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From the Euclidean Group to the Poincaré Group via Osterwalder–Schrader Positivity

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Abstract. Given a continuous representation of the Euclidean group in $n+1$ dimensions, together with a covariant system of subspaces, which satisfies Osterwalder–Schrader positivity, we construct a continuous unitary representation of the orthochronous Poincaré group in $n+1$ dimensions satisfying the spectral condition. A similar result holds for the covering groups of the Euclidean and Poincaré group.

Osterwalder–Schrader positivity allows the analytic continuation of a theory in imaginary time to a quantum theory in real time (Osterwalder and Schrader [6], Klein and Landau [3], Glimm and Jaffe [1]).

When this analytic continuation transforms a field theory covariant with respect to the Euclidean group into one covariant with respect to the Poincaré group, the procedure has been to analytically continue the Schwinger functions of the Euclidean theory, which are distributions invariant under the action of the Euclidean group, into other distributions called Wightman functions, which are then shown to be invariant under the action of the Poincaré group by analytic continuation of the partial differential equations that express the Euclidean invariance of the Schwinger functions (Nelson [5], Osterwalder and Schrader [6]).

In this article we show how Osterwalder–Schrader positivity allows the construction of a unitary representation of the Poincaré group directly from a representation of the Euclidean group with a covariant system of subspaces. The new tool that makes that possible is our theory of symmetric local semigroups (Klein and Landau [4]; see also Fröhlich [10]).

We start by considering the analytic continuation of unitary representations of the Euclidean group. Next we prove similar results for the covering groups of the Euclidean and Poincaré groups. Finally, we describe the extension to representations of the Euclidean group (or its covering group) on topological vector

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spaces; this extension is needed in the application of these results to the Osterwalder–Schrader axioms.

Similar results have been independently obtained by Fröhlich, Osterwalder and Seiler [11].

I. Analytic Continuation of Unitary Representations of the Euclidean Group

Definition. An Osterwalder–Schrader positive unitary representation of the $n+1$ dimensional Euclidean group $IO(n+1)$ with a covariant system of orthogonal projections consists of:

(i) a Hilbert space $\mathscr{K}$;

(ii) a map E from open sets in $\mathbb{R}^{n+1}$ to orthogonal projections in $\mathscr{K}$, such that $A \subset B$ implies $E(A) \leqq E(B)$ and $A_n \uparrow A$ implies $E(A_n) \to E(A)$ strongly;

(iii) a continuous unitary representation U of $IO(n+1)$ such that $U(g)E(A)U(g)^{-1} = E(gA)$ for all open sets $A \subset \mathbb{R}^{n+1}$ and $g \in IO(n+1)$;

(iv) Osterwalder–Schrader positivity: let $\theta \in IO(n+1)$ be time reversal, i.e., $\theta(s,x) = (-s,x)$, let $R = U(\theta)$ and $E_+ = E((0,\infty) \times \mathbb{R}^n)$, then $E_+ R E_+ \geqq 0$. □

Given such a OS-positive representation of $IO(n+1)$, we will construct a unitary representation V of the orthochronous Poincaré group $IO^{\uparrow}(1,n)$ on a Hilbert space $\mathscr{H}$. This will be done in four steps:

1. The Hilbert space $\mathscr{H}$ will be constructed from $\mathscr{K}$ by OS-positivity.
2. A unitary representation of the group of translations of (relativistic) $\mathbb{R}^{n+1}$ will be constructed on $\mathscr{H}$ from the unitary representation of the group of translations of (Euclidean) $\mathbb{R}^{n+1}$ by analytic continuation from imaginary time to real time.
3. A unitary representation of the orthochronous Lorentz group $O^{\uparrow}(1,n)$ will be constructed on $\mathscr{H}$ from the unitary representation of the rotation group $O(n+1)$ on $\mathscr{K}$ by analytic continuation from the trigonometric angle of a rotation towards the imaginary time axis to the hyperbolic angle of a Lorentz boost in real time.
4. The representations constructed in steps 2 and 3 will be combined to give a unitary representation of the orthochronous Poincaré group $IO^{\uparrow}(1,n)$ satisfying the spectral condition.

Steps 1 and 2 are standard (Osterwalder and Schrader [6], Klein [2], Glimm and Jaffe [1]). Step 3 is new; we use symmetric local semigroups to construct the infinitesimal generators of Lorentz boosts. Steps 2 and 3 are totally independent.

Remark 1. We start with a representation of the full rotation group $O(n+1)$ because we need time reversal to define OS-positivity, and $O(n+1)$ has only two connected components. Thus space reflections are necessarily included and we obtain a representation of the orthrochronous Lorentz group.

Remark 2. If the Hilbert space $\mathscr{K}$ is the complexification of a real Hilbert space which is left invariant by the representation U and by the covariant projections $E(A)$, we will obtain a representation of the full Poincaré group $IO(1,n)$ in which time reversal is represented by an anti-unitary operator.

We now proceed with the construction.

1. Construction of the Hilbert Space $\mathscr{H}$

Let $\mathscr{V} = E_+RE_+$ and $\mathscr{H}_0 = \mathscr{V}\mathscr{K} = \mathscr{V}\mathscr{K}_+$, where $\mathscr{K}_+ = E_+\mathscr{K}$. On $\mathscr{H}_0$ we define a sesquilinear form $\langle | \rangle$ by

$$\langle \mathscr{V}F | \mathscr{V}G \rangle = (F, RG)_{\mathscr{K}} \quad \text{for } F, G \in \mathscr{K}_+ .$$

Osterwalder–Schrader positivity makes $\langle | \rangle$ a positive definite inner product. We let $\mathscr{H}$ be the Hilbert space completion of $\mathscr{H}_0$ with the inner product $\langle | \rangle$.

By construction $\mathscr{V} : \mathscr{K}_+ \to \mathscr{H}$ is a contraction with dense range.

2. Construction of the Representation of the Group of Translations on $\mathscr{H}$.

Let $\tau(s,x)$ denote translation by (s,x) in (Euclidean) $\mathbb{R}^{n+1}$, then $\tau(s,x) \in IO(n+1)$ so we let $U(s,x) = U(\tau(s,x))$ be the corresponding unitary operator on $\mathscr{K}$.

Notice that $U(s,x)\mathscr{K}_+ \subset \mathscr{K}_+$ if $s \geqq 0$ and $U(0,x)$ commutes with both R and E_+.

Define $V(0,x)$ on $\mathscr{H}_0$ by

$$V(0,x)\mathscr{V} = \mathscr{V}U(0,x) \quad \text{on } \mathscr{K}_+ .$$

It follows that $V(0,x)$ is a well defined isometry of $\mathscr{H}_0$ onto itself and thus extends to a unitary operator on $\mathscr{H}$, the group property in x being obvious. Here $V(0,x)$ is clearly strongly continuous on $\mathscr{H}_0$ and hence on $\mathscr{H}$. We denote by $\mathbf{P}$ the self-adjoint generators of space translation:

$$V(0,x) = e^{ix \cdot \mathbf{P}} .$$

Consider now $V(s,0)$. For $s \geqq 0$ we define $P(s)$ on $\mathscr{H}_0$ by

$$P(s)\mathscr{V} = \mathscr{V}U(s,0) \quad \text{on } \mathscr{K}_+ .$$

It is easy to see that $P(s)$ is well defined on $\mathscr{H}_0$ and that $(P(s), \mathscr{D}_s = \mathscr{H}_0, \infty)$ form a symmetric local semigroup (Klein and Landau [4]). Hence there exists a unique self-adjoint operator H on $\mathscr{H}$ such that $P(s)$ is the restriction of e^{-sH} to $\mathscr{H}_0$. Since

$$\| e^{-sH}\mathscr{V}F \|_{\mathscr{H}} = \| P(s)\mathscr{V}F \|_{\mathscr{H}} = \| \mathscr{V}U(s,0)F \|_{\mathscr{H}} \leqq \| F \|_{\mathscr{K}}$$

for all $s \geqq 0$ and $F \in \mathscr{K}_+$, it follows that $H \geqq 0$.

