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Weak Convergence of a Random Walk in a Random Environment

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Abstract. Let $\pi_i(x)$, $i=1,\ldots,d$, $x\in Z^d$, satisfy $\pi_i(x)\geqq\alpha>0$, and $\pi_1(x)+\ldots+\pi_d(x)=1$. Define a Markov chain on Z^d by specifying that a particle at x takes a jump of $+1$ in the i^{th} direction with probability $\frac{1}{2}\pi_i(x)$ and a jump of -1 in the i^{th} direction with probability $\frac{1}{2}\pi_i(x)$. If the $\pi_i(x)$ are chosen from a stationary, ergodic distribution, then for almost all π the corresponding chain converges weakly to a Brownian motion.

1. Introduction

Let Z^d be the integer lattice and let e_i, $i=1,\ldots,d$, denote the unit vector whose i^{th} component is equal to 1. Let

$$S=\{(p_1,\ldots,p_d)\in\mathbb{R}^d:p_i\geqq 0, p_1+\ldots+p_d=1\},$$

and suppose we have a function $\pi:Z^d\to S$. Then a Markov chain $X_\pi(j)$ on Z^d is generated with transition probability

$$P\{X_\pi(j+1)=x\pm e_i\,|\,X_\pi(j)=x\}=\tfrac{1}{2}\pi_i(x), \tag{1.1}$$

and generator

$$L_\pi g(x)=\sum_{i=1}^{d}\tfrac{1}{2}\pi_i(x)\{g(x+e_i)+g(x-e_i)\}.$$

If the function π is chosen from some probability distribution on S, this gives an example of a random walk in a random environment.

For any π, we can consider the limiting distribution of the process X_π satisfying $X_\pi(0)=0$ and (1.1). Let $\alpha>0$ and set

$$S^\alpha=\{(p_1,\ldots,p_d)\in S:p_i\geqq\alpha\},$$

and let C^α be the set of functions $\pi:Z^d\to S^\alpha$. The main result of this paper is:

Theorem 1. *Let μ be a stationary ergodic measure on C^α. Then there exists $b\in S^\alpha$ such*

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that for μ—almost all $\pi \in C^\alpha$, the processes

$$X_\pi^{(n)}(t) = \frac{1}{\sqrt{n}} X_\pi([nt])$$

converge in distribution to a Brownian motion with covariance $(b_i\, \delta_{ij})$

A special case of this theorem occurs when the $\pi(x)$ are independent, identically distributed random variables taking values in S^α.

A similar theorem for diffusion processes with random coefficients was proved by Papanicolaou and Varadhan [3], and a considerable portion of this paper is only a restating of their proof in the context of discrete random walk. The crucial new step is Lemma 4, which replaces Lemma 3.1 of their paper. This is a discrete version of an *a priori* estimate for solutions of uniformly elliptic equations. The ideas of Krylov [2] are used in the proof of Lemma 4; properties of concave functions are used to estimate solutions to a discrete Monge–Ampere equation.

2. An Ergodic Theorem on the Space of Environments

Fix an environment $\pi \in C^\alpha$, and assume $X_\pi(0) = 0$. Let $Z_j = (Z_j^1, \ldots, Z_j^d) = X_\pi(j) - X_\pi(j-1)$, and let $\mathscr{T}_j = \sigma\{Z_1, \ldots, Z_j\}$. Let $Y_j = \pi(X_\pi(j))$. Then Y_j is measurable with respect to $\mathscr{T}_j$, and

$$P\{Z_j = e_i | \mathscr{T}_{j-1}\} = P\{Z_j = -e_i | \mathscr{T}_{j-1}\} = \tfrac{1}{2} Y_j^i.$$

Then $X_\pi(n) = \sum_{j=1}^{n} Z_j$ is a martingale and

$$\mathscr{E}(Z_j^{i_1} Z_j^{i_2} | \mathscr{T}_{j-1}) = \begin{cases} 0 & i_1 \neq i_2 \\ Y_{j-1}^{i_1} & i_1 = i_2 \end{cases}.$$

