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A Short Proof of a Kupershmidt-Wilson Theorem

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Abstract. This is a short elementary proof of a statement originally observed by Adler, then pursued by the author, by Kupershmidt and Wilson, and in a more general setting by Drinfeld and Sokolov.

Gardner, Zakharov, and Faddeev have found a Hamiltonian structure for the Korteweg-de Vries equation. Gelfand and the author constructed much more general Hamiltonian structures and integrable equations which were Hamiltonian in those structures. Adler [1] supposed that there is also a second Hamiltonian structure for those equations and gave without proof an expression for this second symplectic form. This conjecture was confirmed by Gelfand and the author [2]. Kupershmidt and Wilson [3] showed that "the second symplectic form" is equivalent to a very simple symplectic form of the Gardner-Zakharov-Miura type and this equivalence is given by a "general Miura transformation" introduced by these authors. This also follows from a recent paper of Drinfeld and Sokolov [4].

Since the theorem of Kupershmidt and Wilson is very important we think that it is useful to give a very simple proof of this theorem by a direct calculation.

We remind the reader the formulation of this theorem. Let R be a ring of formal operators $\sum_{-\infty}^{m} a_i\partial^i$ where a_i belong to a differential algebra $\mathscr{a}$ and ∂^i are symbols; the multiplication is defined by the rules

$$\partial^i\partial^j=\partial^{i+j}, \qquad \partial^i f=f\partial^i+\binom{i}{1}f'\partial^{i-1}+\binom{i}{2}f''\partial^{i-2}+\dots, \qquad f\in \mathscr{a}\,.$$

Let R_+ be a subring of "differential operators" $\sum_0^m a_i\partial^i$, and let R_- be a subring of "Volterra's integral operators," $\sum_{-\infty}^{-1} a_i\partial^i$. We denote

$$\left(\sum_{-\infty}^{m} a_i\partial^i\right)_+=\sum_0^m a_i\partial^i, \qquad \left(\sum_{-\infty}^{m} a_i\partial^i\right)_-=\sum_{-\infty}^{-1} a_i\partial^i, \qquad \operatorname{res}\sum_{-\infty}^{m} a_i\partial^i=a_{-1}\,.$$

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We define the integral $\tilde{f}=\int f\,dx$ in a formal sense as the projection $\mathcal{A}\to\tilde{\mathcal{A}}=\mathcal{A}/\mathcal{A}'$; $\int f\,dx$ is an equivalence class of $f\in\mathcal{A}$ modulo exact derivatives g', where $g\in\mathcal{A}$. It is easy to show that $\int \mathrm{res}\,AB\,dx=\int \mathrm{res}\,BA\,dx$; $A,B\in R$. Now we assume that $\mathcal{A}$ consists of differential polynomials in u_i which are the coefficients of a differential operator $L=\sum_0^n u_i\partial^i$, $u_n=1$. For $\tilde{f}\in\tilde{\mathcal{A}}$ let X_f denote $\sum_0^{n-1}\partial^{-i-1}\delta f/\delta u_i$ $\left[\text{where } \delta f/\delta u_i=\sum_0^\infty(-d/dx)^k\partial f/\partial u_i^{(k)} \text{ are variational derivatives}\right]$. The Poisson bracket of the second kind is defined as

$$\{\tilde{f},\tilde{g}\}=\int \mathrm{res}\{L(X_fL)_+-(LX_f)L\}X_g\,dx. \tag{1}$$

If L is written in a multiplicative form

$$L=\sum_0^n u_i\partial^i=(\partial-v_n)(\partial-v_{n-1})\dots(\partial-v_1), \tag{2}$$

then u_i will be expressed in terms of v_i. This will be called the "Miura transformation." Note that v_i do not belong to $\mathcal{A}$. If $\mathcal{A}_v$ is a differential algebra generated by $\{v_i\}$ the Miura transformation specifies an imbedding $\mathcal{A}$ in $\mathcal{A}_v$.