Notice that $V(0,x)$ and $P(s)$ commute, since

$$\begin{aligned} V(0,x)P(s)\mathscr{V} &= \mathscr{V}U(0,x)U(s,0) = \mathscr{V}U(s,x) = \\ &= \mathscr{V}U(s,0)U(0,x) = P(s)V(0,x)\mathscr{V} . \end{aligned}$$

Thus H and $\mathbf{P}$ commute, in particular

$$e^{ix \cdot \mathbf{P}} e^{itH} = e^{itH} e^{ix \cdot \mathbf{P}} = e^{itH + ix \cdot \mathbf{P}}$$

for all $t \in \mathbb{R}$, $x \in \mathbb{R}^n$.

Let $\tau(t,x)$ denote translation by (t,x) in (relativistic) $\mathbb{R}^{n+1}$. Then

$$V(\tau(t,x)) = V(t,x) = e^{itH + ix \cdot \mathbf{P}}$$

gives a continuous unitary representation of the group of translations of $\mathbb{R}^{n+1}$ on $\mathscr{H}$.

3. Construction of the Representation of the Orthochronous Lorentz Group on $\mathscr{H}$

A. The Analytic Continuation from $O(n+1)$ to $O^{\uparrow}(1,n)$

Let $\hat{s}=(1,0)\in\mathbb{R}^{n+1}$, we identify the subgroup of $O(n+1)$, leaving $\hat{s}$ invariant with $O(n)$ in the usual way.

Given a unit vector $\hat{u}\in\mathbb{R}^n\cong\{0\}\times\mathbb{R}^n$ and $\theta\in\mathbb{R}$, we denote by $r(\hat{u},\theta)$ the rotation by the angle θ from $\hat{u}$ to $\hat{s}$, i.e.,

$$\begin{aligned} r(\hat{u},\theta)\hat{s} &= \cos\theta\,\hat{s}-\sin\,\theta\hat{u},\\ r(\hat{u},\theta)\hat{u} &= \cos\theta\hat{u}+\sin\theta\,\hat{s},\\ r(\hat{u},\theta)x &= x \quad \text{if } x \text{ is perpendicular to } \hat{s} \text{ and } \hat{u}. \end{aligned}$$

The $(n+1)\times(n+1)$ matrix corresponding to $r(\hat{u},\theta)$ is

$$\begin{aligned} r(\hat{u},\theta)_{i,j} = \delta_{i,j} &+ (\delta_{i,0}\delta_{0,j}+u_iu_j)(\cos\theta-1)+\\ &+(\delta_{i,0}u_j-\delta_{0,j}u_i)\sin\theta, \end{aligned} \tag{1}$$

where $i,j=0,1,\ldots,n$, the index 0 corresponding to the $\hat{s}$ axis, and $\hat{u}=\sum_{i=1}^{n}u_i\hat{x}_i$, $\hat{x}_i$ denoting the unit vector in the x_i direction.

Notice that if $g\in O(n+1)$, then

$$g\hat{s}=\cos\theta\hat{s}-\sin\theta\hat{u}=r(\hat{u},\theta)\,\hat{s}$$

for some θ and $\hat{u}$, which implies

$$g=r(\hat{u},\theta)G \tag{2}$$

for some $\theta\in\mathbb{R}$, $\hat{u}$ unit vector in $\mathbb{R}^n$, $G\in O(n)$.

If $k(\hat{u})\in O(n)$ is a rotation taking $\hat{u}$ into $\hat{x}_1$,

$$r(\hat{u},\theta)=k(\hat{u})^{-1}r(\hat{x}_1,\theta)k(\hat{u}). \tag{3}$$

If $\hat{u}\neq-\hat{x}_1$, we may take $k(\hat{u})$ as the rotation that takes $\hat{u}$ into $\hat{x}_1$ and leaves invariant vectors perpendicular to $\hat{u}$ and $\hat{x}_1$. In this case $k(\hat{u})$ has the matrix

$$k(\hat{u})_{i,j}=\delta_{i,j}-(1+u_1)^{-1}(u_j+\delta_{1,j})(u_i+\delta_{i,1})+2u_j\delta_{i,1}, \tag{4}$$

$i,j=0,\ldots,n$. Notice that $(k(\hat{u})^{-1})_{i,j}=k(\hat{u})_{j,i}$.

The group property is expressed by

$$r(\hat{u}_1,\theta_1)r(\hat{u}_2,\theta_2)=r(\hat{u},\theta)G(\theta_1,\theta_2), \tag{5}$$

where

$$\cos\theta=\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\cos\phi, \tag{6}$$

and

$$\hat{u}=(\sin\theta)^{-1}[(\cos\theta_2\sin\theta_1-\sin\theta_2(1-\cos\theta_1)\cos\phi)\hat{u}_1+\sin\theta_2\hat{u}_2] \tag{7}$$

if $\sin\theta \neq 0$. We have set $\cos\phi = \hat{u}_1 \cdot \hat{u}_2$. Thus $G(\theta_1, \theta_2)$ is in $O(n)$ and is given by

$$G(\theta_1, \theta_2) = r(\hat{u}, -\theta) r(\hat{u}_1, \theta_1) r(\hat{u}_2, \theta_2). \tag{8}$$

A similar discussion applies to the orthochronous Lorentz group $O^{\uparrow}(1, n)$. If $\hat{t} = (1, 0)$, we again identify the subgroup of $O^{\uparrow}(1, n)$ leaving $\hat{t}$ invariant with $O(n)$ in the usual way.

Given a unit vector $\hat{u} \in \mathbb{R}^n \cong \{0\} \times \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we let $b(\hat{u}, \alpha)$ denote the Lorentz boost with hyperbolic angle α in the $\hat{u}, \hat{t}$ plane, i.e.,

$$\begin{aligned} b(\hat{u}, \alpha)\hat{t} &= \cosh\alpha \hat{t} + \sinh\alpha \hat{u}, \\ b(\hat{u}, \alpha)\hat{u} &= \cosh\alpha \hat{u} + \sinh\alpha \hat{t}, \\ b(\hat{u}, \alpha)x &= x \text{ if } x \text{ is perpendicular to } \hat{u} \text{ and } \hat{t}. \end{aligned}$$

The corresponding $(n+1) \times (n+1)$ matrix is

$$\begin{aligned} b(\hat{u}, \alpha)_{i,j} = \delta_{i,j} &+ (\delta_{i,0}\delta_{0,j} + \hat{u}_i \hat{u}_j)(\cosh\alpha - 1) + \\ &+ (\delta_{i,0}\hat{u}_j + \delta_{0,j}\hat{u}_i)\sinh\alpha, \end{aligned} \tag{9}$$

$i, j = 0, 1, \ldots, n$.

Notice that given the matrix $b(\hat{u}, \alpha)$, $\cosh\alpha$ and $\sinh\alpha\, \hat{u}$ are uniquely determined, so that if $\alpha \neq 0$, α and $\hat{u}$ are determined up to a sign: $b(\hat{u}, \alpha) = b(-\hat{u}, -\alpha)$.