Let $V_n^i = \sum_{j=0}^{n-1} Y_j^i$. Then the invariance principle for martingales (see e.g. Theorem 4.1 of [1]) states that $W_n(t) = (W_n^1(t), \ldots, W_n^d(t))$ converges in distribution to the standard Brownian motion on $\mathbb{R}^d$, where

$$W_n^i(t) \equiv (V_n^i)^{-1/2} \sum_{j=1}^{[nt]} Z_j^i.$$

Now suppose there exists a $b \in S^\alpha$ such that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \pi(X_\pi(j)) = b \text{ a.s.}.$$

Then by the above argument we can conclude that

$$X^{(n)}(t) = \frac{1}{\sqrt{n}} X_\pi([nt])$$

converges in distribution to a Brownian motion with covariance $(b_i\, \delta_{ij})$. Therefore, in order to prove Theorem 1 it is sufficient to prove:

Theorem 2. *Let μ be a stationary ergodic probability measure on C^α. Then there exists $b \in S^\alpha$ such that for μ—almost all $\pi \in C^\alpha$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \pi(X_\pi(j)) = b \text{ a.s..} \tag{2.1}$$

This is clearly an ergodic theorem and the idea of Papanicolaou and Varadhan [3] is to find a measure on C^α so that a standard ergodic argument can be used.

We define the canonical Markov chain with state space C^α to be the chain whose generator $\mathscr{L}$ is given by

$$\mathscr{L}g(\pi) = \sum_{i=1}^{d} \tfrac{1}{2}\pi_i(0)\{g(\tau_{e_i}\pi) + g(\tau_{-e_i}\pi)\},$$

where $\tau_x \pi(y) = \pi(y-x)$. In this chain, the "particle" stays fixed at the origin and allow the environment to change around it (rather than having the particle move around a fixed environment). If we define $g_0: C^\alpha \to \mathbb{R}^d$ by $g_0(\pi) = \pi(0)$, and let $\mathscr{L}^j\pi$ denote the (random) environment at the j^{th} step of this chain, then (2.1) is equivalent to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} g_0(\mathscr{L}^j \pi) = b \text{ a.s. } \mu. \tag{2.2}$$

By standard ergodic theory we can prove (2.2), and hence (2.1), if we prove:

Theorem 3. *Let μ be a stationary ergodic probability measure on C^α. Then there exists an ergodic probability measure λ on C^α which is mutually absolutely continuous with μ and which is invariant under the canonical Markov chain $\mathscr{L}$.*

Clearly,

$$b = \int_{C^\alpha} g_0(\pi)\, d\lambda(\pi).$$

To prove Theorem 3 we need some lemmas. For each $n > 0$, let T_n denote the elements of Z^d under the equivalence relation

$$(z_1, \ldots, z_d) \sim (w_1, \ldots, w_d) \text{ if } \frac{1}{2n}(z_i - w_i) \in Z \text{ for each } i.$$

Then $|T_n| = (2n)^d$. If $\pi: T_n \to S^\alpha$, we may think of π as a periodic environment in C^α. Let C_n^α denote the set of such periodic environments. For $\pi \in C_n^\alpha$, let R_π^n denote the resolvent operator

$$R_\pi^n g(x) = \sum_{j=0}^{\infty} \left(1 - \frac{1}{n^2}\right)^j L_\pi^j g(x).$$

If $g: T_n \to \mathbb{R}$ we define the usual L^p norms (with respect to normalized counting measure on T_n),

$$\|g\|_p = |(2n)^{-d} \sum_{x \in T_n} (g(x))^p|^{1/p}$$

$$\|g\|_\infty = \sup_{x \in T_n} |g(x)|$$

Lemma 4. *There exists a constant c_1 (depending only on d and α) such that for every $\pi \in C_n^\alpha$, $g: T_n \to R$,*

$$\| R^n g \|_\infty \leqq c_1 n^2 \| g \|_d.$$

The proof of this lemma is delayed until Sect. 3. The next lemma follows from our assumption that μ is stationary (see Parthasarathy [4]).

