \oplus T_K(r)$ at all points $K \in R_N(p)$ where $T_K(r)$ is the tangent space of the ray $r(v) = vK$ at K.*

Theorem 4.4 *Let M be a space–time. If K is in the regular null T-conjugate locus and* $\mathrm{ord}(K) = k > 1$, *then* $\dim T(K) = k$.

5. The M-Conjugate Locus

In this section we consider the physical space–time caustic, i.e. the M-conjugate locus. We say that the M-conjugate locus has a *cusp* at $q = \exp(V)$ if $V \in T(p) - R(p)$ (i.e., if V is a singular conjugate point) or if $V \in R(p)$ but $\exp|R(p)$ is not an immersion of $R(p)$ at V. Thus q is a cusp if $\exp^{-1}(q)$ is not a regular T-conjugate point or if the null space of exp at $\exp^{-1}(q)$ has some direction tangential to $R(p)$. If V is a degenerate T-conjugate point, then the M-conjugate locus is necessarily cuspidal at $\exp(V)$ (cf. the discussion preceding Theorem 3.2).

To be able to use the results of the previous sections, we need to impose a condition which ensures that focusing always occurs to the past of a particular space-time event. One such condition is that (see Tipler [11])

$$\int_{-\infty}^{a} F(v)dv = +\infty$$

for all null geodesics with $\gamma(a) = p$, where $F(v) = \mathbf{Ric}(K, K)$ and $K = \gamma'$ (**Ric** denotes the Ricci tensor). However, it may happen that some or all null geodesics with $\gamma(a) = p$ are past incomplete. In that case the singularity could come "before" the geodesics have had time to refocus. We say that the *omni-directional focusing* *condition* is satisfied at p if any past directed null geodesic γ through p contains a point which is conjugate to p along γ.

Next we show that the null M-conjugate locus in a 4-dimensional space-time can never be a smooth 2-sphere. Intuitively, this follows from the observation that the geodesics touch the caustic. Hence their tangents form an everywhere non-vanishing vector field on the caustic. However, no such field exists on a 2-sphere. This argument is made rigorous in the following theorem:

Theorem 5.1. *Let M be a 4-dimensional space-time. If the omni-directional focusing condition is satisfied at $p \in M$, then there is a cusp in the first null M-conjugate locus to the past of p.*

To prove the theorem we make use of a Euclidean metric on T_p defined as follows. Choose a Lorentzian basis L_a for T_p, i.e. $\langle L_a, L_b \rangle = \eta_{ab}$ where $\eta_{ab} = \text{diag}(-1, +1, +1, +1)$. The Euclidean metric is then defined by $E(L_a, L_b) = \delta_{ab}$. Now let U_p^- be the set of past directed null vectors K in T_p with $E(K, K) = 1$. Then U_p^- is diffeomorphic to a 2-sphere and can be regarded as a representation of the celestial sphere at p. Any point K in N_p^-, the set of past directed null vectors in T_p, can be projected into U_p^- by sliding it along the ray through K. Let this projection be denoted by P. Then P is a C^∞ map and its differential P_* can be use to project elements of the tangent bundle of N_p^- into the tangent bundle of U_p^-.

Proof of Theorem 5.1. We show that the first null T-conjugate locus to the past of p, denoted by C_1^-, contains at least one point which is degenerate, i.e. the multiplicity > 1. Suppose to the contrary that the conjugate order is 1 throughout C_1^-. Then by Theorem 4.3, C_1^- is a smooth submanifold of N_p (it is in fact a 2-sphere since $P|C_1^-$ is a diffeomorphism onto U_p^-).