Every $h \in O^{\uparrow}(1, n)$ may be written

$$h = b(\hat{u}, \alpha) H \tag{10}$$

for some unit vector $\hat{u} \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, and $H \in O(n)$.

Again,

$$b(\hat{u}, \alpha) = k(\hat{u})^{-1} b(\hat{x}_1, \alpha) k(\hat{u}),$$

as in (3).

The group property is now expressed by

$$b(\hat{u}_1, \alpha_1) b(\hat{u}_2, \alpha_2) = b(\hat{v}, \alpha) H(\alpha_1, \alpha_2), \tag{11}$$

where

$$\cosh\alpha = \cosh\alpha_1 \cosh\alpha_2 + \sinh\alpha_1 \sinh\alpha_2 \cos\phi, \tag{12}$$

and

$$\hat{v} = (\sinh\alpha)^{-1}[(\cosh\alpha_2 \sinh\alpha_1 + \sinh\alpha_2(\cosh\alpha_1 - 1)\cos\phi)\hat{u}_1 + \sinh\alpha_2 \hat{u}_2], \tag{13}$$

if $\alpha \neq 0$, where $\cos\phi = \hat{u}_1 \cdot \hat{u}_2$ as before.

Notice that the matrix $r(\hat{u}, z)$, given by (1) with the complex variable z replacing θ, is an entire function of z, and if we take $z = -i\alpha$ purely imaginary,

$$T r(\hat{u}, -i\alpha) T^{-1} = b(\hat{u}, \alpha), \tag{14}$$

where

$$T = \begin{pmatrix} -i & 0 \\ 0 & I \end{pmatrix}, \tag{15}$$

I being the $n \times n$ identity matrix.

The analytic continuation from the rotation group $O(n+1)$ to the orthochronous Lorentz group $O^{\uparrow}(1,n)$ is based on Eq. (14). We will show that Eq. (5) analytically continues to Eq. (11) as the complex variable z goes from $z=\theta$ to $z=-i\alpha$ (inserting the matrix T as in Eq. (14); the insertion of the matrix T accounts for taking imaginary time to real time, i.e., s goes to $-it$).

Let ϕ be a fixed angle (as in (6) or (12)) and define the entire function $W(z_1, z_2)$ of two complex variables by

$$\begin{aligned} W(z_1, z_2) &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \cos \phi \\ &= a \cos(z_1 + z_2) + b \cos(z_1 - z_2), \end{aligned} \tag{16}$$

where $a = \cos^2 \phi/2$, $b = \sin^2 \phi/2$.

Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be given such that not both are zero, and let $0 < \theta_0 < \pi/4$. We now choose $\theta_1, \theta_2 \in \mathbb{R}$ so that $0 < |\theta_1|, |\theta_2| < \theta_0$ and $\theta_1 - \theta_2$ is non-zero with the same sign as $\alpha_1 - \alpha_2$ and $\theta_1 + \theta_2$ is non-zero with the same sign as $\alpha_1 + \alpha_2$.

Define the path $\gamma:[0,1] \to \mathbb{C} \times \mathbb{C}$ by $\gamma(\tau) = (z_1(\tau), z_2(\tau))$, where $z_j(\tau) = (1-\tau)\theta_j - i\tau\alpha_j, j = 1, 2$. It is easily checked that along γ $W(z_1, z_2) \neq \pm 1$ and $\operatorname{Re} W(z_1, z_2) > 0$.

We may then analytically continue the function

$$\theta = \arccos W(\theta_1, \theta_2), \quad -\pi/2 < \theta < \pi/2,$$

along γ as

$$z = \theta - i\alpha = \arccos W(z_1, z_2), \quad -\pi/2 < \operatorname{Re} z < \pi/2, \tag{17}$$

to

$$-i\alpha = \arccos W(-i\alpha_1, -i\alpha_2). \tag{18}$$

But (18) says that

$$\alpha = \operatorname{arc\,cosh}(a \cosh(\alpha_1 + \alpha_2) + b \cosh(\alpha_1 - \alpha_2)),$$

Where α is real since $-\pi/2 < \operatorname{Re}(-i\alpha) < \pi/2$.

We have thus analytically continued from Eq. (6) to Eq. (12).

Remark. More generally, it follows from (16) and the convexity of the image under the cosine of the region $|\operatorname{Re} z| < \varepsilon$, that if $|\theta_1 \pm \theta_2| < \varepsilon$, then $|\operatorname{Re} z| < \varepsilon$ along the path γ, z given by (17). This holds for any $\varepsilon < \pi/2$.

We now show that Eq. (7) also analytically continues to Eq. (13) along the path γ. Let

$$u(z_1, z_2) = (\sin z)^{-1}[(\cos z_2 \sin z_1 - \sin z_2(1 - \cos z_1)\cos\phi)\hat{u}_1 + \sin z_2 \hat{u}_2], \tag{19}$$

where z is given by (17). Since $\sin z \neq 0$ along the path $\gamma(W(z_1, z_2) \neq \pm 1$ along γ), (19) gives an analytic continuation from $\hat{u} = u(\theta_1, \theta_2)$ (Eq. (7)) to $\hat{v} = u(-i\alpha_1, -i\alpha_2)$ (Eq. (13)) along γ.

We can now analytically continue Eq. (5) along the path γ as $r(\hat{u}_1, z_1)r(\hat{u}_2, z_2) = r(u, z)G(z_1, z_2)$, where z is given by (17), u by (19), and $G(z_1, z_2)$ by analytic continuation of (8).

It this way we have analytically continued Eq. (5) to Eq. (11) (inserting the matrix T as in Eq. (14)). Since the *matrix* $b(\hat{v},\alpha)$ is uniquely determined by $b(\hat{u}_1,\alpha_1)$ and $b(\hat{u}_2,\alpha_2)$, it follows that the resulting α and $\hat{v}$ are unique up to sign.

B. The Analytic Continuation of the Unitary Representation of $O(n+1)$ on $\mathscr{K}$ to a Unitary Representation of $O^{\uparrow}(1,n)$ on $\mathscr{H}$

If $g\in O(n)$, then $U(g)$ commutes with both R and E_+. Thus if we define $V(g)$ on $\mathscr{H}_0$ by

$$V(g)\mathscr{V}=\mathscr{V}U(g)\quad \text{on } \mathscr{K}_+,$$

it follows that $V(g)$ is a well defined isometry of $\mathscr{H}_0$ onto itself and $V(g)$ extends to a continuous unitary representation of $O(n)$ on $\mathscr{H}$.

We now construct the unitary operators corresponding to Lorentz boosts $b(\hat{u},\alpha)$. For $0\leqq\theta<\pi/2$, let C_θ denote the open cone in $\mathbb{R}^{n+1}$ having its axis in the $\hat{s}$ direction and with half-angle $\pi/2-\theta$:

$$C_\theta=\{x\in\mathbb{R}^{n+1};\quad x\cdot\hat{s}>|x|\sin\theta\}.$$

Note that $C_\theta\subset(0,\infty)\times\mathbb{R}^n$ and $r(\hat{u},\theta')C_\theta\subset C_{\theta-|\theta'|}$ if $|\theta'|\leqq\theta$ for all $\hat{u}$.