Lemma 5. *For each n, there exists $\pi_n \in C_n^\alpha$ such that if μ_n is the probability measure on C^α which assigns measure $(2n)^{-d}$ to $\tau_x \pi_n$ for each $x \in T_n$, then*

$$\mu_n \to \mu \quad \textit{weakly.}$$

Proof of Theorem 3. Let $\pi_n \in C_n^\alpha$ be a sequence as in Lemma 5 with $\mu_n \to \mu$. Let ϕ_n be the density, with respect to normalized counting measure on T_n, of an invariant probability measure on T_n for π_n, i.e. $L_{\pi_n} \phi_n = \phi_n$ and $\| \phi_n \|_1 = 1$. If $R_n = R_{\pi_n}^n$ is the resolvent corresponding to π_n, then $R_n \phi_n = n^2 \phi_n$. If we consider R_n as a map from $L^d(T_n)$ to $L^\infty(T_n)$, then Lemma 4 states that the map is bounded by $c_1 n^2$. Therefore $R_n^*: L^1(T_n) \to L^{d/(d-1)}(T_n)$ is also bounded by $c_1 n^2$. Since $R_n^* \phi_n = n^2 \phi_n$, we get

$$n^2 \| \phi_n \|_{d/(d-1)} \leqq c_1 n^2 \| \phi_n \|_1 = c_1 n^2,$$
$$\| \phi_n \|_{a/(d-1)} \leqq c_1.$$

Let λ_n be the probability measure on C_n^α,

$$\lambda_n(\tau_x \pi_n) = (2n)^{-d} \phi_n(x).$$

Then λ_n is invariant under the canonical Markov chain $\mathscr{L}$ and

$$\left\| \frac{d\lambda_n}{d\mu_n} \right\|_{d/(d-1)} \leqq c_1.$$

Since $\mu_n \to \mu$ weakly, standard arguments give that λ_n has a subsequence converging to a probability measure λ which is invariant under $\mathscr{L}$. Also $\lambda \ll \mu$ and, in fact,

$$\int_{C^\alpha} \left| \frac{d\lambda}{d\mu} \right|^{d/(d-1)} d\mu \leqq c_1^{d/(d-1)}.$$

Let $E = \{d\lambda/d\mu = 0\}$. Since λ is invariant, $\lambda(\mathscr{L}E) = \lambda(E) = 0$, and hence $\mathscr{L}E \subset E$ (a.s. μ). Since μ is ergodic and $\lambda \ll \mu$, $\mu(E) = 0$, and hence $\mu \ll \lambda$. Since μ and λ are mutually absolutely continuous and μ is ergodic, λ is ergodic.

Example. Let $d = 2$ and μ be product measure with $\mu\{\pi(x) = \alpha\} = \mu\{\pi(x) = 1 - \alpha\} = \frac{1}{2}$, where $0 < \alpha < \frac{1}{2}$. Then μ is not invariant under $\mathscr{L}$, if $B = \{\pi(e_1) = \alpha\}$, then $\mu(B) = \frac{1}{2}$, but $\mu(\mathscr{L}B) = \frac{3}{8} + \frac{\alpha}{4}$. Although it is not easy to describe λ in this case, symmetry considerations give that $b = (\frac{1}{2}, \frac{1}{2})$.

