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Gauge Potentials and Bundles Over the 4-Torus

C. Nash

Department of Mathematical Physics, St. Patrick's College, Maynooth, Ireland

Abstract. The construction of principal bundles over a four dimensional torus is considered. The class of groups considered is $SU(n)/Z_n$, and for this class the Pontrjagin class has even integer values.

1. Introduction

This paper considers principal fibre bundles over a four-dimensional torus. Physically a four-dimensional torus corresponds to space-time being a kind of Euclidean box with periodic boundary conditions. Fibre bundles enter when one considers non-Abelian gauge fields inside this box. This physical picture has been considered by a number of people, cf., for example, [1] and references cited therein.

In [1] it is argued that the gauge groups $SU(n)/Z_n$, $n = 2, 3, \ldots$ are physically important (Z_n stands for the centre of the group $SU(n)$, hence for each n, Z_n is isomorphic to the n^{th} roots of unity). The topology of space-time is $S^1 \times S^1 \times S^1 \times S^1$, where S^1 is a unit circle. We shall denote space-time by T^4. Underlying the non-Abelian gauge field is a fibre bundle and so we are led to the construction of all $SU(n)/Z_n$ bundles over T^4. We describe, in what follows, a method for carrying out this construction. In Sect. 2 we treat the case $n = 2$, and in Sect. 3 the case $n > 2$. An important mathematical tool in the calculations will be the generalised cohomology theory known as K-theory.

2. The $n = 2$ Case

When $n = 2$ there is the well known result, of a kind typical for Lie groups of low dimension, that $SU(2)/Z_2 \simeq SO(3)$. Thus we wish to construct all SO(3)-bundles over T^4. In contrast to the case where the base space is a sphere S^k the calculation is not completely straightforward. It turns out to be most easily accomplished by resorting to a well known mathematical tool of bundle theory known as K-theory. K-theory is a kind of generalised cohomology theory defined for vector bundles. For an introduction to K-theory, cf. the works cited in [2]. The K-theory for T^4 considers all vector bundles E over T^4 and assembles them together into equivalence classes—two bundles E and F are equivalent if the addition of a trivial bundle I^j to each of them renders them isomorphic [2]: $E \oplus I^j \simeq F \oplus I^k$. Although the K-theory over T^4 considers all vector bundles over T^4 of all possible ranks, we shall nevertheless be able to pin down those bundles with SO(3) as their structure group and identify their corresponding principal bundles. We shall use the notation of Husemoller [2]. Since SO(3) is an orthogonal group, the corresponding K-theory is denoted by $\tilde{K}O$—the so-called reduced real K-theory [2]. In general $\tilde{K}O(M)$ for some M forms a ring, with multiplication and addition provided by tensor product and direct sum respectively. If $M = T^4$, then $\tilde{K}O(T^4)$ forms a group G and a certain subgroup H of G provides us with the SO(3)-bundles we seek.

Before constructing H we need some general results about the construction of bundles. If one wishes to construct G-bundles over a compact manifold M, then one needs a space B_G known as the classifying space for bundles with group G. This space B_G is the base space of a certain bundle W_G called a universal G-bundle. Then for a map f

$$f: M \rightarrow B_G, \tag{2.1}$$

$f^* W_G$ is a bundle over M known as the pull-back of W_G by f. All G-bundles over M arise as $f^* W_G$ for some f, also if f and g are homotopic maps the $f^* W_G$ and $g^* W_G$ are isomorphic. Thus all G-bundles over M are given [2] by all homotopy classes of maps $f: M \rightarrow B_G$ by $[M, B_G]$. Now we choose $G = \mathrm{SO(3)}$ and $M = T^4$ so that we wish to know $[T^4, B_{\mathrm{SO(3)}}]$. Next we may use a result of James et al [3] to characterise $[T^4, B_{\mathrm{SO(3)}}]$ in terms of $\tilde{K}O(T^4)$. To this end we calculate $\tilde{K}O(T^4)$. This calculation presents some difficulties which may be circumvented by replacing T^4 by X where X is a space of the same homotopy type as T^4 so that $\tilde{K}O(T^4) = \tilde{K}O(X)$. Such a space X is a given by [4]

