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A Direct Method for Minimizing the Yang-Mills Functional over 4-Manifolds

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Abstract. A direct method is employed to minimize the Yang-Mills functional over a 4-dimensional manifold. The limiting connection is shown to be Yang-Mills, but in a possibly new bundle. We show that a topological invariant of the bundle is preserved by the minimizing process. This implies the existence of an absolute minimum of the Yang-Mills functional in a wide class of bundles.

Introduction

We examine the limiting behavior of a minimizing sequence of connections for the Yang-Mills functional in a principal bundle over a compact 4-manifold. A limiting connection is found, but possibly in a new bundle. It is natural to ask for some invariant of the bundle which is preserved by this procedure. In the minimizing process, there are a finite number of points where curvature collects. When we take the limit, we lose control of the bundle at these points. So an invariant which will survive through the limit should be determined by the bundle with a finite number of fibers removed. If the invariant is to be in cohomology, we see that we want classes which are determined by their restriction to the manifold with finitely many points removed. For 4 manifolds, this is satisfied by 2- and 3-dimensional cohomology classes. Uhlenbeck makes a conjecture in [17] that the first chern class of a unitary bundle is preserved under the minimizing process. Our results show that the conjecture is true, although this case does not seem the most important application of our results.

In his paper [13] Taubes shows existence of self-dual Yang-Mills fields on many oriented 4-manifolds. The principle bundles to which his method applies must have an invariant in dimension 2 cohomology vanishing. This invariant is the obstruction to lifting the structure group of a principle bundle to the universal covering group of the structure group. In this paper we show that this obstruction is preserved by our process, so we obtain Yang-Mills fields in bundles with nontrivial obstruction.

This obstruction also arises in 't Hooft's [14] construction of bundles over a 4-dimensional torus with structure group $SU(n)$ modulo its center. By explicit

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computations he produces Yang-Mills fields in bundles with a nonzero obstruction to lifting the structure group to $SU(n)$.

In dimension four the Yang-Mills equations are conformally invariant. This happens in dimension two for the harmonic map problem. There are strong similarities between the two problems. For details on the harmonic map problem see Schoen and Yau [12], Lemaire [6], and Sacks and Uhlenbeck [10].

Section 1. Preliminaries

Let P be a principle fiber bundle over a compact 4-dimensional Riemannian manifold M. The structure group G is assumed to be a compact Lie group with Lie algebra $\mathscr{G}$. A will denote a connection on P and F_A its curvature. Let $\operatorname{ad}\mathscr{G}$ be the adjoint bundle of P, D_A the exterior covariant derivative induced by A, and d the exterior derivative which is defined on $\Lambda^p M\otimes\mathscr{G}$. We shall denote the induced inner product on any $\Lambda^p M\otimes\operatorname{ad}\mathscr{G}$ by $(\,,)$, the norm by $|\ |$. Recall the definition of the Yang-Mills functional

$$\mathscr{A}(A)=\int_M |F_A|^2 . \tag{1.1}$$

Integration over M is via the density induced by the Riemannian metric. Suppose $\sigma: U\subset M\to P$ is a section. Then σ^*A is a $\mathscr{G}$-valued one form on U, and if we trivialize $\Lambda^p U\otimes\operatorname{ad}\mathscr{G}$ via σ we have $\mathscr{G}$-valued p forms. In particular F_A over U is a $\mathscr{G}$-valued 2 form, and we have $F_A=d\sigma^*A+[\sigma^*A,\sigma^*A]$. Here $[\,,]$ is induced from the Lie algebra multiplication in $\mathscr{G}$. If U is a coordinate chart and ψ is a section of $\Lambda^p U\otimes\operatorname{ad}\mathscr{G}\simeq\Lambda^p U\otimes\mathscr{G}$; we write $D_A\psi=d\psi+\sigma^*A\psi$, where $\sigma^*A\psi$ involves only multiplication. For more details see [5].

We assume some knowledge of Sobolev spaces. Let L_k^p denote the space of functions with weak derivatives through order k in L^p; $\|\ \|_{p,k}$ denotes the norm in L_k^p. Let $\rightharpoonup$ denote weak convergence, $\to$ strong convergence. Recall that $\to$ in L_0^p implies pointwise convergence almost everywhere. For more details see [8].

We need to define $L_k^p(U,G)$, where U is a coordinate chart. Since G is compact it may be viewed as a group of matrices, so sits naturally in some $\mathbb{R}^n$. We say $f: U\to G$ is in L_k^p iff each of its components are. Since the group operations are now just matrix operations, we see that the usual multiplication theorems in Sobolev spaces hold. Recall that $f\in L_0^p$ iff $|f|\in L_0^p$, where $|\ |$ means the norm in $\mathbb{R}^n$. If f is C^1, then df is tangent to $G\subset\mathbb{R}$. If we had a Riemannian metric with norm $\|\ \|$ on G, we could compute the L_0^p norm of df. Of course, one metric comes from $\mathbb{R}^n$ itself. Using the fact that any 2 Riemannian metrics on a compact manifold are uniformly equivalent, we see that $|df|$ is in L_0^p iff $\|df\|$ is. In particular we may use an invariant metric on G to define $L_k^p(U,G)$.

An L_k^p section of P is a section of the form $\sigma\cdot g$, where σ is a C^∞ section, $g\in L_k^p(U,G)$. A connection A is L_k^p means there exists an open cover $\{U_\alpha\}$ of M by coordinate charts and C^∞ sections $\sigma_\alpha: U_\alpha\to P$ for which σ_α^*A is L_k^p (see [5]).

