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All Massless, Scalar Fields with Trivial S-Matrix are Wick-Polynomials

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Abstract. We extend a result about non-interacting fields given by Buchholz and Fredenhagen. Consider a massless, scalar field ϕ in $3+1$ dimensional space-time which does not interact. The corresponding Hilbert space is assumed to be the Fockspace H of the free massless field A. This implies – as we show in the first part – that all n-point-functions are rational functions of their arguments. In the second part we use this fact to construct a symmetric, traceless tensorfield $\phi^{\mu_1 \cdots \mu_n}$, relatively local to the original field ϕ, and connecting the vacuum with the one particle states. In the last part we prove $\phi^{\mu_1 \cdots \mu_n}$ to be relatively local to the free field A.

0. Introduction

In a series of papers Buchholz establishes a frame for a scattering theory for massless particles in $3+1$ dimensional space-time [1]:

Let $A(x)$ be the free, massless, scalar field acting in the Fockspace H. Let $\phi(x)$ be a real, scalar field which transforms under the same unitary representation of the Poincaré group as $A(x)$. The corresponding Hilbert space is assumed to be the Fockspace H. We identify $A(x)$ with the incoming field $\phi^{\text{in}}(x)$, respectively the outgoing field $\phi^{\text{out}}(x)$. In [1] Buchholz shows that

$$[\phi^{\text{in}}(x), \phi(y)] = 0 \quad \text{for} \quad y - x \in V^- \text{ (backward cone)}$$

and

$$[\phi^{\text{out}}(x), \phi(y)] = 0 \quad \text{for} \quad y - x \in V^+ \text{ (forward cone)}.$$

We want to prove the following:

Theorem. *If $\phi(x)$ has a trivial S-matrix, then $\phi(x)$ is relatively local to the free field $A(x)$.*

This theorem extends the result given by Buchholz and Fredenhagen [2]. In their paper they show first that ϕ can be decomposed into a finite sum of fields ϕ_d with

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dimension d. The technical assumption $P_1\phi(x)\Omega=A(x)\Omega$ ensures that $\phi_1(x)$ equals $A(x)$. Then they conclude from the locality of ϕ that all ϕ_d are relatively local to $\phi_1\equiv A$. And for this second step it is crucial that A shows up in the above decomposition of ϕ. But the example $\phi=:A^3:$ shows that one should modify the proof to get rid of this technical assumption. This turned out to be quite difficult. Our new proof is based on a paper [3] by Buchholz.

I. The Structure of the *n*-Point-Functions

To prove our theorem we assume that ϕ has a trivial S-matrix – i.e. $\phi^{\text{in}}(x)=A(x)=\phi^{\text{out}}(x)$ and therefore we have

$$\phi \text{ is weakly local relative to } A \text{ (see [9, Chap. VII])} \tag{1.1}$$

and

$$[A(x),\phi(y)]=0 \quad \textit{for} \quad (y-x)^2>0\,. \tag{1.2}$$

As shown in [2] we have a decomposition

$$\phi(x)=\sum_{d=0}^{D}\phi_d(x)\,, \tag{1.3}$$

where each field ϕ_d transforms under dilation like

$$D(\lambda)\phi_d(x)D(\lambda)^{-1}=\lambda^d\phi_d(\lambda x), \qquad \lambda>0\,, \tag{1.4}$$

and $D(\lambda)$ denotes the dilation operator acting on $A(x)$ according to

$$D(\lambda)A(x)D(\lambda)^{-1}=\lambda A(\lambda x), \qquad \lambda>0\,. \tag{1.5}$$

Furthermore we want to rely upon the following theorem given by Buchholz [3] which, under the above assumptions, relates interaction to commutation relations for timelike distances:

Theorem. *ϕ does not interact if and only if*

$$[\phi(x),\phi(y)]=0 \quad \textit{for} \quad (y-x)^2>0\,.$$

Therefore we get

$$[\phi(x),\phi(y)]=0 \quad \text{for} \quad (y-x)^2\neq 0\,. \tag{1.6}$$

Lemma 1. *$[\phi(x),\phi(y)]=0$ for $(y-x)^2\neq 0$ implies $(y-x)^{2N}[\phi(x),\ \phi(y)]\Omega\equiv 0$ for some $N\in\mathbb{N}$ in the sense of vector-valued distributions.*