$$X = S^4 v(\underbrace{S^3 v S^3 v \ldots v S^3}_{\text{4-times}}) v (\underbrace{S^2 v S^2 \ldots v S^2}_{\text{6-times}}) v (\underbrace{S^1 v \ldots v S^1}_{\text{4-times}}), \tag{2.2}$$

where $A v B$ denotes the disjoint union of A and B with base points identified. (Alternatively, instead of introducing X, one may calculate $\tilde{K}O(T^4)$ via the properties of $\tilde{K}O^{-P}(T^4)$, where $\tilde{K}O^{-P} = \tilde{K}O(S^P M)$ and $S^P M$ is the P-fold suspension of M.) We then have

$$\tilde{K}O(X) = \tilde{K}O(T^4) = \tilde{K}O(S^4) \oplus \underbrace{\tilde{K}O(S^3)}_{\text{4-times}} \oplus \underbrace{\tilde{K}O(S^2)}_{\text{6-times}} \oplus \underbrace{\tilde{K}O(S^1)}_{\text{4-times}}. \tag{2.3}$$

The right-hand side of 2.3 is well known [2] so that we obtain

$$\tilde{K}O(T^4) = Z \oplus \underbrace{(Z_2)}_{\text{10-times}}, \tag{2.4}$$

where Z_2 denotes the group of integers modulo 2. Next we utilise Theorem 1.6 of [3] which says that the map

$$[T^4, B_{\mathrm{SO(3)}}] \rightarrow [T^4, B_{\mathrm{SO}}] = \tilde{K}O(T^4) \tag{2.5}$$

is injective, and that under this map the elements of $[T^4, B_{SO(3)}]$ correspond to a subgroup H of $\tilde{K}O(T^4)$: namely those elements of $\tilde{K}O(T^4)$ with vanishing 4th-Stiefel–Whitney class W_4. (In 2.5 B_{SO} is the classifying space for all principal SO(n) bundles and SO denotes the infinite special orthogonal group.) The subgroup H is then given by

$$H = 2\mathbf{Z} \oplus \underset{\text{6-times}}{(Z_2)}. \tag{2.6}$$

We can now describe the various SO(3)-bundles over T^4. To do this requires the notion of an induced or pullback bundle: if f is a map from T^4 to M and E is a bundle over M, then f^*E, the pullback bundle, is a bundle over T^4. The six Z_2 summands in H correspond to the following pullbacks. Project first from T^4 to T^2 (the 2-torus). This can clearly be done in six possible ways. Denote the six projections by $\pi_1, \ldots \pi_6$:

$$\begin{gathered}\pi_i : T^4 \to T^2.\\ i = 1, \ldots 6\end{gathered} \tag{2.7}$$

Now consider the Hopf bundle S^3 over S^2 and a map $f: T^2 \to S^2$; if we denote the Hopf bundle by ξ, then $f^*\xi$ is the pullback of ξ to T^2 and $(f \circ \pi_i)^*\xi$ is the pullback to T^4. These six bundles contain the twist $\eta_{\mu\nu}$ referred to by 't Hooft [1]. They also have zero Pontrjagin number p_1. This is because

$$\begin{aligned}H^4(S^2; Z) = 0 \text{ so that } p_1(\xi) = 0 \text{ and}\\ p_1\{(f \circ \pi_i)^*\xi\} = (f \circ \pi_i)^* p_1(\xi)\\ = 0,\end{aligned} \tag{2.8}$$