Define $\mathscr{K}_\theta=E(C_\theta)\mathscr{K}=E(C_\theta)\mathscr{K}_+\subset\mathscr{K}_+$ and let $\mathscr{D}_\theta=\mathscr{V}\mathscr{K}_\theta\subset\mathscr{H}_0$. Now define the linear operator $P(\hat{u},\theta)$ on $\mathscr{H}_0$ having domain $\mathscr{D}_\theta$ by

$$P(\hat{u},\theta)\mathscr{V}=\mathscr{V}U(r(\hat{u},\theta))\quad \text{on } \mathscr{K}_\theta.$$

Here $P(\hat{u},\theta)$ is well defined by the same argument as in the proof of Lemma 8.2 in Klein and Landau [3]. Furthermore, $P(\hat{u},\theta)$ is symmetric on $\mathscr{D}_\theta$: let $F,G\in\mathscr{K}_\theta$, then

$$\begin{aligned}\langle\mathscr{V}F|P(\hat{u},\theta)\mathscr{V}G\rangle&=(F,RU(r(\hat{u},\theta))G)=(F,U(r(\hat{u},-\theta))RG)\\&=(U(r(\hat{u},\theta))F,RG)=\langle P(\hat{u},\theta)\mathscr{V}F|\mathscr{V}G\rangle.\end{aligned}$$

Since $U(r(\hat{u},\theta'))$ is strongly continuous in θ' for fixed $\hat{u}$, it follows that $P(\hat{u},\theta')$ is strongly continuous in $\mathscr{D}_\theta$ for $0\leqq\theta'\leqq\theta$. Notice also that

$$\bigcup_{0<\theta<\pi/2}\mathscr{D}_\theta=\mathscr{V}\left(\bigcup_{0<\theta<\pi/2}\mathscr{K}_\theta\right)$$

and hence is dense in $\mathscr{H}$.

We can thus conclude that, for fixed $\hat{u}$, $(P(\hat{u},\theta),\mathscr{D}_\theta,\pi/2)$ form a symmetric local semigroup, so there is a unique self-adjoint operator $L(\hat{u})$ on $\mathscr{H}$ such that $\mathscr{D}_\theta$ is contained in the domain of $e^{-\theta L(\hat{u})}$ and $P(\hat{u},\theta)$ is the restriction of $e^{-\theta L(\hat{u})}$ to $\mathscr{D}_\theta$ (Klein and Landau [4]).

In addition, on $\mathscr{K}_\theta$ we have that

$$P(-\hat{u},\theta/2)P(\hat{u},\theta/2)\mathscr{V}=\mathscr{V}U(r(-\hat{u},\theta/2)r(\hat{u},\theta/2))=\mathscr{V},$$

since $r(-\hat{u},\theta/2)=r(\hat{u},-\theta/2)$. It follows that

$$L(-\hat{u})=-L(\hat{u}). \tag{20}$$

Now let $h\in O^{\uparrow}(1,n)$, then we can write $h=b(\hat{u},\alpha)H$ (see (10)). Recall that the matrices of the Lorentz boost $b(\hat{u},\alpha)$ and of $H\in O(n)$ are uniquely determined, $\hat{u}$ and α

being determined up to sign: $b(-\hat{u},\alpha)=b(\hat{u},-\alpha)$. Thus

$$V(h)=e^{i\alpha L(\hat{u})}V(H)$$

gives a well defined map from $O^{\uparrow}(1,n)$ to unitary operators on $\mathscr{H}$ by (20).

It follows from the construction of $L(\hat{u})$ and Eq. (3) that

$$e^{i\alpha L(\hat{u})}=V(k(\hat{u})^{-1})\exp(i\alpha L(\hat{x}_1))V(k(\hat{u})). \tag{21}$$

We must still show that V gives a representation of $O^{\uparrow}(1,n)$; to do so it suffices to prove that (compare with Eq. 11))

$$\exp(i\alpha_1 L(\hat{u}_1))\exp(i\alpha_2 L(\hat{u}_2))=e^{i\alpha L(\hat{v})}V(H), \tag{22}$$

where $\alpha,\hat{v},H$ are given by Eq. (12), (13), and (11), respectively.

So let $0<\varepsilon<\pi/2$, and choose θ_1 and θ_2 so that $|\theta_1|+|\theta_2|<\varepsilon$. From Eq. (5) and the construction of $L(\hat{u})$, we have that the equality

$$\exp(-\theta_1 L(\hat{u}_1))\exp(-\theta_2 L(\hat{u}_2))=e^{-\theta L(\hat{u})}V(G) \tag{23}$$

holds in $\mathscr{D}_\varepsilon$, where $\theta,\hat{u}$, and G are given by Eqs. (6), (7) and (8), respectively, and θ_1,θ_2 are chosen so that $|\theta|<\varepsilon$.

We now rewrite (23) using (21):

$$\exp(-\theta_1 L(\hat{u}_1))\exp(-\theta_2 L(\hat{u}_2))=V(k(\hat{u})^{-1})\exp(-\theta L(\hat{x}_1))V(k(\hat{u}))V(G), \tag{24}$$

again holding in $\mathscr{D}_\varepsilon$.

So let us analytically continue in the complex variables $z_j=\theta_j-i\alpha_j, j=1,2$, from $z_j=\theta_j$ to $z_j=-i\alpha_j, j=1,2$, as in subsection A, and obtain (22) from (24). To do so, having fixed α_1, α_2, we choose θ_1,θ_2 satisfying the above conditions and the conditions after Eq. (15), and do the analytic continuation along the path γ defined below Eq. (15).

To do this notice that the closed subspace $\mathscr{K}_\varepsilon$ is invariant under $U(g)$ for $g\in O(n)$. Since $O(n)$ is a compact group, $\mathscr{K}_\varepsilon$ can be decomposed into finite-dimensional representations of $O(n)$. Let $F'_j, j=1,\dots,N'$, and $F''_\ell, \ell=1,\dots,N''$, be basis vectors of two such finite-dimensional sub-representations, and let $f'_j=\mathscr{V}F'_j, f''_\ell=\mathscr{V}F''_\ell$. Then Eq. (24) implies that

$$\langle\exp(-\theta_1 L(\hat{u}_1))f'_j|\exp(-\theta_2 L(\hat{u}_2))f''_\ell\rangle=$$
$$\sum_{j'=1}^{N'}\sum_{\ell'=1}^{N''}\Gamma'(k(\hat{u})^{-1})_{j',j}\Gamma''(k(\hat{u})G)_{\ell,\ell'}\langle f'_{j'}|\exp(-\theta L(\hat{x}_1))f''_{\ell'}\rangle, \tag{25}$$

where $\Gamma'(g)_{j',j}$, $\Gamma''(g)_{\ell,\ell'}$ are polynomials in the matrix element of $g\in O(n)$.

We may now analytically continue Eq. (25) along the path γ. This analytic continuation is based on three facts:

(i) $\sin z\neq 0$ along γ.

(ii) $|\mathrm{Re}\, z_1|+|\mathrm{Re}\, z_2|<\varepsilon$ and $|\mathrm{Re}\, z|<\varepsilon$ along γ by the construction of the path γ and the remark following Eq. (18).

(iii) Using (21) it suffices to prove (22) with $\hat{u}_1=\hat{x}_1$. In this case, if $\hat{u}_2\neq\pm\hat{x}_1$ (in which case (22) is obvious), it follows that $\hat{x}_1 \cdot u \neq -1$($u$ is given by (19)) along the path γ and so $k(u)$ is analytic along γ.

Since the above construction can be done for every $0 < \varepsilon < \pi/2$, we conclude that (22) holds on $\mathscr{H}_0$ and hence on $\mathscr{H}$.