Proof. The vector-valued tempered distribution

$$\psi(u,v):=\left[\phi\left(\frac{u+v}{2}\right),\phi\left(\frac{u-v}{2}\right)\right]\Omega \tag{1.7}$$

vanishes for $v^2\neq 0$. Because of temperedness there is a $N\in\mathbb{N}$ such that

$$(v^2)^N\psi(u,v)\equiv 0\,. \tag{1.8}$$

By the Edge of the Wedge theorem we get for the 4-point-function W_4 of ϕ:

Lemma 2. *$W_4(\xi_1, \xi_2, \xi_3)\xi_3^{2N}$ can be analytically continued to all points (ξ_1, ξ_2, ξ_3) in a complex neighbourhood of $\tau_2^+ \times \mathbb{R}^4$.*

This is the basic assumption for a series of papers – initiated by Schlieder and Seiler [4] – on Wilson-Zimmermann-Expansions. We refer to [5] for the proof of the following property of the n-point-function W_n of ϕ:

Lemma 3. *For every $n \geqq 2$ the functions*

$$F_n(\underline{\xi}) := W_n(\underline{\xi}) \prod_{i=1}^{n-1} \xi_i^{2N} \prod_{1 \leqq i < j \leqq n-1} (\xi_i + \ldots + \xi_j)^{2N}$$

can be analytically continued to $\underline{\xi} \in \mathbb{C}^{4(n-1)}$ with $\|\underline{\xi}\| < R_n$, where $\|\underline{\xi}\|$ denotes the Euclidean norm.

So for all $\underline{\xi} \in \mathbb{C}^{4(n-1)}$ with $\|\underline{\xi}\| < R_n$ the power series

$$F_n(\lambda\underline{\xi}) = \sum_{l=0}^{\infty} a_\ell(\underline{\xi})\lambda^\ell \tag{1.9}$$

is absolutely convergent for $|\lambda| < 1$ and the coefficients $a_\ell(\underline{\xi})$ are polynomials in $\underline{\xi}$. Now we want to use the fact that ϕ is a finite sum of fields with integer dimensions to show that $F_n(\underline{\xi})$ is a polynomial.

For $\underline{\xi} \in \tau_{n-1}^+$ and $0 < \lambda \in \mathbb{R}$ we have

$$\begin{aligned} F_n(\lambda\underline{\xi}) &= \mathscr{W}_n(\lambda z_1, \ldots, \lambda z_n) \prod_{1 \leqq i < j \leqq n} (\lambda z_j - \lambda z_i)^{2N} \\ &= (\Omega, \phi(\lambda z_1) \ldots \phi(\lambda z_n)\Omega) \prod_{1 \leqq i < j \leqq n} (\lambda z_j - \lambda z_i)^{2N} \\ &= (\Omega, D(\lambda)^{-1}\phi(\lambda z_1)D(\lambda) \ldots D(\lambda)^{-1}\phi(\lambda z_n)D(\lambda)\Omega) \prod_{1 \leqq i < j \leqq n} (\lambda z_j - \lambda z_i)^{2N}, \end{aligned} \tag{1.10}$$

and because of

$$\begin{aligned} &D(\lambda)^{-1}\phi_d(\lambda x)D(\lambda) = \lambda^{-d} \cdot \phi_d(x) \\ &= \left(\Omega, \left[\sum_{d=0}^{D} \lambda^{-d}\phi_d(z_1)\right] \ldots \left[\sum_{d=0}^{D} \lambda^{-d}\phi_d(z_n)\right]\Omega\right) \lambda^{Nn(n-1)} \prod_{1 \leqq i < j \leqq n} (z_j - z_i)^{2N}. \end{aligned}$$

Therefore $F_n(\lambda\underline{\xi})$ is a polynomial in λ and as shown in [5] we can take $N = D$. Now the intersection of τ_{n-1}^+ with $\{\underline{\xi} \mid \|\underline{\xi}\| < R_n\}$ is open so all but finitely many $a_\ell(\underline{\xi})$ vanish identically. Therefore $F_n(\underline{\xi})$ is a polynomial and we get the following representation:

Lemma 4. *The n-point-functions have the form*

$$\mathscr{W}_n(z_1, \ldots, z_n) = \frac{P_n(z_1, \ldots, z_n)}{\prod_{1 \leqq i < j \leqq n} (z_j - z_i)^{2D}},$$

where P_n is a polynomial in $z_1, \ldots, z_n$.