We will often use the following construction. Given a collection A_k of sequences, we take a subsequence of A_1-hence a map $\ell_1:\mathbb{N}\to\mathbb{N}$. Next take a subsequence of $A_2\circ\ell_1$, and get a map ℓ_2. Continuing, we get maps $\ell_1,\ell_2,\dots$. If we define ℓ by $\ell(i)=\ell_1\circ\dots\circ\ell_1(i)$, then $A_k\circ\ell$ is a subsequence of A_k which for sufficiently large i is a subsequence of $A_k \circ \ell_1 \circ \ell_2 \dots \ell_k$. In particular, if the original subsequences converged, then $A_k \circ \ell$ converges as well, and to the same limit. The above process is called diagonalization, and will be used frequently in this paper. We shall be taking so many subsequences that we will not notationally distinguish a sequence and a subsequence obtained from it.

Section 2. The Obstruction $\eta(P)$

Let $\tilde{G}$ be a Lie group with a homomorphism $\pi:\tilde{G}\to G$ which is a surjective covering map. Let e be the identity and set $K=\pi^{-1}(e)$. Here K is discrete since π is a covering map. We assume further that K is in the center of $\tilde{G}$. This is automatic if $\tilde{G}$ is connected. In particular K is abelian. We say we have a lift of P to a principle $\tilde{G}$ bundle $\tilde{P}$ if there exists $f:\tilde{P}\to P$ such that $f(pg)=f(p)\pi g$ and f is the identity on M.

Following Greub and Petry [3] we define the obstruction η using Čech cohomology. Let $\{U_\alpha\}$ be an open cover of M with all finite intersections $U_{\alpha_0}\cap \dots \cap U_{\alpha_n}$ contractible. Such a cover is called simple. Let $\sigma_\alpha: U_\alpha \to P$ be local sections with $g_{\alpha\beta}:U_\alpha\cap U_\beta\to G$ defined by $\sigma_\alpha=\sigma_\beta g_{\beta\alpha}$. Recall that the $g_{\alpha\beta}$ satisfy the cocycle condition $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}=e$. Choose any lifts $\tilde{g}_{\alpha\beta}:U_\alpha\cap U_\beta\to\tilde{G}$ with $\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\alpha}=e$, $\pi\tilde{g}_{\alpha\beta}=g_{\alpha\beta}$. Define

$$f_{\alpha\beta\gamma}:U_\alpha\cap U_\beta\cap U_\gamma\to K \quad \text{by} \quad f_{\alpha\beta\gamma}=\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}.$$

If $f_{\alpha\beta\gamma}=e$ then the $\tilde{g}_{\alpha\beta}$ actually define a bundle. In any case, $f_{\alpha\beta\gamma}\in K$ since

$$\pi f_{\alpha\beta\gamma}=(\pi\tilde{g}_{\alpha\beta})(\pi\tilde{g}_{\beta\gamma})(\pi\tilde{g}_{\gamma\alpha})=g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}=e.$$

Definition 2.1. $\eta(P)$ is the element in $H^2(M,K)$ defined by $\{f_{\alpha\beta\gamma}\}$.

To justify this definition we need to check a few details. First, by changing the lifts $\tilde{g}_{\alpha\beta}$ to $\tilde{g}_{\alpha\beta}\cdot\gamma_{\alpha\beta}$, where $\gamma_{\alpha\beta}:U_\alpha\cap U_\beta\to K$ must satisfy $\gamma_{\alpha\beta}^{-1}=\gamma_{\beta\alpha}$, we see that $f_{\alpha\beta\gamma}$ may be varied by $\gamma_{\beta\gamma}\gamma_{\alpha\gamma}^{-1}\gamma_{\alpha\beta}$.

This is actually a coboundary in the Čech theory. Recall that the Čech cohomology theory proceeds as follows [4]. For an open set V let $\Gamma(V)$ be the abelian group of continuous functions from V to K. Let $\{U_\alpha\}$ be any open cover of M. Define groups $C^p\{U_\alpha\},K)$ for $p\geqq 0$ by $C^p(\{U_\alpha\},K)=$ the group of functions f mapping $p+1$-tuples $\alpha_0\alpha_q\dots\alpha_p$ to $\Gamma(U_{\alpha_0}\cap\dots\cap U_{\alpha_p})$. Let $f_{\alpha_0\dots\alpha_p}$ denote the value of f at $\alpha_0\dots\alpha_p$.

Addition is pointwise. Define $d:C^p\to C^{p+1}$ by

$$(df)_{\alpha_0\dots\alpha_{p+1}}=f_{\alpha_1\alpha_2\dots\alpha_{p+1}}f^{-1}_{\alpha_0\alpha_2\dots\alpha_p}\dots f^{(-1)^{p+1}}_{\alpha_0\dots\alpha_p}.$$

We have $d^2=0$, so can form $\check{H}^p(\{U_\alpha\},K)$ from this complex. The Čech group $\check{H}^p(M,K)$ is defined by taking the limit of $\check{H}^p(\{U_\alpha\},K)$ over all covers $\{U_\alpha\}$. For $\{U_\alpha\}$ simple we have

$$\check{H}^p(\{U_\alpha\},K)\simeq\check{H}^p(M,K)\simeq H^p(M,K)$$

(see [4, 18]).

It is easy to check that $f=\{f_{\alpha\beta\gamma}\}$ satisfies $df=0$, and that $\gamma_{\beta\gamma}\gamma_{\alpha\gamma}^{-1}\gamma_{\alpha\beta}=(d\gamma)_{\alpha\beta\gamma}$ for $\gamma=\{\gamma_{\alpha\beta}\}\in C^1$. Finally, it is clear that $\eta(P)=0$ iff there exists a lift $\tilde{P}$ of P.

Lemma 2.2. *Let P, P' be principle G bundles, $\{U_\alpha\}$ a simple cover with corresponding transition functions $g_{\alpha\beta}, h_{\alpha\beta}$. Then $\eta(P)=\eta(P')$ iff there exist lifts $\tilde{g}_{\alpha\beta}, \tilde{h}_{\alpha\beta}$ such that $\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}=\tilde{h}_{\alpha\beta}\tilde{h}_{\beta\gamma}\tilde{h}_{\gamma\alpha}$.*

Proof. Immediate from the preceding discussion.