Theorem.

$$\{\tilde{f},\tilde{g}\}=-\int\sum_{i=1}^n\frac{\delta f}{\delta v_i}\left(\frac{\delta g}{\delta v_i}\right)'dx. \tag{3}$$

Proof. The expression of $\delta f/\delta v_i$ in terms of $\delta f/\delta u_i$ can be obtained from

$$\delta\tilde{f}=\int\sum_0^{n-1}\frac{\delta f}{\delta u_i}\delta u_i\,dx=\int\sum_1^n\frac{\delta f}{\delta v_i}\delta v_i\,dx.$$

We have $\int\sum\delta f/\delta u_i\cdot\delta u_i\,dx=\int \mathrm{res}\,X_f\delta L\,dx$. Denoting $\partial-v_i=\partial_i$ we obtain from (2) an equality

$$\begin{aligned}\int \mathrm{res}\,X_f\delta L\,dx&=-\int \mathrm{res}\,X_f\sum_i\partial_n\dots\partial_{i+1}\delta v_2\partial_{i-1}\dots\partial_1\,dx\\&=-\int \mathrm{res}\sum_i\delta v_i\partial_{i-1}\dots\partial_1X_f\partial_n\dots\partial_{i+1}\,dx,\end{aligned}$$

whence

$$\frac{\delta f}{\delta v_i}=-\mathrm{res}\,\partial_{i-1}\dots\partial_1X_f\partial_n\dots\partial_{i+1}. \tag{4}$$

Now

$$\begin{aligned}\int\sum\frac{\delta f}{\delta v_i}\left(\frac{\delta g}{\delta v_i}\right)'dx&=\int\sum \mathrm{res}(\partial_{i-1}\dots X_f\dots\partial_{i+1})(\mathrm{res}(\partial_{i-1}\dots X_g\dots\partial_{i+1}))dx\\&=\int \mathrm{res}\{\textstyle\sum\partial_{i-1}\dots X_f\dots\partial_{i+1}[\partial,\mathrm{res}(\partial_{i-1}\dots X_g\dots\partial_{i+1})]\}dx.\end{aligned}$$

We can replace here ∂ by ∂_i. From the obvious relations

$$\mathrm{res}\,A=(\partial A_-)_+=(A_-\partial)_+=(\partial_iA_-)_+=(A_-\partial_i)_+$$

we obtain

$$\int \sum \frac{\delta f}{\delta v_i} \left(\frac{\delta g}{\delta v_i} \right)' dx = \int \operatorname{res} \{ \sum \partial_{i-1} \ldots X_f \ldots \partial_i ((\partial_{i-1} \ldots X_g \ldots \partial_{i+1})_- \partial_i)_+ - \sum \partial_i \ldots X_f \ldots \partial_{i+1} (\partial_i (\partial_{i-1} \ldots X_g \ldots \partial_{i+1})_-)_+ \} dx .$$

The subscripts $-$ can be omitted: if we replace them by $+$, the right-hand side will be zero (in this case external subscripts $+$ can be dropped and both the terms mutually cancel). We get

$$\begin{aligned} \int \sum \frac{\delta f}{\delta v_i} \left(\frac{\delta g}{\delta v_i} \right)' dx &= \int \operatorname{res} \{ \sum \partial_{i-1} \ldots X_f \ldots \partial_i (\partial_{i-1} \ldots X_g \ldots \partial_i)_+ \\ &\quad - \sum \partial_i \ldots X_f \ldots \partial_{i+1} (\partial_i \ldots X_g \ldots \partial_{i+1})_+ \} dx \\ &= \int \operatorname{res} \{ X_f \partial_n \ldots \partial_1 (X_g \partial_n \ldots \partial_1)_+ - \partial_n \ldots \partial_1 X_f (\partial_n \ldots \partial_1 X_g)_+ \} dx \\ &= \int \operatorname{res} \{ X_f L (X_g L)_+ - L X_f (L X_g)_+ \} dx \\ &= \int \operatorname{res} \{ L (X_g L)_+ - (L X_g)_+ L \} X_f dx = \{ g, f \} \end{aligned}$$

as stated. □

Remark. We could restrict ourselves by operators L with $u_{n-1} = 0$. In this case $X_f = \sum_0^{n-1} X_i \partial^{-i-1}$, where $X_i = \delta f / \delta u_i$ only for $i \leqq n-2$. Here X_{n-1} is determined from $\operatorname{res}[L, X_f] = 0$ (see [2]). Thus v_i are no longer independent, $\sum v_i = 0$. We define $\delta f / \delta v_i$ to be such a_i that $\delta \tilde{f} = \int a_i \delta v_i dx$ and $\sum a_i = 0$. Then the theorem remains valid. It is only necessary to prove that the sum of right-hand side of (4) is zero. It is sufficient to prove that the derivative of this sum is zero, which can be done by analogy with the proof of the above theorem.

References

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