Let T be the mapping $K \to N(K)$ which assigns the null space of exp to any point K in C_1^-. Then T is C^∞ in the sense that the matrix equation $(\partial F_i/\partial x^k)X^k = 0$, where $F = \exp$ in some coordinates x^k, locally has a solution X which is a C^∞ vector field along C_1^- with $X \in N(K)$ wherever X is defined. Now $N(K)$ is always perpendicular to K because of the Gauss lemma (Cheeger and Ebin [7, p. 8] or Beem and Ehrlich [8, p. 262]). Then since $N(K)$ is never parallel to K, T can be projected by P_* to a C^∞ distribution on U_p^-. However, U_p^- is a 2-sphere and hence does not admit a 1-dimensional distribution. This contradiction shows that our assumption that all points of C_1^- have conjugate order 1 is false. Thus C_1^- contains at least one degenerate conjugate point. []

Corollary 5.2. *In a 4-dimensional space–time M, the null M-conjugate locus cannot be a smooth 2-sphere.*

Note 5.3. Since no even-dimensional sphere admits an everywhere non-vanishing vector field, Theorem 5.1 is valid for any even-dimensional space-time with the corollary that the null M-conjugate locus cannot be a smooth $(n-2)$-sphere.

The observational significance of a degenerate conjugate point is that the observed distortion in a given direction becomes infinite in more than one direction, i.e. a given object at the caustic is infinitely distorted both longitudinally and latitudinally on the celestial 2-sphere in the case of a 4-dimensional space-time. Therefore we also refer to this as degenerate focusing.

Theorem 5.4. *Let M be a 4-dimensional space–time with a point $p \in M$ such that an observer at p sees degenerate focusing in all directions. Then the space-time is closed in the sense that it admits a compact (topology $\mathbb{S}^3$) slice.*

Proof. As before we denote the first past null conjugate locus by C_1^-. In a 4-dimensional space-time, the conjugate order for a null geodesic can take on the values 1 and 2. Since the focusing is degenerate in all directions the conjugate order is 2 throughout C_1^-. Now let $K \in C_1^-$ and let (x, y) be coordinates on some neighbourhood U of K in C_1^-. Then, since the null space of exp is tangential to C_1^- by Theorem 4.4, we have

$$\exp_*(\partial/\partial x|L) = \exp_*(\partial/\partial y|L) = 0 \in T_{\exp(L)}$$

for all $L \in U$, which shows that exp is constant on U. But then exp is constant on the whole of C_1^-. Thus, exp (C_1^-) is a single point in M. Then by [3, Theorem 4.6] the space-time admits a smooth spacelike 3-sphere. [ ]

As stated, Theorem 5.4 is only true for 4-dimensional space-times. However, if we require the focusing to be maximally degenerate instead of just degenerate (i.e. the conjugate order $= n-2$), then the theorem is valid in any dimension $\geqq 4$ (the topology of the spacelike slice will be $\mathbb{S}^{n-1}$ in general).

The requirement of degenerate focusing in all directions is not as unrealistic as it may seem at first glance. Such behaviour is actually present in the closed Friedmann–Robertson–Walker (FRW) universes (cf. [3], [4]). One may use Theorem 5.4 to explain the fact that there are no open (i.e. not admitting a compact spacelike slice) geodesically complete FRW models (with or without cosmological constant). For that purpose, consider any isotropic space–time satisfying the omni-directional focusing condition. Then some observer would see a non-empty C_1^- and by isotropy (see [4]), all points of C_1^- would be degenerate. Hence we can apply Theorem 5.4 to establish the existence of a compact slice so that the space–time is closed. The power of this argument is that it is grounded on purely geometric reasoning. It is therefore valid regardless of the field equations.

Appendix 1. Timelike Index Theory

For a treatment of Riemannian index theory, see Cheeger and Ebin [7]. The Lorentzian index theory can be found in Beem and Ehrlich [8], [12], [13]. Let γ: $[a, b] \to M$ be a timelike geodesic with $\gamma(a) = p$ and $\gamma(b) = q$. If X and Y are vector fields in $\chi_b(\gamma)$, then the *timelike index form* is defined by ([8, p. 251])

$$I(X, Y) = -\int_a^b (\langle X', Y' \rangle + \langle \mathbf{R}(X, K)Y, K \rangle) dv, \tag{A.1.1}$$