Remark. On a compact Riemannian manifold there exists $r>0$ such that any cover by geodesic balls of radius $<r$ is simple.

Lemma 2.3. *If $f: N \to M$, then $\eta(f^*P)=f^*\eta(P)$.*

Proof. See Greub and Petry [3].

A particular case of the preceding occurs when G is connected and $\tilde{G}$ is the universal covering group of G. Recall that $K \simeq \pi_1 G$ in this case, so we have $\eta(P) \in H^2(M, \pi_1 G)$. This is the obstruction Taubes [13] needs to vanish and is the "twist" appearing in 't Hooft's paper [14].

Theorem 2.4. a) *For $G=O(n)$ or $SO(n)$ we have $\eta(P)=W_2(P) \in H^2(M, \mathbb{Z}_2)$. Here W_2 is the second Stiefel-Whitney class of the vector bundle associated to P by the representations $O(n) \to GL(n)$, $SO(n) \to GL(n)$, and we try to lift to $pin(n)$ and $spin(n)$ respectively.*

b) *For $G=U(n)$ we have $\eta(P)=c_1(P) \in H^2(M, \mathbb{Z})$ where c_1 is the first chern class of the vector bundle associated to P by the representation $U(n) \to GL(n, \mathbb{C})$. Here we try to lift to $\mathbb{S}UN(n)=\mathbb{R} \times \mathbb{S}U(n)$.*

Proof. See Greub and Petry [3].

For G connected, it is well known that bundles over S^2 are classified by $\pi_1 G$. See Atiyah and Bott [1]. As a simple example, if we take a bundle specified by $C:[0,1] \to G$, $C(0)=e$, lifting the structure group to the universal cover $\tilde{G}$ amounts to lifting C to $\tilde{C}:[0,1] \to \tilde{G}$ with $\tilde{C}(0)=e$. However, the lifted curve won't be a loop unless C was 0 in $\pi_1 G$. Indeed, the lifted curve will have $\tilde{C}(1) \in K$. This is the usual correspondence between $\pi_1 G$ and K, and in our case gives $\eta(P) H^2(M, \pi_1 G) \simeq \pi_1 G$ for S^2. Therefore η classifies such bundles over S^2. Of course, we don't have to try to lift all the way to the universal cover – if we don't we get less information from η.

More generally, the appendix shows that the map $P \to \eta(P)$ from bundles to obstructions is onto. Moreover, the obstruction $\eta(P)$ determines P over the 3-skeleton of M.

Section 3. Weak Compactness

Remarks. In this section we show that a bound on the curvature of a sequence of connections implies convergence of a subsequence. However, the limiting object is some kind of L_1^2 "connection" in an L_2^2 "bundle." Since we are working in dimension 4 these objects may not even be continuous, so should be treated cautiously. Theorem 3.1 is really more general than is needed for the sequel. We state it in this generality in order to make it applicable to coupled Yang-Mills equations, which we do not treat here. For details on coupled equations see Parker [8] and Jaffe and Taubes [5].

Theorem 3.1. *Let $\{A_i\}$ be a sequence of C^∞ connections in principal bundles $\{P_i\}$ over M with $\mathscr{A}(A_i) \leqq B$. Then there exists a subsequence, a countable set of arbitrarily small geodesic balls $\{U_\alpha\}$ covering $M-\{x_1, \ldots, x_\ell\}$, C^∞ sections*

$$\sigma_\alpha(i): U_\alpha \to P, \quad A_\alpha \in L_1^2(U_\alpha, \Lambda^1 U_\alpha \otimes \mathscr{G}), \quad \textit{and} \quad g_{\alpha\beta} \in L_1^4(U_\alpha \cap U_\beta, G)$$

such that:

a) $d^*A_\alpha(i)$ *is* 0 *for i sufficiently large,*

b) $d^*A_\alpha=0$,

c) $g_{\alpha\beta}(i)\rightharpoonup g_{\alpha\beta}(L_1^4)$,

d) $F_\alpha(i)\rightharpoonup F_\alpha(L_0^2)$,

e) $A_\alpha(i)\rightharpoonup A_\alpha(L_1^2)$,
$A_\alpha(i)\to A_\alpha(L_0^2)$,

f) $A_\alpha=g_{\alpha\beta}^{-1}A_\beta g_{\alpha\beta}+g_{\alpha\beta}^{-1}dg_{\alpha\beta}$.

Here $A_\alpha(i)=\sigma_\alpha^*(i)A_i$, $F_\alpha=dA_\alpha+[A_\alpha,A_\alpha]$, $F_\alpha(i)=dA_\alpha(i)+[A_\alpha(i),A_\alpha(i)]$ *and* d^* *is the adjoint of* d *in the flat metric on* U_α *induced by normal coordinates.*

The idea of the proof is obvious. Use the bound on curvature to get an L_1^2 bound on the connections, then use weak compactness to get a limiting connection. The difficulty comes from the fact that we can only do this locally and in a particular gauge. More precisely, we have a theorem of Uhlenbeck [15].

Theorem 3.2. *Let D be a geodesic ball. There exist constants R, S depending on the geometry of M such that if* $\int_D |F_A|^2 \leqq R$, $A\in L_1^2$, *then there exists a section* $\sigma: D\to P$ *such that:*

a) $d^*(\sigma^*A)=0$,

b) $\|\sigma^*A\|_{2,1}\leqq RS$,

σ *is* L_2^2 *in general; if A is* C^∞ *so is* σ.

Proof. See Uhlenbeck [15]. Here d^* is the adjoint of d in the flat metric arising from normal coordinates.

What we need next are covers $\{C_j\}$ of M with the following properties:

a) The elements of C_j are balls of radius $r_j\to 0$.

b) There exists h independent of j such that any $h+1$ balls of C_j have empty intersection – in particular C_j is finite.