Thus

$$V(h) = e^{i\alpha L(\hat{u})} V(H) = V(k(\hat{u})^{-1}) \exp(i\alpha L(\hat{x}_1)) V(k(\hat{u})H) \tag{26}$$

for $h = b(\hat{u}, \alpha)H$ defines a unitary representation of $O^{\uparrow}(1,n)$ on $\mathscr{H}$.

The continuity of the representation follows. For let $h = b(\hat{u}, \alpha)H \in O^{\uparrow}(1,n)$ with $\alpha \neq 0$, $\hat{u} \neq -\hat{x}_1$. Then we can find a neighborhood of h in $O^{\uparrow}(1,n)$ such that these conditions hold and $\alpha, \hat{u}, H$ are continuous functions of h. The strong continuity of V in this neighborhood then follows from (26). The continuity of V around an arbitrary $h \in O^{\uparrow}(1,n)$ now follows from the group property.

4. Combining the Lorentz Group with Translations

A. The Spectral Condition

Since U is a representation of the inhomogeneous $IO(n+1)$,

$$U(r(\hat{u}, \theta)) U(\tau(x)) U(r(\hat{u}, -\theta)) = U(\tau(\hat{u}, \theta)x)), \tag{27}$$

where as before $\tau(x)$ denotes translation by $x \in \mathbb{R}^{n+1}$.

Taking $x = (s/\cos\theta, 0)$ in (27) gives

$$\begin{aligned} U(r(\hat{u}, \theta)) U(\tau(s/\cos\theta, 0)) U(r(\hat{u}, -\theta)) = \\ = U(\tau(s, -s(\tan\theta)\hat{u})). \end{aligned} \tag{28}$$

Let $0 < \varepsilon < \pi/2$, $f \in \mathscr{D}_\varepsilon$ and $|\theta| \leqq \varepsilon$. Then (28) leads to

$$\langle e^{-\theta L} f | e^{-(s/\cos\theta)H} e^{\theta L} f \rangle = \langle e^{-(s/2)H} f | e^{-is(\tan\theta)P} e^{-(s/2)H} f \rangle, \tag{29}$$

where $L = L(\hat{u})$, $P = \mathbf{P} \cdot \hat{u}$.

Since $f \in \mathscr{D}_\varepsilon \subset \mathscr{D}(e^{-\varepsilon L}) \cap \mathscr{D}(e^{\varepsilon L})$, $e^{(\theta - i\alpha)L}$ is an analytic vector-valued function of $\theta - i\alpha$ for $|\theta| < \varepsilon$. Moreover, e^{-zH} is operator-norm analytic for $\mathrm{Re}\, z > 0$, because $H \geqq 0$. Since $\mathrm{Re}[\cos(\theta - i\alpha)]^{-1} > 0$ for $|\theta| < \pi/2$, it follows that $e^{-s[\cos(\theta - i\alpha)]^{-1} H}$ is operator-norm analytic for $|\theta| < \pi/2$. Therefore the left-hand side of (29) has an analytic continuation $G(\theta - i\alpha)$ for $|\theta| < \varepsilon$,

$$G(\theta - i\alpha) = \langle e^{-(\theta + i\alpha)L} f | e^{-s[\cos(\theta - i\alpha)]^{-1} H} e^{(\theta - i\alpha)L} f \rangle.$$

In particular,

$$G(-i\alpha) = \langle e^{-i\alpha L} f | e^{-s(\cosh\alpha)^{-1} H} e^{-i\alpha L} f \rangle. \tag{30}$$

Since $\tan(\theta - i\alpha)$ is a one-to-one conformal mapping of the strip $|\theta| < \varepsilon$ onto an open neighborhood of $\{z = iy, -1 < y < 1\}$, it follows from (29) and the lemma in the Appendix that $e^{-(s/2)H} f \in \mathscr{D}(e^{-i(s/2)(\tan(\theta - i\alpha))P})$ for $|\theta| < \varepsilon$, and

$$G(\theta - i\alpha) = \langle e^{-(s/2)H} f | e^{-is(\tan(\theta - i\alpha))P} e^{-(s/2)H} f \rangle.$$

In particular,

$$G(-i\alpha) = \langle e^{-(s/2)H} f | e^{-s(\tanh\alpha)P} e^{-(s/2)H} f \rangle, \tag{31}$$

and $e^{-(s/2)H} f \in \mathscr{D}(e^{-(s/2)(\tanh\alpha)P})$.

Combining (30) and (31) we conclude that

$$\| e^{-(s/2)(\tanh\alpha)P} e^{-(s/2)H} f \| \leqq \| f \|$$

for all $f \in \mathscr{D}_\varepsilon$, $0 < \varepsilon < \pi/2$, and $\alpha \in \mathbb{R}$.

Since $\bigcup_{0<\varepsilon<\pi/2} \mathscr{D}_\varepsilon$ is dense in $\mathscr{H}$, it follows that (recall $L = L(\hat{u})$, $P = \mathbf{P} \cdot \hat{u}$)

$$H + (\tanh\alpha)\mathbf{P} \cdot \hat{u} \geqq 0 \tag{32}$$

for all $\alpha \in \mathbb{R}$.

Since (32) holds for all unit vectors $\hat{u} \in \mathbb{R}^n$, we can conclude the *spectral condition*: $H \geqq |P|$.

B. The Orthochronous Poincaré Group.

Using the spectral condition (32), we may now analytically continue (30) and (31) in s to $-\mathrm{it}(\cosh\alpha)$, getting

$$\langle f | e^{i\alpha L} e^{itH} e^{-i\alpha L} f \rangle = \langle f | e^{it[(\cosh\alpha)H + (\sinh\alpha)P]} f \rangle,$$

for all $f \in \mathscr{H}$. It follows that

$$e^{i\alpha L} e^{itH} e^{-i\alpha L} = e^{it[(\cosh\alpha)H + (\sinh\alpha)P]} \tag{33}$$

for all $t, \alpha \in \mathbb{R}$.

Replacing t with $r(\sinh\alpha)^{-1}$ in (33), and rearranging (33), we get

$$e^{i\alpha L} e^{irP} e^{-i\alpha L} = e^{ir[(\cosh\alpha)P + (\sinh\alpha)H]}. \tag{34}$$

Combining (33) and (34) we conclude that

$$V(b(\hat{u}, \alpha)) V(\tau(t, r\hat{u})) V(b(\hat{u}, -\alpha)) = V(\tau(b(\hat{u}, -\alpha)(t, r\hat{u}))). \tag{35}$$

Equation (35) together with the immediate equation

$$V(g) V(\tau(t, x)) V(g)^{-1} = V(\tau(s, gx))$$

for $g \in O(n)$ shows that the representation of the orthochronous Lorentz group $O^\uparrow(1, n)$ combines with the representation of the group of translations on $\mathbb{R}^{n+1}$ to give a representation of the orthochronous Poincaré-group $IO^\uparrow(1, n)$.

This finishes the construction of the continuous unitary representation V of $IO^\uparrow(1, n)$ on $\mathscr{H}$.

Remark. If $\mathscr{K}$ is the complexification of a real Hilbert space $\mathscr{K}'$, and both the representation U of $IO(n+1)$ and the covariant orthogonal projections $E(A)$ leave

$\mathscr{K}'$ invariant, we may define an anti-unitary operator T on $\mathscr{H}$ corresponding to time reversal. For if we let $\mathscr{H}'$ be the closure of $\mathscr{V}\mathscr{K}'_+$, then $\mathscr{H}'$ is a real Hilbert space of which $\mathscr{H}$ is the complexification. By construction e^{-tH}, $V(\tau(0,x))$, $e^{-\theta L(\hat{u})}$ and $V(g)$ for $g \in O(n)$ leave $\mathscr{H}'$ invariant. Thus if T is conjugation with respect to the decomposition $\mathscr{H} = \mathscr{H}' + i\mathscr{H}'$, it follows that T is time reversal.