We remark that this is exactly the structure which the n-point-functions of the Wick polynomials of a massless free field exhibit.

II. Local Operator Products

In this section we shall construct a local field which is relatively local to the original field ϕ and connects the vacuum with the one particle states. Of course one can formulate conditions on the set of complex functions $\{W_n|n=0,1,\ldots\}$ which are equivalent to the Wightman axioms – i.e. there exist fields such that the given W_n's are the n-point-functions of these fields (see [9]).

Consider an expression like

$$\phi(x_1)\ldots\phi(x_\ell)\prod_{1\leqq i<j\leqq\ell}(x_j-x_i)^{2D} \tag{2.1}$$

which defines an operator valued distribution. We want to show that after applying a differential operator D_x acting on $x_1,\ldots,x_\ell$ and putting $x_1=\ldots=x_\ell=x$ we still have a well defined operator-valued distribution.

For the proof we start with the $n\cdot\ell$-point-function of ϕ in the analyticity domain

$$(\Omega,\phi(z_1^{(1)})\ldots\phi(z_1^{(\ell)})\ldots\phi(z_n^{(1)})\ldots\phi(z_n^{(\ell)})\Omega) \tag{2.2}$$

and multiply it with the necessary factors $\prod_{1\leqq i<j\leqq\ell}(z_k^{(j)}-z_k^{(i)})^{2D}$. Then we apply on each group the differential operator D_x and put within each group the arguments equal to each other. So we end up with the expression

$$\Big\{D_{z_1}\ldots D_{z_n}(\Omega,\phi(z_1^{(1)})\ldots\phi(z_1^{(\ell)})\ldots\phi(z_n^{(1)})\ldots\phi(z_n^{(\ell)})\Omega)\prod_{1\leqq i<j\leqq\ell}(z_1^{(j)}-z_1^{(i)})^{2D}\ldots(z_n^{(j)}-z_n^{(i)})^{2D}\Big\}\Big|_{\underline{z}^{(1)}=\ldots=\underline{z}^{(\ell)}=\underline{z}} \tag{2.3}$$

Because of the structure of the $n\cdot\ell$-point-function (see Lemma 4) and by simple limiting arguments it is easy to see that this defines a n-point-function. The transformation properties under the Lorentz group depend on the operator D_x. If we take a covariant expression we get in general a tensorfield – let's call it ϕ^D. Along the same lines we can prove ϕ^D to be relatively local to the original field ϕ. With the free field A we get the commutation relation (1.2.) because

$$\Big[A(x),\phi(y_1)\ldots\phi(y_\ell)\prod_{1\leqq i<j\leqq\ell}(y_j-y_i)^{2D}\Big]=0 \tag{2.4}$$

as long as all $(y_i-x)^2>0, i=1,\ldots,\ell$ or repeating the analysis given by Buchholz in his fundamental paper [1]. Now for some $\ell\in\mathbb{N}$

$$(\Omega,A(x)\phi(y_1)\ldots\phi(y_\ell)\Omega)\not\equiv 0 \tag{2.5}$$

by asymptotic completeness. But

$$(\Omega,A(\bar{z})\phi(z_1)\ldots\phi(z_\ell)\Omega)\prod_{1\leqq i<j\leqq\ell}(z_j-z_i)^{2D} \tag{2.6}$$

is analytic for $z\in\tau^+$ and small $\|z_i\|$, $i=1,\ldots,\ell$, so we can make a Taylor expansion around $z_1=\ldots=z_\ell=0$, and because of (2.5) there must be a tensorfield $\phi^{\mu_1\ldots\mu_n}$ such that

$$P_1\phi^{\mu_1\ldots\mu_n}(y)\Omega \neq 0\,, \tag{2.7}$$

where P_1 denotes the projection operator onto the asymptotic one particle states. It is no restriction to assume that

$$P_1\partial_{\mu_i}\phi^{\mu_1\ldots\mu_n}(x)\Omega\equiv 0 \quad \text{for all} \quad i \tag{2.8}$$