Such covers certainly exist on compact Riemannian manifolds. We would like to apply Theorem 3.2 to each element of the cover C_j, but the hypothesis needn't be met on all balls. However, we have an upper bound on the number of balls on which it fails:

$$hB\geqq h\int_M |F_{A_i}|^2 \geqq \sum_{D\in C_j}\int_D |F_{A_i}|^2\geqq N_{ij}R,$$

where $N_{ij}=\#$ of $D\in C_j$ for which $\int_D |F_{A_i}|^2\geqq R$.

Remark. We will call a ball D bad for A_i if $\int_D |F_{A_i}|^2\geqq R$, otherwise it is called good.

Proposition 3.3. *There is a subsequence of* $\{A_i\}$ *for which the bad balls in a given* C_j *are independent of i for i sufficiently large.*

Proof. By diagonalization it suffices to find for any fixed C_j a subsequence of $\{A_i\}$ for which the bad balls are fixed. This is easy. Looking at the centers of the bad balls we get at most hB/R sequences in M. Compactness gives us convergent subsequences of the centers, and the finiteness of C_j then implies the centers are actually fixed.

We now toss out the "eventually bad" balls in each cover C_j and throw what's left into a set $\{U_\alpha\}$ of balls. Since $\bigcup\{U_\alpha\}\supset M-\{$at most hB/R balls of radius $r_j\}$

and $r_j \to 0$, we see that $\bigcup\{U_\alpha\} = M - \{x_1 \dots x_\ell\}$, where $\ell \leqq \frac{hB}{R}$. By construction we see that any U_α is eventually good – that is, for i large enough, U_α is good for all A_i. Of course "large enough" depends on α.

Lemma 3.4. *There is a subsequence of $\{A_i\}$ which for every α has C^∞ sections $\sigma_\alpha(i): U_\alpha \to P$ such that:*

a) $\sigma_\alpha(i)^* A_i \rightharpoonup A_\alpha$ *in* L_1^2,

b) $d^* \sigma_\alpha(i)^* A_i = 0$ *for sufficiently large i.*

Proof. By using Proposition 3.3, Theorem 3.2 and taking i sufficiently large we get $\sigma_\alpha(i)$ for which b) is true and for which $\|\sigma_\alpha(i)^* A_i\|_{2,1}$ is bounded. We then apply weak compactness in $L_1^2(U_\alpha)$ to get a convergent subsequence $\sigma_\alpha(i)^* A \rightharpoonup A_\alpha$ in L_1^2 for each α. Diagonalization over the countable cover $\{U_\alpha\}$ then gives us our subsequence.

Lemma 3.5. *The $\sigma_\alpha(i)$ give rise to transition functions $g_{\alpha\beta}(i)$. There is a subsequence of $\{A_i\}$ for which*

$$g_{\alpha\beta}(i) \rightharpoonup g_{\alpha\beta}(L_1^4).$$

Proof. By diagonalization over the countable collection $\{\alpha\beta\}$ it suffices to produce a convergent subsequence for fixed $\alpha\beta$. We show $\|g_{\alpha\beta}(i)\|_{4,1}$ is bounded. Recall that

$$\sigma_\beta^*(i) A_i = g_{\alpha\beta}(i)^{-1} \sigma_\alpha^*(i) A_i g_{\alpha\beta}(i) + g_{\alpha\beta}(i)^{-1} d g_{\alpha\beta}(i).$$

Taking L_0^4 norms and using the invariant metric on G we get:

$$\begin{aligned}
\|dg_{\alpha\beta}(i)\|_{4,0} &= \|g_{\alpha\beta}(i)^{-1} dg_{\alpha\beta}(i)\|_{4,0} \\
&\leqq \|\sigma_\beta^*(i) A_i\|_{4,0} + \|\sigma_\alpha^*(i) A_i\|_{4,0} \\
&\leqq C(\|\sigma_\alpha^*(i) A_i\|_{2,1} + \|\sigma_\alpha^*(i) A_i\|_{2,1}).
\end{aligned}$$

Here we used the fact that L_1^2 imbeds continuously into L_0^4 in dimension 4 [8]. Finally, since G is compact it is automatic that $\|g_{\alpha\beta}(i)\|_{4,0}$ is uniformly bounded.

Lemma 3.6. *There exists a subsequence such that: $F_\alpha(i) \rightharpoonup F_\alpha(L_0^2)$.*

Proof. We have

$$F_\alpha(i) = d\sigma_\alpha^*(i) A_i + [\sigma_\alpha^*(i) A_i, \sigma_\alpha^*(i) A_i].$$

Since $\sigma_\alpha^*(i) A_i \rightharpoonup A_\alpha$ in L_1^2, we get $d\sigma_\alpha^*(i) A_i \rightharpoonup dA_\alpha$ in L_0^2.

Using the continuity of the imbedding $L_1^2 \hookrightarrow L_0^4$, the continuity of multiplication from $L_0^4 \otimes L_0^4$ to L_0^2 and the fact that $\|A_\alpha(i)\|_{2,1}$ is bounded we see that $\|[A_\alpha(i), A_\alpha(i)]\|_{2,0}$ is bounded. Since we have pointwise convergence almost everywhere, we get $[A_\alpha(i), A_\alpha(i)] \rightharpoonup [A_\alpha, A_\alpha]$ in L_0^2.

Lemma 3.7. $d^* A_\alpha = 0$.

Proof. It suffices to notice that d^* involves only one derivative, which implies that $d^* A_\alpha(i) \rightharpoonup d^* A_\alpha(L_0^2)$ since $A_\alpha(i) \rightharpoonup A_\alpha(L_1^2)$. To finish, recall that $d^* A_\alpha(i) = 0$ for i sufficiently large.

Proof of Theorem 3.1. This is almost immediate from the preceding discussion.

Remark. We will call the collection $\{A_\alpha\}$ a connection A_∞ in the bundle $P_\infty = \{g_{\alpha\beta}\}$.