where $\mathbf{R}$ is the curvature tensor and $K = \gamma'$. The index form is a symmetric bilinear form on $\chi_b(\gamma)$. Let us choose a sequence $a = t_0 < t_1 < \ldots < t_k = b$ such that $X|[t_i, t_{i+1}]$ and $Y|[t_i, t_{i+1}]$ are smooth for all i. Then the index form can be integrated by parts to give

$$I(X,Y)=\int_a^b (\langle X''+\mathbf{R}(X,K)K,Y\rangle)dv+\sum_i \Delta_i\langle X',Y\rangle, \tag{A.1.2}$$

where

$$\Delta_i\langle X',Y\rangle=\lim_{t\to t_i+}\langle X',Y\rangle-\lim_{t\to t_i-}\langle X',Y\rangle.$$

If X and Y are smooth, then the index form reduces to

$$I(X,Y)=\int_a^b \langle X''+\mathbf{R}(X,K)K,Y\rangle dv. \tag{A.1.3}$$

From this expression one sees clearly the connection with Jacobi fields. We say that X is in the null space of I if $I(X,Y)=0$ for all $Y\in\chi_b(\gamma)$. Then $X\in\chi_b(\gamma)$ is in the null space of I if and only if X is a Jacobi field. The *index*, ind(I), of the index form is defined to be the maximum dimension of a subspace of $\chi_b(\gamma)$ on which I is positive definite. The index at $V\in T_p$ is defined as the index of the index form along the geodesic $\gamma:[0,1]\to M$, where $\gamma(v)=\exp(vV)$. The main result of index theory is the Morse index theorem which relates the index to the number of conjugate points on a geodesic.

Theorem A.1.1 (Timelike Morse Index Theorem) (Beem and Ehrlich [8, Theorem 9.27]). *Let M be a space–time and γ: $[a,b]\to M$ a timelike geodesic segment. If I is the timelike index form on $\gamma|[a,b]$, then the index of I is finite and equal to the number of conjugate points to $p=\gamma(a)$ along $\gamma|[a,b)$ counted according to their multiplicities. The null space of I is zero unless $q=\gamma(b)$ is conjugate to p along γ and in that case its dimension equals the conjugate order of q.*

Appendix 2. Null Index Theory

The index theory of null geodesics differs from that of timelike geodesics. To see why let $\gamma:[a,b]\to M$ be a null geodesic segment. We first observe that unbroken Jacobi fields in $\chi_b(\gamma)$ are always perpendicular to γ by Lemma 2.1. Therefore, defining

$$\chi_b^\perp(\gamma)=\{X\in\chi_b(\gamma):\langle X,\gamma'\rangle\equiv 0\},$$

the index form can always be restricted to vector fields in $\chi_b^\perp(\gamma)$ without loss of generality. Now consider vector fields along γ of the form $f(v)\gamma'$, where $f(a)=f(b)=0$. If Z is any field in $\chi_b^\perp(\gamma)$, then $I(f\gamma',Z)=0$, showing that $f\gamma'$ is in the null space of I. But vector fields of the form $f\gamma'$ never give rise to conjugate points. Hence, the definiteness of the index form cannot be used to characterize conjugate points. However, this difficulty can be resolved by working with a quotient bundle of $\chi_b(\gamma)$, where vector fields parallel to γ' are considered to be zero.

We make the following definitions

$$\begin{aligned} N(\gamma(v))&=\{X\in T_{\gamma(v)}:\langle X,\gamma'(v)\rangle=0\},\\ [\gamma'(v)]&=\{\lambda\gamma'(v):\lambda\in\mathbb{R}\},\\ G(\gamma(v))&=N(\gamma(v))/\gamma'(v). \end{aligned} \tag{A.2.1}$$

Further we define the quotient bundle by $G(\gamma)=\cup G(\gamma(v))$. The space of piecewise smooth sections of $G(\gamma)$ is denoted by $\mathfrak{X}(\gamma)$. The elements of $\mathfrak{X}(\gamma)$ are called *vector classes* along γ. We shall distinguish vector classes from vector fields by putting a bar over symbols for vector classes. The space of vector classes along γ which vanish at $\gamma(a)$ and $\gamma(b)$ is denoted by $\mathfrak{X}_b(\gamma)$, i.e.