3. Proof of Lemma 4

It remains to prove Lemma 4. Let

$$D_n = \{(z_1, \ldots, z_d) \in Z^d : |z_1| + \ldots + |z_d| \leqq n\},$$

$$\partial D_n = \{z \in D_n : |z_1| + \ldots + |z_d| = n\},$$

$$\operatorname{int} D_n = D_n / \partial D_n.$$

Let $\pi \in C_n^\alpha$. If $f: D_n \to [0, \infty)$ with $f(x) = 0$ for $x \in \partial D_n$, let

$$Qf(x) = E_x \sum_{j=0}^{\tau} f(X_\pi(j)),$$

where $\tau = \inf\{j : X_\pi(j) \in \partial D_n\}$, and E_x denotes expectation assuming $X_\pi(0) = x$. We will prove the following:

Lemma 6. *There exists a constant c_2 (depending only on d and α) such that for every $f: D_n \to [0, \infty)$,*

$$\|Qf\|_\infty \leqq c_2 n^2 \|f\|_d,$$

where

$$\|f\|_d^d = \frac{1}{|D_n|} \sum_{x \in D_n} (f(x))^d.$$

To get Lemma 4 from Lemma 6 is routine using the fact that the expected time until hitting ∂D_n is of order n^2.

Fix n, and write $D = D_n$. If $u: D \to \mathbb{R}$, we define the second difference operators on $\operatorname{int} D$ by

$$\Delta_i u(x) = u(x + e_i) + u(x - e_i) - 2u(x).$$

We will call u *concave* on D if $\Delta_i u(x) \leqq 0$ for all $x \in \operatorname{int} D$ and all i (note this is weaker than the usual definition of concave). We define the *discrete Monge–Ampere operator M* on $\operatorname{int} D$ by

$$Mu = \prod_{i=1}^{d} \Delta_i u.$$

we will prove the following:

Lemma 7. *Let $f: D \to [0, \infty)$ be a function with $f \equiv 0$ on ∂D. Then there exists a concave function $z: D \to [0, \infty)$ such that*

(i) $z \equiv 0$ *on* ∂D,

(ii) $(-1)^d Mz = f^d$ *on* $\operatorname{int} D$.

Moreover, there exists a constant c_3 (depending only on d) such that

(iii) $\|z\|_\infty \leqq c_3 n^2 \|f\|_d$.

Suppose that we have Lemma 7, and let us derive Lemma 6. Fix $x \in \operatorname{int} D$, and let

$X_\pi(j)$ be the Markov chain induced by π with $X_\pi(0)=x$. Then

$$\begin{aligned} E(z(X_\pi(1))-z(X_\pi(0)) &= \sum_{i=1}^{d} \tfrac{1}{2}\pi_i(x)\Delta_i z(x) \\ &\leqq -\tfrac{1}{2}\alpha|Mz(x)|^{1/d} \\ &= -\tfrac{1}{2}\alpha f(x). \end{aligned}$$

Here we have used the inequality $(a_1b_1+\ldots+a_db_d)^d \geqq (a_1\ldots a_d)\ (b_1\ldots b_d)$. Continuing as above we may deduce

$$E[z(X_\pi(j\wedge\tau))-z(X_\pi(0))+\tfrac{1}{2}\alpha\sum_{k=0}^{(j-1)\wedge\tau} f(X_\pi(k))] \leqq 0.$$

Letting j go to infinity,

$$\tfrac{1}{2}\alpha Qf(x)=E_x\tfrac{1}{2}\alpha\sum_{k=0}^{\tau} f(X_\pi(k)) \leqq z(x).$$

and Lemma 7 then gives the required bound.

To prove Lemma 7, let $\mathscr{A}$ be the set of all concave functions u on D satisfying

(i) $u\equiv 0$ on ∂D,

(ii) $(-1)^d Mu \geqq f^d$ on $\operatorname{int} D$.

We first note that $\mathscr{A}$ is non-empty: let $h: D\to[0,\infty)$ by

$$h(x)=n(n+1)-|x|(|x|+1),$$

where $|(x_1,\ldots,x_d)|=|x_1|+\ldots+|x_d|$. One can check that $(-1)^d Mh \geqq 2^d$ and hence $\beta h\in\mathscr{A}$ for β sufficiently large.