Section 4. Existence of a Yang-Mills Minimum

Definition. Suppose G and $\tilde{G}$ are given and there exists a G-bundle P over M with $\eta(P)=\eta\in H^2(M,K)$. We set $m(\eta)=\inf\{\mathscr{A}(A)|A$ is a C^∞ connection on a G bundle with obstruction $\eta\}$.

Take any sequence $\{A_i\}$ of connections with $\mathscr{A}(A_i)\to m(\eta)$. Since this implies that there exists B with $\mathscr{A}(A_i)\leqq B$, we can apply Theorem 3.1 to get the existence of a weak limiting connection A_∞ in a bundle P_∞ over $M-\{x_1,\dots,x_\ell\}$.

Theorem 4.1. *For each α A_α is a weak Yang-Mills field. More precisely:*

$$(D_{A_\alpha}\phi, F_{A_\alpha})=0 \quad \textit{for all} \quad \phi\in C_0^\infty(U_\alpha, \Lambda^1 U_\alpha\otimes\mathscr{G}). \tag{4.1}$$

Here $(\,,)$ denotes the inner product in L_0^2.

Proof. Suppose (4.1) is false. We construct a sequence of C^∞ connections $\tilde{A}_i$ such that $\mathscr{A}(\tilde{A}_i)\to\tilde{m}<m(\eta)$, giving a contradiction. To start, take α,ϕ for which $(D_{A_\alpha}\phi, F_{A_\alpha})<0$. Let $t\phi_i$ be that section of $\Lambda^1 M\otimes\mathrm{ad}\,\mathscr{G}(i)$ which over U_α is trivialized by $\sigma_\alpha(i)$ to be $t\phi$. Since $\operatorname{supp}\phi\subset U_\alpha$ we have $t\phi_i$ globally defined on M and C^∞. Set $\tilde{A}_i=A_i+t\phi_i$ by regarding $t\phi_i$ as a $\mathscr{G}$-valued one form on P. Since $\int_M = \int_{M-U_\alpha} + \int_{U_\alpha}$ and $\tilde{A}_i=A_i$ over $M-U_\alpha$, it suffices when comparing $\mathscr{A}(\tilde{A}_i)$ and $\mathscr{A}(A_i)$ to work over U_α. Therefore we may trivialize everything using $\sigma_\alpha(i)$ and write:

$$F_{\tilde{A}_{\alpha(i)}}=F_{\alpha(i)}+tD_{A_{\alpha(i)}}\phi+\frac{t^2}{2}[\phi,\phi]. \tag{4.2}$$

This implies:

$$\begin{aligned}\mathscr{A}(\tilde{A}_i)=\mathscr{A}(A_i)&+2t(D_{A_{\alpha(i)}}\phi, F_\alpha(i))\\ &+t^2(F_\alpha(i),[\phi,\phi])+t^3(D_{A_{\alpha(i)}}\phi,[\phi,\phi])\\ &+\frac{t^4}{4}\|[\phi,\phi]\|_{2,0}^2.\end{aligned} \tag{4.3}$$

We now examine the terms as $i\to\infty$. Observe that $D_{A_{\alpha(i)}}\phi\to D_{A_\alpha}\phi$ in L_0^2. This follows immediately from the fact that $A_\alpha(i)\to A_\alpha$ in L_0^2 which is part of Theorem 3.1. This theorem also gives $F_\alpha(i)\rightharpoonup F_\alpha$ in L_0^2. This immediately implies that the last 4 terms converge to the obvious limits.

To handle the second term write:

$$(D_{A_\alpha}\phi, F_\alpha)-(D_{A_{\alpha(i)}}\phi, F_\alpha(i))=(D_{A_\alpha}\phi, F_\alpha-F_\alpha(i))+(D_{A_\alpha}\phi-D_{A_{\alpha(i)}}\phi, F_\alpha(i)). \tag{4.4}$$

The first term approaches 0 since $F_\alpha-F_\alpha(i)\rightharpoonup 0$ in L_0^2. The second term approaches 0 since $D_{A_\alpha}\phi-D_{A_{\alpha(i)}}\phi\to 0$ in L_0^2 and $\|F_\alpha(i)\|_{2,0}$ is bounded. So we see that:

$$\mathscr{A}(\tilde{A}_i)\to m(\eta)+2t(D_{A_\alpha}\phi, F_\alpha)+\text{higher order terms}. \tag{4.5}$$

Taking t to be a sufficiently small positive number, we get our contradiction.

Proposition 4.2. *A_α is a smooth solution to the Yang-Mills equation. Moreover, the $g_{\alpha\beta}$ are C^∞.*

Proof. By Lemma 3.7 $d^*A_\alpha=0$. Since A_α also satisfies the Yang-Mills equation on U_α by Theorem 4.1, it is a smooth solution. See Parker [9], Taubes [13], and

Uhlenbeck [15]. The proof that $g_{\alpha\beta}$ is C^∞ is similar to the proof of Lemma 3.5, see [15].

Theorem 4.3. *P_∞ extends to a C^∞ bundle over M, and A_∞ extends to a C^∞ Yang-Mills Connection in the extended bundle.*

Proof. See Parker [9] and Uhlenbeck [16]. From now on A_∞ and P_∞ will denote the extended connection and bundle.

Remark. We have $\int_{U_\alpha} |F_\alpha|^2 \leqq \lim \int_{U_\alpha} |F_\alpha(i)|^2$ which implies

$$\mathscr{A}(A_\infty) \leqq \lim \mathscr{A}(A_i) = m(\eta).$$

Here we used the weak convergence of $F_\alpha(i)$ to F_α and the fact that $\| \;\|_{2,0}$ is lower semicontinuous.

Section 5. The Obstruction is Preserved

We first prove a technical lemma on cohomology.

Lemma 5.1. *Let* $\dim M \geqq 3$, $Q = \{x_1, \ldots, x_\ell\}$ *and* $J: M - Q \to M$ *be the inclusion. If* P, P' *are bundles over* M *with structure group* G, *then* $\eta(J^*P) = \eta(J^*P')$ *implies* $\eta(P) = \eta(P')$.