so the bundles over T^4 have vanishing p_1 also. These bundles $(f \circ \pi_i)^*\xi$ are SO(3)-bundles by virtue of the embedding of U(1), the group of ξ, in SO(3); they correspond to the generators of the six Z_2 summands in H. Further U(1)-bundles may be formed as we shall see below shortly. The summand $2\mathbf{Z}$ in H is generated by pulling back a certain bundle ζ over S^4 to T^4 under a map $g: T^4 \to S^4$. The bundle ζ has total space $\mathbf{CP}^3$ and base space $\mathbf{HP}^1$, where $\mathbf{HP}^1$ stands for one dimensional quaternionic projective space, and in fact $\mathbf{HP}^1 \simeq S^4$. The fibration is as follows: $\mathbf{CP}^3$ has four homogeneous coordinates $[z_1, \ldots z_4]$, a quaternion q may be regarded as being given by a pair of complex numbers a, b so that $q = a + bj$. The projection p of the bundle ζ projects $[z_1, \ldots z_4]$ onto $[z_1 + z_2 j, z_3 + z_4 j]$, which is an element of $\mathbf{HP}^1$. The pullback $g^*\zeta$ is an SO(3)-bundle over T^4. Further $p_1(g^*\zeta)$ is always even. This is because in general we have

$$p_1(\zeta) \bmod 2 = W_2^2(\zeta), \tag{2.9}$$

where $W_2(\zeta) \in H^2(S^4 \mathbf{Z}/2)$ is the second Stiefel–Whitney class of ζ. Since $H^2(S^4 : \mathbf{Z}/2) = 0$, then $p_1(\zeta)$ is even. Now if the map $g: T^4 \to S^4$ has degree k, we have

$$\begin{aligned}p_1(g^*(\zeta)) &= g^* p_1(\zeta)\\ &= k p_1(\zeta)\end{aligned} \tag{2.10}$$

so that $p_1(g^*\zeta)$ is also even, in fact $p_1(\zeta)=2$, and thus $p_1(g^*\zeta)=2k$. We have now identified the bundles that correspond to the generators of H. The operations of $\otimes$ and $\oplus$ which provide $\widetilde{KO}(T^4)$ with its ring structure provide a source of further bundles.

In general we define $\xi_i=(f\circ\pi_i)^*\xi$ and $\xi_j=(f\circ\pi_i)^*\xi$; the tensor product $\xi_i\otimes\xi_j$ remains a U(1)-bundle and will have Pontrjagin number given by, (cf. appendix)

$$p_1(\xi_i\otimes\xi_j)=C\varepsilon_{\mu\nu\alpha\beta}\nu^{\mu\nu}\nu^{\alpha\beta}, \tag{2.11}$$

where the integer $\eta^{\mu\nu}$ is the twist of $\xi_i\otimes\xi_j$ and $C=\frac{1}{4}$, so that $p_1(\xi_i\otimes\xi_j)$ is an even integer. Now if we set $\zeta_{ij}=\zeta_i\otimes\zeta_j$ and form $\tau=\zeta_{ij}\oplus g^*\zeta$, then we have, since $H^4(T^4;\mathbf{Z})$ contains no elements of order 2,

$$\begin{aligned} p_1(\tau)&=p_1(\zeta_{ij}\oplus g^*\zeta)\\ &=p_1(\zeta_{ij})+p_1(g^*\zeta)\\ &=2k+C\varepsilon_{\mu\nu\alpha\beta}\eta^{\mu\nu}\eta^{\alpha\beta}. \end{aligned}\tag{2.12}$$

Compare this with Eq. (1.1) of ref. 5, cf. also Van Baal [6], where the definition of p_1 used in ref. 5 is corrected.

3. The $n>2$ Case

When $n>2$ the group to be considered is $SU(n)/Z_n$, which we write as PU(n). Here PU(n) is the projective unitary group, if G is any group with centre Z then $PG=G/Z$; note that PU(n) = PSU(n). The essentials of our problem will again be reduced to the calculation over spheres S^i via 2.2; now G-bundles over S^i are classified by the homotopy group $\pi_{i-1}(G)$, i.e. we have an isomorphism