II. The Covering Groups

A similar construction can be carried out in the case of the spinorial two-fold covering groups of the Euclidean and Poincaré groups. This case is appropriate when dealing with fermions.

We begin with a discussion of the covering groups and their relation via analytic continuation. These groups are most easily considered as subgroups of a complex Clifford algebra (e.g., Atiyah, Bott and Shapiro [8]).

Definition. For $n = 1, 2, 3, \cdots$ the complex Clifford algebra C_{n+1} is generated (over $\mathbb{C}$) by $e_0, e_1, \ldots, e_n$ which satisfy the relations

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i,j = 0, 1, 2, \ldots, n.$$

Thus an independent basis for C_{n+1} is

$$\{e_{i_1} \cdots e_{i_r} \mid 0 \leqq i_1 < i_2 < \cdots < i_r \leqq n;\ r = 1, 2, \ldots, n\} \cup \{1\}.$$

□

For any element $z = (z^0, x) = (z^0, z^1, \ldots, z^n) \in \mathbb{C}^{n+1}$, we write

$$e(z) = \sum_{j=0}^{n} z^j e_j.$$

We now define two $n+1$ dimensional real-linear subsets of C_{n+1} which will correspond to Euclidean or relativistic space-time:

$$E^{n+1} = \{e(s,x);\quad (s,x) \in \mathbb{R}^{n+1}\},$$
$$E^{1,n} = \{e(-it,x);\quad (t,x) \in \mathbb{R}^{n+1}\}.$$

We also define subsets $S^{n+1} \subset E^{n+1}$, $S^{1,n} \subset E^{1,n}$ by

$$S^{n+1} = \{e(s,x);\quad s^2 + x^2 = 1\},$$
$$S^{1,n} = \{e(-it,x);\quad -t^2 + x^2 = 1\}.$$

Note that the elements of $S^{n+1}(S^{1,n})$ are invertible in $S^{n+1}(S^{1,n})$:

$$e(s,x)^{-1} = e(-s,-x), e(-it,x)^{-1} = e(-it,-x).$$

Definition. $\mathrm{Pin}(n+1)$ is the group consisting of finite products of elements from S^{n+1} and $\mathrm{Spin}(n+1)$ is the subgroup consisting of products of an even number of elements from S^{n+1}. Similarly $\mathrm{Pin}(1,n)$ is the group generated by $S^{1,n}$ and $\mathrm{Spin}(1,n)$ is the subgroup with an even number of elements from $S^{1,n}$. □

Now each element $e(z)\in S^{n+1}$ defines a linear mapping of E^{n+1} by

$$e(z)e(w)e(z)=e(\phi^E(z)W).$$

Then ϕ^E extends to a homomorphism of $\mathrm{Pin}(n+1)$ into linear mappings of E^{n+1}. Indeed ϕ^E is a $2-1$ homomorphism of $\mathrm{Pin}(n+1)$ onto $O(n+1)$. This follows from the fact that only ± 1 are mapped to the identity, and that $\phi^E(z)$ is reflection through the hyperplane perpendicular to z, while $O(n+1)$ is generated by reflections. Thus ϕ^E is also a $2-1$ homomorphism of $\mathrm{Spin}(n+1)$ onto $\mathrm{SO}(n+1)$. It also follows that $\mathrm{Pin}(n+1)$ and $\mathrm{Spin}(n+1)$ are compact groups since they are $2-1$ coverings of compact groups.

Similarly, each element $e(z)\in S^{1,n}$ defines a linear mapping of $E^{1,n}$ by

$$e(z)e(w)e(z)=e(\phi^R(z)w),$$

and ϕ^R extends to a $2-1$ homomorphism of $\mathrm{Pin}(1,n)$ onto $O^{\uparrow}(1,n)$ and of $\mathrm{Spin}(1,n)$ onto $\mathrm{SO}(1,n)$. This follows from the fact that every element of $O^{\uparrow}(1,n)$ can be written as a product (10) and the Lorentz boosts will be obtained explicitly in the following, and that each element of $S^{1,n}$ preserves the sign of the $\hat{t}$ component of a time-like w.

We now construct explicit elements of $\mathrm{Spin}(n+1)$ and $\mathrm{Spin}(1,n)$ which cover $r(\hat{u},\theta)$ and $b(\hat{u},\alpha)$ defined in Sect. 3.

Definition.
$$\begin{aligned}R(\hat{u},\theta)&=-e\left(\sin\frac{\theta}{4},\cos\frac{\theta}{4}\hat{u}\right)e\left(-\sin\frac{\theta}{4},\cos\frac{\theta}{4}\hat{u}\right)\\&=\left(\cos\frac{\theta}{2}\right)+\left(\sin\frac{\theta}{2}\right)e(\hat{u})e_0,\\B(\hat{u},\alpha)&=-e\left(-i\sinh\frac{\alpha}{4},\cosh\frac{\alpha}{4}\hat{u}\right)e\left(i\sinh\frac{\alpha}{4},\cosh\frac{\alpha}{4}\hat{u}\right)\\&=\left(\cosh\frac{\alpha}{2}\right)+\left(\sinh\frac{\alpha}{4}\right)e(\hat{u})[-ie_0].\end{aligned}$$
□

Then for fixed $\hat{u}$, $R(\hat{u},\theta)$ and $B(\hat{u},\alpha)$ are one-parameter groups and

$$\begin{aligned}R(-\hat{u},-\theta)&=R(\hat{u},\theta),\\B(-\hat{u},-\alpha)&=B(\hat{u},\alpha),\\R(\hat{u},\theta+2\pi)&=-R(\hat{u},\theta),\\R(\hat{u},\theta+4\pi)&=R(\hat{u},\theta),\\\phi^E(R(\hat{u},\theta))&=r(\hat{u},\theta),\\\phi^R(B(\hat{u},\alpha))&=b(\hat{u},\alpha).\end{aligned}$$

Since every element of $O(n+1)$ has the form (2), it follows that every element of $\mathrm{Pin}(n+1)$ has the form

$$R(\hat{u},\theta)\mathcal{G} \tag{36}$$

for some $\theta\in\mathbb{R}$, $\hat{u}$ a unit vector in $\mathbb{R}^n$, $\mathcal{G}\in\mathrm{Pin}(n)$. We also have (as in (3))

$$R(\hat{u},\theta)=K(\hat{u})^{-1}R(\hat{x}_1,\theta)K(\hat{u}), \tag{37}$$

where $K(\hat{u}) \in \mathrm{Spin}(n)$ covers $k(\hat{u}) \in \mathrm{SO}(n)$. We may take

$$K(\hat{u}) = [2(1 + \hat{u}\cdot x_1)]^{-1/2}[1 - e(\hat{u})e_1],$$
$$K(\hat{u})^{-1} = [2(1 + \hat{u}\cdot \hat{x}_1)]^{-1/2}[1 - e_1 e(\hat{u})]. \tag{38}$$

In a similar way every element of $\mathrm{Pin}(1, n)$ has the form

$$B(\hat{u}, \alpha)\mathscr{H} \tag{39}$$

for some $\alpha \in \mathbb{R}$, $\hat{u}$ a unit vector in $\mathbb{R}^n$, $\mathscr{H} \in \mathrm{Pin}(n)$, and

$$B(\hat{u}, \alpha) = K(\hat{u})^{-1} B(\hat{x}_1, \alpha) K(\hat{u}). \tag{40}$$

With these results, the analytic continuation from $\mathrm{Pin}(n+1)$ to $\mathrm{Pin}(1, n)$ is carried out along the path γ in the parameter space (z_1, z_2) from $z_j = \theta_j$ to $z_j = -i\alpha_j$ as in Sect. 3A in Part I.