Proof. Since $\eta(J^*P) = J^*\eta(P)$, it suffices to show that J^* is injective. This follows from the exact sequence of the pair $(M, M-Q)$ and excision:

$$\ldots \to H^2(M, M-Q) \to H^2(M) \xrightarrow{J^*} H^2(M-Q) \to \ldots.$$

Take small disjoint balls $B_1, \ldots, B_\ell$ around the points $x_1, \ldots, x_\ell$. Set $B = \bigcup_1^\ell B_i$, Excision of $M-B$ gives $H^*(M, M-Q) \simeq H^*(B, B-Q)$. The exact sequence of the pair $(B, B-Q)$ is:

$$\ldots \to H^1(B-Q) \to H^2(B, B-Q) \to H^2(B) \to \ldots.$$

But since $B-Q$ is homotopy equivalent to ℓ spheres of dimension $\geqq 2$, and B to ℓ points, we see that: $0 = H^2(B, B-Q) \simeq H^2(M, M-Q)$ which implies J^* is injective.

Proposition 5.2. *Assume $\tilde{G}$ is compact and that the hypotheses of Theorem 3.1 are satisfied. Then there is a subsequence of $\{A_i\}$ with lifts $\tilde{g}_{\alpha\beta}(i) \rightharpoonup \tilde{g}_{\alpha\beta}$ in L_1^4 where $\pi\tilde{g}_{\alpha\beta} = g_{\alpha\beta}$. Here $\tilde{g}_{\alpha\beta}(i)$ may be any lift of $g_{\alpha\beta}(i)$.*

Proof. By Theorem 3.1 we get a subsequence of the $\{A_i\}$ and existence of $g_{\alpha\beta}$ for which $g_{\alpha\beta}(i) \rightharpoonup g_{\alpha\beta}(L_1^4)$. Let $\tilde{g}_{\alpha\beta}(i)$ be any lift of $g_{\alpha\beta}(i)$. The crucial point here is that $\|\tilde{g}_{\alpha\beta}(i)\|_{4,1}$ is bounded. However, this follows from the fact that

$$\|d\tilde{g}_{\alpha\beta}(i)\|_{4,0} = \|\pi d\tilde{g}_{\alpha\beta}(i)\|_{4,0} = \|dg_{\alpha\beta}(i)\|_{4,0}$$

since we can use an invariant metric on $\tilde{G}$ for which π is an isometry. We have $\|\tilde{g}_{\alpha\beta}(i)\|_{4,0}$ bounded because $\tilde{G}$ is compact – this is the only point that used compactness of $\tilde{G}$.

The argument finishes by using weak compactness in L_1^4 to pick convergent subsequences $\tilde{g}_{\alpha\beta}(i) \rightharpoonup \tilde{g}_{\alpha\beta}$ in L_1^4. Then $\pi\tilde{g}_{\alpha\beta} = g_{\alpha\beta}$ follows from the fact that we have pointwise convergence almost everywhere. Of course, we need to use diagonalization over the pairs $\alpha\beta$ to get our subsequence.

Lemma 5.3. *Suppose $f: U_\alpha \to R$ has only a finite number of values and has a weak derivative in L_1^1. Then f is constant.*

Proof. It suffices to show the weak derivative of f is 0. Let $a_1, \ldots, a_n$ be the values f assumes, let $\phi_i: R \to R$ be C^∞ functions with $\phi_i(a_i) \equiv a_i$ near a_i, $i = 1, \ldots, n$, and $\phi_i \equiv 0$ away from a_i so that $\phi_i(a_j) = 0$ for $i \neq j$. Then $f = \sum_1^n \phi_i \circ f$. By the weak chain rule (Morrey [7]) we get the weak derivative of f being 0.

Lemma 5.4. *If $\tilde{G}$ is compact and $\tilde{g}_{\alpha\beta}$ is an L_1^4 lift of $g_{\alpha\beta} \in C^\infty$, then $\tilde{g}_{\alpha\beta} \in C^\infty$.*

Proof. We have $\tilde{g}_{\alpha\beta} \in L_1^4$ and $\pi\tilde{g}_{\alpha\beta} = g_{\alpha\beta}$. Let $\bar{g}_{\beta\alpha}$ be any C^∞ lift of $g_{\alpha\beta}$. Then

$$f = \tilde{g}_{\alpha\beta} \cdot \bar{g}_{\beta\alpha} : U_\alpha \cap U_\beta \to K$$

is in L_1^4. Indeed, $\pi(\tilde{g}_{\alpha\beta}\bar{g}_{\beta\alpha}) = g_{\alpha\beta} g_{\beta\alpha} = g_{\alpha\alpha} = e$ so f is K-valued. The multiplication theorems for Sobolev spaces (Palais [8]) imply f is L_1^4. By Lemma 5.3 we see f is constant, so $\tilde{g}_{\alpha\beta}$ is C^∞.

Theorem 5.5. *Assume $\tilde{G}$ is compact and that we have the bundle P_∞ from Theorem 4.3. Then the obstruction is preserved. That is, $\eta(P_\infty) = \eta$.*