$$[S^i,B_G]\simeq[S^{i-1},G]. \tag{3.1}$$

If $G=SU(n)$, then it is important to know that

$$\pi_i(PU(n))=\pi_i(SU(n)),\ i>1,$$

but that

$$\pi_1(PU(n))=Z_n;\ \pi_1(SU(n))=0. \tag{3.2}$$

We shall refer to a PU(n)-bundle as a projective bundle. Projective bundles may be obtained from U(n)-bundles by a procedure that we now describe, however inequivalent U(n)-bundles may give rise to the same projective bundle: If E is a U(n)-bundle over a manifold M, then it gives rise to a projective bundle PE by use of the natural projection p: $U(n)\to PU(n)$. If, however, L is a U(1)-bundle, or line bundle, then $E\otimes L$ is another U(n)-bundle; in general inequivalent to E, but certainly $P(E\otimes L)=PE$. A converse also holds, i.e. if PE and PF are equivalent projective bundles, then there exists a line bundle L such that

$$E\simeq F\otimes L. \tag{3.3}$$

There is therefore a one to one correspondence between projective bundles PE and equivalence classes of U(n)-bundles, the equivalence relation is denoted by $\sim$ and the equivalence class given by:

$$E \sim F \Leftrightarrow E = F \otimes L \qquad (3.4)$$

for some line bundle L. To calculate projective bundles PE over M one can therefore first calculate all U(n)-bundles over M, and then divide these up into equivalence classes according to (3.4), one then has all the projective bundles PE. We have a specific situation: namely $n > 2$ and $M = T^4$. It is then known that the U(n)-bundles are found by calculating $\tilde{K}U(T^4)$, the reduced complex K-theory of T^4; since $n > 2$, we are in what is known as the stable range and two vector bundles of rank n are isomorphic if and only if they are equivalent in $\tilde{K}U(T^4)$. In other words

$$[T^4, B_{U(n)}] \simeq \tilde{K}U(T^4), \qquad n > 2. \qquad (3.5)$$

The calculation of $\tilde{K}U(T^4)$ is done by exactly similar methods to those used in Sect. 2 and the result is

$$\begin{aligned} KU(T^4) &= KU(S^4) \oplus \underset{4\text{-times}}{KU(S^3)} \oplus \underset{6\text{-times}}{KU(S^2)} \oplus \underset{4\text{-times}}{KU(S^1)} \\ &= \underset{7\text{-times}}{\mathbf{Z} \oplus \mathbf{Z} \oplus \ldots \mathbf{Z}}. \end{aligned} \qquad (3.6)$$

In the right-hand side of (3.6), one copy of $\mathbf{Z}$ comes from the fact that $\tilde{K}U(S^4) = \mathbf{Z}$, the other six come from the fact that $\tilde{K}U(S^2) = \mathbf{Z}$, $\tilde{K}U(S^3)$ and $\tilde{K}U(S^1)$ being zero. The description of the U(n)-bundles over T^4 requires first the giving of the bundles over S^2 and S^4 that correspond to the generators of $KU(S^2)$ and $KU(S^4)$. We denote these bundles by ξ and ζ respectively, ξ is determined by a map

$$\alpha : S^1 \to \mathrm{U}(n) \qquad (3.7)$$

and ζ by a map

$$\beta : S^3 \to \mathrm{U}(n). \qquad (3.8)$$

In fact $\alpha \in \pi_1(\mathrm{U}(n)) = \mathbf{Z}$ and $\beta \in \pi_3(\mathrm{U}(n)) = \mathbf{Z}$ so that only the homotopy classes of α and β matter. The integers, a and b say, that label the homotopy class of α and β respectively are chosen to be unity and are given by

$$\begin{aligned} a &= \int_{S_2} C_1(\xi), \\ b &= \int_{S_4} C_2(\zeta), \end{aligned} \qquad (3.9)$$

where C_1 and C_2 denote Chern classes. To construct U(n)-bundles over T^4 we simply need to pull-back ξ and ζ to T^4, i.e. to construct the bundles $(f \circ \pi_i)^* \xi$ and $g^* \zeta$.