We now consider the analytic continuation of a unitary representation of $\mathrm{Pin}(n+1)$ on $\mathscr{K}$ to a unitary representation of $\mathrm{Pin}(1, n)$ on $\mathscr{H}$.

Let U be the unitary representation of $\mathrm{Pin}(n+1)$ on $\mathscr{K}$. The covariance of the orthogonal system of propections in expresed by

$$U(g)E(A)U(g)^{-1} = E(\phi^E(g)A)$$

for all $g \in \mathrm{Pin}(n+1)$ and open sets $A \subset \mathbb{R}^{n+1}$.

We note that $U(-1)$ belongs to the center of the representation and commutes with $E(A)$ for all $A \subset \mathbb{R}^{n+1}$.

Osterwalder–Schrader positivity holds with respect to the reflection operator $R: E_+ R E_+ \geqq 0$.

The operator R cannot be $U(e_0)$, since if

$$E_+ U(e_0) E_+ \geqq 0,$$

it must be self-adjoint and hence taking adjoints yields

$$E_+ U(e_0) E_+ = E_+ U(-e_0) E_+ = U(-1) E_+ U(e_0) E_+.$$

From this it follows that $V(-1) = I$ on $\mathscr{H}$. In other words the two-fold covering of $O(n+1)$ is extraneous and is lost in the construction of the representation of $\mathrm{Pin}(1, n)$ on $\mathscr{H}$, which is then actually a representation of $O^{\uparrow}(1, n)$. To avoid this we suppose $R = CU(e_0)$, where C is a unitary operator commuting with $U(g)$ for all $g \in \mathrm{Pin}(n+1)$ and with $E(A)$ for all $A \subset \mathbb{R}^{n+1}$, so R will act by conjugation as time-reversal. Also, $C^2 = U(-1)$, which implies that R is a unitary involution: $R^2 = I$.

C can be interpreted as charge conjugation (Osterwalder and Schrader [7]).

We now proceed as in Sect. 3B, defining $P(\hat{u}, \theta)$ by $P(\hat{u}, \theta)\mathscr{V} = \mathscr{V}(U(R(\hat{u}, \theta)))$ on $\mathscr{K}_\theta$. Then there is a unique self-adjoint operator $L(\hat{u})$ on $\mathscr{H}$ such that $P(\hat{u}, \theta)$ is the restriction of $e^{-\theta L(\hat{u})}$ to $\mathscr{D}_\theta$. As in (20), $L(-\hat{u}) = -L(\hat{u})$. The remainder of the argument follows as before.

The inhomogeneous spinorial covering group $\mathrm{IPin}(n+1)$ of the Euclidean group is the semidirect product of $\mathrm{Pin}(n+1)$ with the translation group, satisfying

$$U(R(\hat{u}, \theta))U(\tau(x))U(R(\hat{u}, -\theta)) = U(\tau(r(\hat{u}, \theta)x))$$

(compare (27)). As a result, the representation of IPin$(1,n)$ and the spectral condition follow as in Sect. 4.

As a final remark, we note that the above construction goes through unchanged if we assume only a unitary representation of ISpin$(n+1)$ together with $E(A)$ for $A \subset \mathbb{R}^{n+1}$ and the reflection R with the appropriate properties. We obtain a representation of ISpin$(1,n)$ satisfying the spectral condition.

III. Analytic Continuation of Representations on Topological Vector Spaces

Let $\mathscr{E}$ denote either IO$(n+1)$, ISO$(n+1)$, IPin$(n+1)$ or ISpin$(n+1)$, and $\mathscr{P}$ denote IO$^{\uparrow}(1,n)$, ISO$(1,n)$, IPin$(1,n)$ or ISpin$(1,n)$, respectively. By ϕ we denote either the covering map of IPin$(n+1)$ onto IO$(n+1)$ or the identity map on IO$(n+1)$ or their restrictions to ISpin$(n+1)$ or ISO$(n+1)$ according to the case. We let Θ be either in Pin$(n+1)$ or $O(n+1)$ so that $\phi(\Theta)=\theta$ (time reversal). Notice that $\Theta\,\mathscr{E}\,\Theta^{-1}=\mathscr{E}$ for any of the above choices of $\mathscr{E}$ and the corresponding choice of Θ.

Definition. An Osterwalder–Schrader positive representation of $\mathscr{E}$ with a covariant system of subspaces consists of:

(i) a topological vector space $\mathscr{K}$;

(ii) a map $A \to \mathscr{K}(A)$ from open sets in $\mathbb{R}^{n+1}$ to subspaces of $\mathscr{K}$, such that $A \subset B$ implies $\mathscr{K}(A) \subset \mathscr{K}(B)$ and $A_n \uparrow A$ implies that $\bigcup_n \mathscr{K}(A_n)$ is dense in $\mathscr{K}(A)$;

(iii) a strongly continuous representation U of $\mathscr{E}$ on $\mathscr{K}$ such that $U(g)\mathscr{K}(A) = \mathscr{K}(\phi(g)A)$ for all $g \in \mathscr{E}$ and open $A \subset \mathbb{R}^{n+1}$;

(iv) Osterwalder–Schrader positivity: there is a continuous positive semi-definite inner product $\langle\,|\,\rangle$ on $\mathscr{K}_+ = \mathscr{K}((0,\infty)\times\mathbb{R}^n)$ such that

(a) if $g \in \mathscr{E}$ and $F, G, U(g)F$ and $U(\Theta g^{-1}\Theta^{-1})G$ are in $\mathscr{K}_+$, then

$$\langle G|U(g)F\rangle = \langle U(\Theta g^{-1}\Theta^{-1})G|F\rangle,$$

(b) for each $F \in \mathscr{K}_+$ and $c>0$ $e^{-cs}\langle F|U(\tau(s,0))F\rangle$ is bounded for $s \geqq 0$. □