Proof. By Proposition 5.1 it suffices to work over $M - \{x_1, \ldots, x_\ell\}$. By hypothesis we have $\eta(P_i) = \eta$, so we can choose lifts $\tilde{g}_{\alpha\beta}(i)$ for which

$$f_{\alpha\beta\gamma}(i) = \tilde{g}_{\alpha\beta}(i)\tilde{g}_{\beta\gamma}(i)g_{\gamma\alpha}(i) = f_{\alpha\beta\gamma}(1)$$

say. By Theorem 5.2 we can pick a subsequence of $\{A_i\}$ for which $\tilde{g}_{\alpha\beta}(i) \rightharpoonup \tilde{g}_{\alpha\beta}$. The $\tilde{g}_{\alpha\beta}$ are legitimate C^∞ lifts by Lemma 5.4. So we have: $f_{\alpha\beta\gamma}(\infty) = \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha}$. Since $\tilde{g}_{\alpha\beta}(i)$ converges to $\tilde{g}_{\alpha\beta}$ almost everywhere, we can select $x \in U_\alpha \cap U_\beta \cap U_\gamma$ for which all 3 terms in $f_{\alpha\beta\gamma}(i)$ converge:

$$\begin{aligned} f_{\alpha\beta\gamma}(i)(x) &= \tilde{g}_{\alpha\beta}(i)(x) \cdot \tilde{g}_{\beta\gamma}(i)(x) \cdot \tilde{g}_{\gamma\alpha}(i)(x) \\ &\to \tilde{g}_{\alpha\beta}(x) \cdot \tilde{g}_{\beta\gamma}(x) \cdot \tilde{g}_{\gamma\alpha}(x) = f_{\alpha\beta\gamma}(\infty)(x). \end{aligned}$$

Since $f_{\alpha\beta\gamma}(i) = f_{\alpha\beta\gamma}(1)$, and both sides are constant functions, we see that $\eta = \eta(P_1) = \eta(P_\infty)$ on $M - \{x_1, \ldots, x_\ell\}$.

Corollary 5.6. $\mathscr{A}(A_\infty) = m(\eta)$.

Proof. We remarked earlier that $\mathscr{A}(A_\infty) \leqq m(\eta)$. But A_∞ is a C^∞ connection in P_∞. By Theorem 5.5 we have $\eta(P_\infty) = \eta$, so we get $m(\eta) \leqq \mathscr{A}(A_\infty)$.

Section 6. Relaxing the Compactness of $\tilde{G}$

In the previous theorems we required $\tilde{G}$ to be compact. It is possible to relax this assumption.

Theorem 6.1. *The conclusions of Theorems 5.5 and 5.6 hold for any* $\tilde{G}$.

Proof. Since K is a quotient group of $\pi_1 G$ which is finitely generated by the compactness of G, we have $K = F \oplus T$ where F is a finitely generated free group and T is a finite torsion group. Given $\eta \in H^2(M, K)$ and K' a subgroup of K, we may use the map $K \to K/K'$ to define classes $\eta_{K'} \in H^2(M, K/K')$. It is easy to see that $\eta_{K'}$ is the obstruction to lifting a bundle to G/K' if η is the obstruction to lifting to $\tilde{G}$. See Greub and Petry [3] for details. We have:

$$H^2(M, K) = H^2(M, F \oplus T) \simeq H^2(M, F) \oplus H^2(M, T),$$

which give $\eta = \eta_1 + \eta_2$. Theorem 5.5 implies that η_K is preserved for K' which satisfy K/K' finite. In particular we may take $K' = F$ to get $\eta_F = \eta_2 \in H^2(M, T)$ preserved. Next take $K' = nF \oplus T$ to conclude that $\eta_{nF \oplus T} \in H^2(M, F/nF)$ is preserved for $n = 1, 2, \ldots$. But clearly $\eta_{nF \oplus T}$ is the image of η_1 in $H^2(M, F/nF)$ under the map induced by $F \to F/nF$. The exact sequence

$$0 \to F \xrightarrow{x_n} F \xrightarrow{\alpha_n} F/nF \to 0$$

gives rise to a long exact sequence:

$$\ldots H^2(M, F) \xrightarrow{x_n} H^2(M, F) \xrightarrow{\alpha_{n*}} H^2(M, F/nF) \to \ldots.$$

To show η_1 is preserved the preceding shows that it suffices to show $\alpha_{n*}(\eta_1) = 0$ for all n implies $\eta_1 = 0$. From the exact sequence we get the existence of $x_n \in H^2(M, F)$ for which $\eta_1 = n x_n$. Since $H^2(M, F)$ is finitely generated we see that $\eta_1 = 0$.

Corollary 6.2. *The first chern class c_1 of a $\mathbb{U}(n)$ bundle is preserved.*

Proof. Recall that c_1 is the obstruction to lifting the structure group to $\mathbb{R} \times \mathbb{S}U(n)$ and apply the theorem.

Section 7. Concluding Remarks

The preceding theorems show that given any $\eta \in H^2(m, K)$ then there exists a bundle P_∞ with $\eta(P_\infty) = \eta$ and a Yang-Mills connection A_∞ in P_∞ which is an absolute minimum of the Yang-Mills functional over all bundles with obstruction η, and which minimizes in P_∞ in particular. We could have taken a fixed bundle P and minimized over connections in P in an attempt to realize $\hat{m}(P) = \inf\{\mathscr{A}(A) | A$ a C^∞ connection in $P\}$. In carrying out this procedure we get a bundle $\hat{P}_\infty$ and Yang-Mills connection $\hat{A}_\infty$. All we know about $\hat{P}_\infty$ is that $\eta(\hat{P}_\infty) = \eta(P)$. As remarked earlier, we also have $\mathscr{A}(\hat{A}_\infty) \leqq \hat{m}(P)$.

If we try our procedure over a manifold with $H^2(M, K) = 0$ then $m(\eta) = 0$, and A_∞ is a flat connection; perhaps the trivial connection in a product bundle. If we minimize in a fixed bundle we still can't really say much, since we know nothing about the resulting bundle $\hat{P}_\infty$. These remarks apply, of course, to S^4 in particular.

Recall that η is not affected by removing finitely many points. Of course

$$p_1(P) = \frac{1}{4\pi^2} \int_M F \wedge F$$

is drastically affected. Removing one point x on S^4 for example, we see that P_1 is completely free since all bundles over $S^4 - x$ are trivial.