[otherwise we go over to the contracted field

$$\psi^{\mu_1\ldots\mu_{n-1}}(x):=\partial_\nu\phi^{\mu_1\ldots\mu_{i-1}\nu\mu_i\ldots\mu_{n-1}}(x) \text{ and so on!}] .$$

Equation (2.8) forces the corresponding asymptotic field to be proportional to $\partial^{\mu_1}\ldots\partial^{\mu_n}A(x)=:A^{\mu_1\ldots\mu_n}(x)$. But $A^{\mu_1\ldots\mu_n}(x)$ is obviously symmetric in the indices and traceless so we can symmetrize $\phi^{\mu_1\ldots\mu_n}(x)$ and subtract out all traces and still get the same asymptotic field. We summarize our construction in

Lemma 5. *There exists a local, symmetric, and traceless tensorfield $\phi^{\mu_1\ldots\mu_n}$ with*

i) $\phi^{\mu_1\ldots\mu_n}$ *relatively local to* ϕ

ii) $[A(x),\phi^{\mu_1\ldots\mu_n}(y)]=0$ *for* $(y-x)^2>0$

iii) $P_1\phi^{\mu_1\ldots\mu_n}(x)\Omega=A^{\mu_1\ldots\mu_n}(x)\Omega$.

By Lemma 5 we have found a field with properties which are very similar to those assumed by Buchholz and Fredenhagen in their paper [2] with the only difference that it is a symmetric, traceless tensorfield instead of a scalar field. In the next section we shall show that $\phi^{\mu_1\ldots\mu_n}$ is necessarily a Wick polynomial in the free field A.

III. Completion of the Proof

Using the same methods as in [2] we show

$$\phi^{\mu_1\ldots\mu_n}(x)=\sum_{\substack{d\in\mathbb{N}\\ \text{finite}}}\phi_d^{\mu_1\ldots\mu_n}(x)\,, \tag{3.1}$$

where each field $\phi_d^{\mu_1\ldots\mu_n}$ carries dimension d. Because of

$$P_1\phi^{\mu_1\ldots\mu_n}(x)\Omega=A^{\mu_1\ldots\mu_n}(x)\Omega \tag{3.2}$$

we know

$$P_1\phi_d^{\mu_1\ldots\mu_n}(x)\Omega\equiv 0 \quad \text{for} \quad d\neq n+1\,. \tag{3.3}$$

We are left with the 2-point-function

$$(\Omega,\phi_d^{\mu_1\ldots\mu_n}(x)(1-P_1)\phi_d^{\nu_1\ldots\nu_n}(y)\Omega)\,. \tag{3.4}$$

In Appendix A we write down the general form of such 2-point-functions given by Oksak and Todorov [6]. If we further specialize this result to homogeneous 2-point-functions we get $d>n+1$ because the projection operator $(1-P_1)$ suppresses the contribution of mass zero fields. Therefore we can identify $A^{\mu_1\cdots\mu_n}$ with $\phi_{n+1}^{\mu_1\cdots\mu_n}$ and all other fields $\phi_d^{\mu_1\cdots\mu_n}$ showing up in the decomposition (3.1) have dimensions greater than or equal to $n+2$.

From locality we get for all $\lambda>0$ and for $(y-x)^2<0$

$$0=D(\lambda)\left[\phi^{\mu_1\dots\mu_n}\left(\frac{x}{\lambda}\right),\phi^{\nu_1\dots\nu_n}\left(\frac{y}{\lambda}\right)\right]D(\lambda)^{-1}$$
$$=\sum_{k=2n+2}^{2N}\lambda^k\sum_{d+d'=k}[\phi_d^{\mu_1\dots\mu_n}(x),\phi_{d'}^{\nu_1\dots\nu_n}(y)]. \tag{3.5}$$

The following lemma, if we use it successively, shows that all $\phi_d^{\mu_1\cdots\mu_n}$ are relatively local to $A^{\mu_1\dots\mu_n}$.