Finally we need to construct PU(n)-bundles over T^4. This is done by giving $P\xi$ and $P\zeta$: let p be the projection $p : \mathrm{U}(n) \to \mathrm{PU}(n)$, then the maps

$$\begin{aligned} \alpha_p &: S^1 \xrightarrow{\alpha} \mathrm{U}(n) \xrightarrow{p} \mathrm{PU}(n), \\ \beta_p &: S^3 \xrightarrow{\beta} \mathrm{U}(n) \xrightarrow{p} \mathrm{PU}(n), \end{aligned} \qquad (3.10)$$

where $\alpha_p = p \circ \alpha$ and $\beta_p = p \circ \beta$ define the projective bundles $P\xi$ and $P\zeta$ which are then pulled back to $(f\circ\pi_i)^*P\xi$ and $g^*P\zeta$ respectively to give projective bundles over T^4. Evidently $\alpha_p\in\pi_1(\mathrm{PU}(n))=Z_n$ and $\beta_p\in\pi_3(\mathrm{PU}(n))=\mathbf{Z}$. Thus the six kinds of bundles $(f\circ\pi_i)^*P\xi$ are classified by a twist $\eta_{\mu\nu}$ defined modulo n and the bundles $g^*P\zeta$ are classified by an integer. The twist $\eta_{\mu\nu}$ is defined modulo n because $\alpha\in\pi_1(\mathrm{U}(n))=\mathbf{Z}$ and $\alpha_p\in\pi_1(\mathrm{PU}(n))=Z_n$, for example, because of this fact, two homotopically inequivalent $\alpha,\alpha':S^1\to \mathrm{U}(n)$ may become homotopically equivalent when composed with p, i.e. we may have $\alpha\not\simeq\alpha'$ but $\alpha_p\simeq\alpha'_p$.

In fact since topologically $\mathrm{U}(n)=\mathrm{U}(1)\times \mathrm{SU}(n)$, then $\pi_1(\mathrm{U}(n))=\pi_1(\mathrm{U}(1))$, since $\pi_1(\mathrm{SU}(n))=0$, so that in (3.10) the map $\alpha\in\pi_1(\mathrm{U}(n))$ is actually determined by an element α' of $\pi_1(\mathrm{U}(1))=\pi_1(S')$ and $\deg\alpha'$, the degree of α', is unity. This means that ξ is again the Hopf bundle of Sect. 2. As a consequence we may again construct the $\mathrm{PU}(n)$-bundle τ where

$$\tau=\zeta_{ij}\oplus g^*P\zeta, \tag{3.11}$$

and $\zeta_{ij}=\xi_i\otimes\xi_j$ and $\xi_i=(f\circ\pi_i)^*P\xi$, $\xi_j=(f'\circ\pi_j)^*P\xi$. Even though τ was originally derived from $\mathrm{U}(n)$-bundles whose characteristic classes are Chern classes, τ may be regarded as having a Pontrjagin class $p_1(\tau)$. This point, also made independently by Van Baal [6], is that $\mathrm{PU}(n)$ is isomorphic to a subgroup G of $\mathrm{SO}(n^2-1)$, indeed any compact Lie group is isomorphic to a subgroup of $\mathrm{O}(n)$ for some n. The isomorphism in the case of $\mathrm{PU}(n)$ is provided by simply taking the adjoint representation of $\mathrm{U}(n)$, the map defining the adjoint representation has, by definition, kernel equal to the centre of $\mathrm{U}(n)$ so that the desired isomorphism $\mathrm{PU}(n)\simeq \mathrm{AdU}(n)$ follows. This being so, a $\mathrm{PU}(n)$-bundle may be regarded as an $\mathrm{SO}(n^2-1)$-bundle whose structure group reduces to G, its appropriate characteristic class can then be taken to be a Pontrjagin class.