Another interesting question concerns the "first pontryagin class" p_1. For this to be nontrivial we need M oriented. We then have the volume form dv and the $*$ operator giving rise to the usual decomposition $F = F^+ + F^-$, where $*F^+ = F^+$, $*F^- = -F^-$. Using the fact that $F \Lambda F = (|F^+|^2 - |F^-|^2) dv$ we may write:

$$p_1(P) = \frac{1}{4\pi^2} \int_M |F^+|^2 - |F^-|^2 .$$

Remark. p_1 is not integer in general. If we use the killing form, then p_1 is the usual first pontryagin number which is an integer. The number $p_1(P)$ is an invariant of the bundle since it is defined in accordance with the Chern-Weil theory of characteristic classes (see [9]).

The next question to ask is about the relationship of $p_1(\hat{P}_\infty)$ to that of $p_1(P)$. First recall that $\mathscr{A}(A) = \int_M |F_A^+|^2 + |F_A^-|^2$. Letting $\{A_i\}$ be the sequence of connections converging to the connection $\hat{A}_\infty$ in $\hat{P}_\infty$ as we first observe:

a) $$\int_M |F_{\hat{A}_\infty}^+|^2 \leqq \lim \int_M |F_{A_i}^+|^2 ,$$

b) $$\int_M |F_{\hat{A}_\infty}^-|^2 \leqq \lim \int_M |F_{A_i}^-|^2 .$$

This follows from $F_{A_i} \rightharpoonup F_{\hat{A}_\infty}$ in L_0^2 and the fact that $*$ is an isometry. So we have:

$$\begin{aligned} 4\pi^2(p_1(P) - p_1(\hat{P}_\infty)) &= \int_M |F_{A_i}^+|^2 - |F_{A_i}^-|^2 - \int_M |F_{\hat{A}_\infty}^+|^2 - |F_{\hat{A}_\infty}^-|^2 \\ &= \int_M |F_{A_i}^+|^2 + |F_{\hat{A}_\infty}^-|^2 - \int_M |F_{A_i}^-|^2 + |F_{\hat{A}_\infty}^+|^2 . \end{aligned}$$

Therefore we have:

$$\begin{aligned} 4\pi^2(p_1(P) - p_1(\hat{P}_\infty)) &\leqq \lim \int_M |F_{A_i}^+|^2 + |F_{\hat{A}_\infty}^-|^2 - \lim \int_M |F_{A_i}^-|^2 + |F_{\hat{A}_\infty}^+|^2 \\ &\leqq \lim \int_M |F_{A_i}^+|^2 + |F_{A_i}^-|^2 - \lim \int_M |F_{A_i}^-|^2 + |F_{\hat{A}_\infty}^+|^2 \\ &= \hat{m}(P) - \lim \int_M |F_{A_i}^-|^2 + |F_{\hat{A}_\infty}^+|^2 \\ &\leqq \hat{m}(P) - \int_M |F_{\hat{A}_\infty}^-|^2 + |F_{\hat{A}_\infty}^+|^2 \\ &\leqq \hat{m}(P) - \hat{m}(P_\infty) . \end{aligned}$$

Switching orientation on M gives:

Theorem 7.1. *Let P be a bundle, $\hat{P}_\infty$ the bundle constructed when minimizing for $\hat{m}(P)$. Then*

$$|p_1(P) - p_1(\hat{P}_\infty)| \leqq \frac{\hat{m}(P) - \hat{m}(P_\infty)}{4\pi^2} .$$

Remarks. If we fix a bundle P it is possible to realize all other bundles with the same obstruction by gluing in a trivial bundle over a disc to P. In this process one uses an element of $\pi_3 G$ to do the gluing. Of course, such an element corresponds to a bundle over S^4. Therefore, we may hope to glue S^4 connections to our minimizing bundles to get connections which are almost Yang-Mills in other bundles, and apply an implicit function theorem to get Yang-Mills connections at least with suitable assumptions on P (see [13]).

Appendix. Classification of Principal Bundles

For G compact it is well known [11] that the universal covering group $\tilde{G} \simeq \mathbb{R}^k \times G_1 \times \ldots \times G_1$ where the G_i are compact simple simply connected groups. This causes the adjoint bundle to split into a k-dimensional trivial bundle plus 1 bundles corresponding to the G_i. Therefore we have a "vector pontry agin number" $p_1 \in \mathbb{Z}^1$. The following theorem gives a complete description of principal bundles.

Theorem. *Isomorphism classes of principal bundles are uniquely determined by η and the vector pontryagin number.*

Remarks. All values of η are realized, but not all values of p_1 occur. The values of p_1 will differ by arbitrary multiples of a specific $m \in \mathbb{Z}^1$. Furthermore, for a fixed η, the values of p_1 define a coset in $\mathbb{Z}^1/m\mathbb{Z}^1$. This coset is determined by η.

The preceding follows from studying the homotopy classes $[M, BG]$ where BG is the classifying space for G. Standard methods in homotopy theory plus the paper of Dold and Whitney [2] give the theorem.

Of course, other invariants of the bundle may be of interest. However, those invariants will be expressible in terms of η and p_1 since they characterize the bundle. For example, given a bundle with group $\mathbb{U}(n+1)/S^1$ can we lift the group to $\mathbb{U}(n+1)$? This corresponds to asking if a bundle with fiber $\mathbb{C}P^n$ arises as a projective bundle of vector bundle with fiber $\mathbb{C}^{n+1}$. This problem gives rise to the Brauer obstruction $b \in H^3(M, \mathbb{Z})$. The exact sequence of groups

$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1 \to 0$$

gives rise to the Bockstein $\beta: H^2(M, S^1) \to H^3(M, \mathbb{Z})$. Now $\mathbb{U}(n+1)/S^1 \simeq \mathbb{S}U(n)/\mathbb{Z}_n$ so that we have $\eta \in H^2(M, \mathbb{Z}_n)$. But $\mathbb{Z}_n \to S^1$ gives a map $i: H^2(M, \mathbb{Z}_n) \to H^2(M, S^1)$. It is not hard to show that $\beta i(\eta) = b$.

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