Given such an Osterwalder–Schrader positive representation of $\mathscr{E}$ we construct as before a unitary representation of the corresponding $\mathscr{P}$ satisfying the spectral condition. The Hilbert space $\mathscr{H}$ is constructed as in Osterwalder and Schrader [6]. If $\mathscr{N} = \{F \in \mathscr{K}_+;\ \langle F|F\rangle = 0\}$, then $\mathscr{N}$ is a closed subspace of $\mathscr{K}_+$ and $\mathscr{H}_0 = \mathscr{K}_+/\mathscr{N}$ is a pre-Hilbert space with inner product $\langle\,|\,\rangle$. We denote by $\mathscr{V}$ the canonical map $\mathscr{K}_+ \to \mathscr{K}_+/\mathscr{N}$ and take $\mathscr{H}$ to be the Hilbert space completion of $\mathscr{H}_0$ (when $\mathscr{K}$ is a Hilbert space this construction is equivalent to the one given in Sect. 1 of Part I). The rest of the construction goes as before with minor modifications. Condition (iv) (b) together with the spectral theorem insures that $H \geqq 0$. Also in Sect. 3B Part I we used the fact that a continuous unitary representation of a compact group on a Hilbert space is a direct sum of finite-dimensional representations. Since $\mathscr{K}_\varepsilon$ is now only a topological vector space, we must modify our argument (notice that the decomposition of a strongly continuous representation of a compact group as a direct sum of finite dimensional representations still holds for representations on quasi-complete barrelled topological vector spaces, see Dieudonne [9]; if we assumed $\mathscr{K}$ to be a barrelled topological vector space we could still use the decomposition of the completion of $\mathscr{K}_\varepsilon$ and carry the argument through). Instead of decomposing $\mathscr{K}_\varepsilon$ we will decompose an enlargement $\tilde{\mathscr{D}}_\varepsilon$ of $\mathscr{D}_\varepsilon$. To do this, recall that $\mathscr{D}_\varepsilon$ is a linear sub-space of the Hilbert space $\mathscr{H}$ that is left invariant by the unitary representation V of the compact group $G = O(n)$ (or $\mathrm{SO}(n)$, or $\mathrm{Pin}(n)$, or $\mathrm{Spin}(n)$). Let $V(\varphi) = \int dg\, \varphi(g) V(g)$ for $\varphi \in C(G)$. Then $V(\varphi)$ does not necessarily leave $\mathscr{D}_\varepsilon$ invariant, so let $\tilde{\mathscr{D}}_\varepsilon$ be the linear span of $\mathscr{D}_\varepsilon \cup \{V(\varphi)f; \varphi \in C(G), f \in \mathscr{D}_\varepsilon\}$. Then $\tilde{\mathscr{D}}_\varepsilon$ is left invariant by both $V(g)$, $g \in G$, and $V(\varphi)$, $\varphi \in C(G)$, and $\mathscr{D}_\varepsilon$ is dense in $\tilde{\mathscr{D}}_\varepsilon$. It follows by the usual proof for unitary representations of compact groups that the unitary representation V of G on the pre-Hilbert space $\tilde{\mathscr{D}}_\varepsilon$ can be decomposed as a direct sum of finite-dimensional representations. Now recall that $\mathscr{D}_\varepsilon \subset \mathscr{D}(e^{-\theta L(\hat{u})})$ for $|\theta| < \varepsilon$ and any unit vector $\hat{u} \in \mathbb{R}^n$, and that

$$e^{-\theta L(\hat{u})} V(g) \mathscr{V} F = \mathscr{V} U(r(\hat{u}, \theta)) U(g) F$$

is a continuous function of $g \in G$ for $F \in \mathscr{K}_\varepsilon$, since the representation U is strongly continuous. Thus, if $f \in \mathscr{D}_\varepsilon$, $\varphi \in C(G)$, $V(\varphi)f = \int dg\, \varphi(g) V(g) f$ and $\int dg\, \varphi(g) e^{-\theta L(\hat{u})} V(g) f$ can be constructed as Riemann integrals. Since $e^{-\theta L(\hat{u})}$ is a closed operator, it follows that $V(\varphi) f \in \mathscr{D}(e^{-\theta L(\hat{u})})$ and $e^{-\theta L(\hat{u})} V(\varphi) f = \int dg\, \varphi(g) e^{-\theta L(\hat{u})} V(g) f$. Hence $\tilde{\mathscr{D}}_\varepsilon \subset \mathscr{D}(e^{-\theta L(\hat{u})})$. Thus if we now take $f'_j, j = 1, \ldots, N'$, and $f''_l, l = 1, \ldots, N''$, to be basis vectors for two finite-dimensional sub-representations of V on $\tilde{\mathscr{D}}_\varepsilon$, Eq. (25) still holds and the rest of the argument goes through as before. Thus the analytic continuation argument of Sect. 3B is still valid.

Remark. The Osterwalder–Schrader axioms give an Osterwalder–Schrader positive representation of the Euclidean group with a covariant system of subspaces (see Osterwalder and Schrader [6]). The above construction can then be applied to give directly the unitary representation of the Poincaré group on the physical Hilbert space.

Appendix. A Technical Lemma

Lemma. *Let $G(x) = \langle f | e^{ixP} f \rangle$ for $x \in \mathbb{R}$, where P is a self-adjoint operator on a Hilbert space $\mathscr{H}$ and $f \in \mathscr{H}$, $f \neq 0$. Suppose there exists a function $G(z)$, analytic in a neighborhood of $I = \{z = iy, -y_1 < y < y_2\}$, where $y_1, y_2 > 0$, such that $g(x) = G(x)$ for $|x| < \varepsilon$, some $\varepsilon > 0$. Then $f \in \mathscr{D}(e^{i(z/2)P})$ for $z \in I' = \{z; -y_1 < \operatorname{Im} z < y_2\}$ and $\langle f | e^{izP} f \rangle$ is an analytic continuation of $G(z)$ to I'.* □

Proof. 1) $f \in \mathscr{D}(P)$:

$$\begin{aligned} \| -ix^{-1}(e^{ixP} - 1) f \|^2 &= x^{-2} \langle f | (2 - e^{ixP} - e^{-ixP}) f \rangle \\ &= x^{-2}(2G(0) - G(x) - G(-x)) \to -G''(0) \end{aligned}$$

as $x \to 0$.

In particular, $\| -ix^{-1}(e^{ixP}-I)f\|$ is bounded as $x \to 0$. It follows that $f \in \mathscr{D}(P)$ and

$$g'(x) = i\langle f|e^{ixP}Pf\rangle, g''(x) = -\langle Pf|e^{ixP}Pf\rangle.$$

2) The hypothesis of the lemma now apply to $g''(x)$ and $G''(z)$. If follows from 1) that $f \in \mathscr{D}(P^2)$ and $g^{(3)}(x) = -i\langle Pf|e^{ixP}P^2f\rangle$ and $g^{(4)}(x) = \langle P^2f|e^{ixP}P^2f\rangle$.

Repeating the argument we get $f \in C_\infty(P)$ and $g^{(n)}(x) = i^n\langle f|e^{ixP}P^nf\rangle$.

3) Since $G(z)$ is analytic in a neighborhood of I, there is a disk of radius r about the origin which is contained in the region of analyticity. It follows that $|G^n(0)| \leqq Cn!/r^n$ for some constant C and hence

$$\|P^nf\| = |G^{(2n)}(0)|^{1/2} \leqq (C(2n)!/r^{2n})^{1/2} \leqq C^{1/2}2^n n!/r^n.$$

It follows that $f \in \mathscr{D}(e^{i(z/2)P})$ for $|z| < r$ and thus $f \in \mathscr{D}(e^{i(z/2)P})$ for $|\mathrm{Im}\, z| < r$.

4) We now repeat the considerations of 1) to 3) with $g(x)$ replaced by

$$g_y(x) = \langle e^{-(y/2)P}f|e^{ixP}e^{-(y/2)P}f\rangle, \quad |y| < r.$$

The disk centered at the origin is replaced by a disk centered at iy.

5) Given any $y', -y_1 < y' < y_2$, say $y' > 0$, we can find $r' > 0$ such that $G(z)$ is analytic in

$$\{z = x + iy; -r' < y < y' + r', \quad |x| < r'\}$$

by compactness. In particular $G(z)$ is analytic in the disk with radius r' centered at iy for $0 \leqq y \leqq y'$. Thus repeating 4) a finite number of times we get that $f \in \mathscr{D}(e^{i(z/2)P})$ for $-r' < \mathrm{Im}\, z < y' + r'$.

Since $y', -y_1 < y' < y_2$ was arbitrary, we conclude that $f \in \mathscr{D}(e^{i(z/2)P})$ for $-y_1 < \mathrm{Im}\, z < y_2$ and the lemma follows. □

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