A general Abelian configuration is given by taking a sum of $(n-1)$-bundles ζ_{ij} which we denote by $\zeta^{(a)}$, $a=1,\dots n-1$. The resulting bundle, ζ say, has group $\mathrm{SO}(2)\times\ldots\times\mathrm{SO}(2)$, ($(n-1)$-times), which corresponds to the maximal Abelian subalgebra for $\mathrm{AdU}(n)\subset\mathrm{SO}(n^2-1)$, $n>2$. For ζ we have

$$\begin{aligned}p_1(\zeta)&=p_1(\zeta^{(1)}\oplus\zeta^{(2)}\ldots\oplus\zeta^{(n-1)})\\&=p_1(\zeta^{(1)})+\ldots p_1(\zeta^{(n-1)})\\&=\sum_{a=1}^{n-1}\frac{\varepsilon_{\mu\nu\alpha\beta}}{4}\hat{\eta}^{(a)}_{\mu\nu}\hat{\eta}^{(a)}_{\alpha\beta}.\end{aligned} \tag{3.12}$$

These $\hat{\eta}^{(\alpha)}_{\mu\nu}$ differ from those of ref. 6 due to a difference in the normalisation of the subalgebra. With the normalisation of ref. 6 we indeed find

$$p_1(\zeta)=\frac{(n-1)}{4}\varepsilon_{\mu\nu\alpha\beta}\eta_{\mu\nu}\eta_{\alpha\beta}+k, \tag{3.13}$$

where $\eta_{\mu\nu}$ is the twist defined modulo n and k is an even integer. The splitting principle [2] guarantees that a general value of p_1 may be obtained with such Abelian configurations in agreement with refs. 1 and 6, and p_1 is also always even [6].

Appendix

Let the bundle ξ_i have twist $\eta_{\mu\nu}$ and ξ_j have twist $\eta_{\alpha\beta}$ as U(1)-bundles. Then we have

$$c_1(\xi_i) = \eta_{\mu\nu} = \int_{T^2} i\frac{\mathbf{F}^i}{2\pi},$$

$$c_2(\xi_j) = \eta_{\alpha\beta} = \int_{T^2} i\frac{\mathbf{F}^j}{2\pi}, \tag{A.1}$$

where $\mathbf{F}^i$ and $\mathbf{F}^j$ are curvature defined on 2-dimensional tori. The Pontrjagin class $p_1(\xi_i \otimes \xi_j)$ is given by [2]

$$\begin{aligned} p_1(\xi_i \otimes \xi_i) &= \{c_1(\xi_i \otimes \xi_j)\}^2 = \{c_1(\xi_i) + c_1(\xi_j)\}^2, \\ c_1^2(\xi_i) + c_1(\xi_i)c_1(\xi_j) &+ c_1(\xi_j)c_1(\xi_i) + c_1^2(\xi_j) = 2c_1(\xi_i)c_1(\xi_j), \end{aligned} \tag{A.2}$$

where we have used the facts that $c_1^2(\xi_i) = c_1^2(\xi_j) = 0$ and $c_1(\xi_i)c_1(\xi_j) = c_1(\xi_j)c_1(\xi_i)$, which follow from naturality and U(1)-valuedness respectively.
Thus

$$p_1(\xi_i \otimes \xi_j) = -\frac{2}{(2\pi)^2} \int_{T^4} (\pi_i^* \mathbf{F}^i) \wedge (\pi_j^* \mathbf{F}^j), \tag{A.3}$$

where $\pi_i^*\mathbf{F}^i$ and $\pi_j^*\mathbf{F}^j$ are the pullbacks of the curvatures $\mathbf{F}^i$ and $\mathbf{F}^j$ to T^4. The right-hand side of (A.3) is evidently proportional to $\varepsilon_{\mu\nu\alpha\beta}\eta_{\mu\nu}\eta_{\alpha\beta}$. If one takes a specific case where $\eta_{\mu\nu} = \eta_{12}$, $\eta_{\alpha\beta} \equiv \eta_{34}$, one finds easily that this constant is $\frac{1}{4}$, so we have

$$p_1(\xi_i \otimes \xi_j) = \tfrac{1}{4}\varepsilon_{\mu\nu\alpha\beta}\eta_{\mu\nu}\eta_{\alpha\beta} \tag{A.4}$$

as desired, and this formula holds for general $\eta_{\mu\nu}$, $\eta_{\alpha\beta}$ defined modulo n.

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