$$\mathfrak{X}_b(\gamma)=\{\bar{X}\in\mathfrak{X}(\gamma):\bar{X}(a)=[\gamma'(a)] \text{ and } \bar{X}(b)=[\gamma'(b)]\}. \tag{A.2.2}$$

Vector fields in $\chi(\gamma)$ may be projected by the natural projection map $\pi:N(\gamma(v))\to G(\gamma(v))$ into vector classes in $\mathfrak{X}(\gamma)$. Also, the Lorentzian metric $\langle,\rangle$ can be projected to a positive definite metric, denoted by $\langle\langle,\rangle\rangle$, on $G(\gamma(v))\times G(\gamma(v))$ by putting $\langle\langle\bar{X},\bar{Y}\rangle\rangle=\langle V,W\rangle$ if $\bar{X},\bar{Y}\in G(\gamma(v))$ and $V,W\in N(\gamma(v))$ satisfy $\pi(V)=\bar{X}$ and $\pi(W)=\bar{Y}$. Covariant differentiation of a vector class $\bar{X}$ along γ is defined by $\bar{X}'=\pi(V')$ if $V\in\chi(\gamma)$ and $\pi(V)=\bar{X}$. The endomorphism $V\to\mathbf{R}(V,\gamma')\gamma'$ of $N(\gamma'(v))$ may be projected to an endomorphism of $G(\gamma(v))$ by setting $\bar{\mathbf{R}}(\bar{X},\gamma')\gamma'=\pi(\mathbf{R}(V,\gamma')\gamma')$ for $\bar{X}\in G(\gamma(v))$ if $V\in N(\gamma(v))$ satisfies $\pi(V)=\bar{X}$. All projection operations introduced here are well defined (see Beem and Ehrlich [8]).

A smooth vector class $\bar{X}\in\mathfrak{X}(\gamma)$ is said to be a *Jacobi class* along γ if

$$\bar{X}''+\bar{\mathbf{R}}(\bar{X},\gamma')\gamma'=[\gamma'], \tag{A.2.3}$$

where $[\gamma']$ denotes the zero element of $G(\gamma)$. Finally we define the *null index form* $\bar{I}$ on $\gamma|[a,b]$ by

$$\bar{I}(\bar{X},\bar{Y})=-\int_a^b(\langle\langle\bar{X}',\bar{Y}'\rangle\rangle-\langle\langle\mathbf{R}(\bar{X},K)K,\bar{Y}\rangle\rangle)dv, \tag{A.2.4}$$

where $K=\gamma'$ and $\bar{X},\bar{Y}\in\mathfrak{X}_b(\gamma)$. As in the timelike case, this expression can be partially integrated to give

$$\bar{I}(\bar{X},\bar{Y})=\int_a^b\langle\langle\bar{X}''+\bar{\mathbf{R}}(\bar{X},K)K,\bar{Y}\rangle\rangle dv+\sum_i\Delta_i\langle\langle\bar{X}',\bar{Y}\rangle\rangle. \tag{A.2.5}$$

From this it follows that

$$\bar{I}(\bar{X},\bar{Y})=\sum\Delta_i\langle\langle\bar{X}',\bar{Y}\rangle\rangle \tag{A.2.6}$$

if $\bar{X}$ is a piecewise smooth Jacobi class in $\mathfrak{X}_b(\gamma)$.

Theorem A.2.1 (Null Morse Index Theorem) (Beem and Ehrlich [8, Theorem 9.77]). *Let M be a space–time and $\gamma:[a,b]\to M$ a null geodesic segment. If $\bar{I}$ is the null index form on $\gamma|[a,b]$, then the index of $\bar{I}$ is finite and equal to the number of conjugate points to $p=\gamma(a)$ along $\gamma|[a,b)$ counted according to their multiplicities. The null space of $\bar{I}$ is zero unless $q=\gamma(b)$ is conjugate to p along γ and in that case its dimension equals the conjugate order of q.*

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