It is easy to check that if $u_1,u_2\in\mathscr{A}$, then $\min(u_1,u_2)\in\mathscr{A}$; in fact, if we let

$$z(x)=\inf_{u\in\mathscr{A}} u(x),$$

one can verify that $z\in\mathscr{A}$. It remains to be shown that $(-1)^d Mz = f^d$. Suppose $(-1)^d Mz(x) > (f(x))^d$ for some $x\in\operatorname{int} D$, i.e.

$$(-1)^d \prod_{i=1}^{d} (z(x+e_i)+z(x-e_i)-2z(x)) > (f(x))^d.$$

Let $\gamma < z(x)$ be such that

$$(-1)^d \prod_{i=1}^{d} (z(x+e_i)+z(x-e_i)-2\gamma) = (f(x))^d.$$

and set

$$v(y)=\begin{cases} z(y) & y\neq x \\ \gamma & y=x \end{cases}.$$

Then again one can check that $v\in\mathscr{A}$, contradicting the minimality of z.

We now wish to estimate z. For $x\in\operatorname{int} D$, let

$$\begin{aligned} I(x)=\{(a_1,\ldots,a_d)\in R^d : z(x+e_i)-z(x) \\ \leqq a_i \leqq z(x)-z(x-e_i)\}. \end{aligned}$$

Note that meas $(I(x)) = (-1)^d Mz(x) = (f(x))^d$. We state the next easily provable fact as a lemma:

Lemma 8. *Let $a \in R^d$, $b > 0$, and let r be the affine function $r(x) = a \cdot x + b$. Suppose $r(x) \geqq z(x)$ for every $x \in D$ and $r(x_0) = z(x_0)$ for some $x_0 \in \operatorname{int} D$. Then $a \in I(x_0)$.*

Now let $\bar{z} = \| z \|_\infty$ and let $\bar{x} \in \operatorname{int} D$ with $z(\bar{x}) = \bar{z}$. Assume $\bar{z} > 0$. Let

$$A = \{a \in \mathbb{R}^d : |a| \leqq \bar{z}/4n\}.$$

Fix $a \in A$. If $b \geqq \frac{3}{2}\bar{z}$, then $a \cdot x + b > \bar{z} \geqq z(x)$ for every $x \in D$. Therefore there exists a least b (depending on a) such that $a \cdot x + b \geqq z(x)$ for all $x \in D$. It is easy to see that $a \cdot x_0 + b = z(x_0)$ for some $x_0 \in D$, and since

$$a \cdot x_0 + b = a \cdot \bar{x} + b + a \cdot (x_0 - \bar{x}) \geqq \tfrac{1}{2}\bar{z} > 0,$$

$x_0 \in \operatorname{int} D$. By Lemma 8, $a \in I(x_0)$. Therefore

$$A \subset \bigcup_{x \in \operatorname{int} D} I(x),$$

$$\begin{aligned} \operatorname{meas}(A) &\leqq \operatorname{meas}\left(\bigcup I(x)\right), \\ &\leqq \sum_{x \in D} ((f(x))^d. \end{aligned}$$

Since meas$(A) = (\bar{z}^d)(c_4 n)^{-d}$ for some c_4, we get

$$\begin{aligned} \bar{z} &\leqq c_4 n \Big[\sum_{x \in D} (f(x))^d \Big]^{1/d} \\ &\leqq c_3 n^2 \| f \|_d. \end{aligned}$$

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References

1. Hall, P., Heyde, C. C.: Martingale limit theory and its application. New York: Academic Press 1980
2. Krylov, N. V.: An inequality in the theory of stochastic integrals. Theor. Prob. Appl. **16**, 438–448 (1971)
3. Papanicolaou, C., Varadhan, S. R. S.: Diffusions with Random coefficients. In: Essays in Honor of C. R. Rao. Amsterdam: North Holland 1982
4. Parthasarathy, K. R.: On the category of ergodic measure. Ill. J. Math. **5**, 648–655 (1961)

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