Lemma 6. *Let* $[A(x),\phi_d^{\nu_1\cdots\nu_n}(y)]=0$ *for* $(y-x)^2>0$ *and*

$$[A^{\mu_1\dots\mu_n}(x),\phi_d^{\nu_1\dots\nu_n}(y)]+[\phi_d^{\mu_1\dots\mu_n}(x),A^{\nu_1\dots\nu_n}(y)]=0 \quad \text{for} \quad (y-x)^2<0,$$

then $[A^{\mu_1\cdots\mu_n}(x),\phi_d^{\nu_1\cdots\nu_n}(y)]=0$ *for* $(y-x)^2<0$.

Proof. Because $[A(x),\phi_d^{\nu_1\cdots\nu_n}(y)]=0$ for $(y-x)^2>0$ it is enough to prove

$$[A^{\mu_1\dots\mu_n}(x),\phi_d^{\nu_1\dots\nu_n}(y)]\Omega=0 \quad \text{for} \quad (y-x)^2<0 \tag{3.6}$$

because the set of all vectors $\{\Omega, A(f_1)\Omega,\dots,A(f_1)\dots A(f_n)\Omega,\dots\}$ with $\operatorname{supp} f_i$ timelike to x and y forms a dense set in H.

a) We consider

$$(\Omega, A(y)[A(x),\phi_d^{\mu_1\dots\mu_n}(0)]\Omega)=:F^{\mu_1\dots\mu_n}(x,y). \tag{3.7}$$

Using spectrum condition we get for the Fourier transform

$$\tilde{F}^{\mu_1\dots\mu_n}(p,q)=\delta_-(q^2)\{\delta_+(p^2)f_+^{\mu_1\dots\mu_n}(p,q)+\delta_-(p^2)f_-^{\mu_1\dots\mu_n}(p,q)\}. \tag{3.8}$$

Lorentz covariance restricts $f_\pm^{\mu_1\cdots\mu_n}(p,q)$ to be covariant polynomials where the coefficients are invariant distributions. Covariance under dilations forces the invariant distributions to be homogeneous and fixes them up to factors – e.g. for n even

$$f_\pm^{\mu_1\dots\mu_n}(p,q)=(pq)^{(d-n-2)/2}P_\pm(p^\mu,q^\mu,g^{\mu\nu}), \tag{3.9}$$

where $P_\pm(p^\mu,q^\mu,g^{\mu\nu})$ denote covariant polynomials homogeneous of degree n and symmetric in the indices $\mu_1,\dots,\mu_n$.

In Appendix B we characterize solutions of the wave equation which vanish for timelike, respectively spacelike, arguments (and this analysis might be of some independent interest!). Because $\Box_x F^{\mu_1\cdots\mu_n}(x,y)=0$ and $F^{\mu_1\cdots\mu_n}(x,y)=0$ for $x^2>0$, we can apply the criterion given in Appendix B which restricts the exponents of pq to be integers. And because $d\geqq n+2$ all these exponents are positive, which implies $F^{\mu_1\cdots\mu_n}(x,y)=0$ for $x^2<0$. The span of $A(f)\Omega$ is dense in P_1H so we have

$$P_1[A(x),\phi_d^{\nu_1\dots\nu_n}(y)]\Omega=0 \quad \text{for} \quad (y-x)^2<0. \tag{3.10}$$

b) Now we consider

$$\left(\psi,(1-P_1)\left[A^{\mu_1\dots\mu_n}\left(-\frac{\xi}{2}\right),\phi^{\nu_1\dots\nu_n}\left(\frac{\xi}{2}\right)\right]\Omega\right). \tag{3.11}$$

But $1-P_1$ projects out vectors with momentum $p^\mu \in L^+ = \{p^2=0, p^0>0\}$ so we only take vectors $\psi \in E(V^+)H$ (and not $\psi \in E(\bar{V}^+)H$!). Let $K \subset V^+$ be a ball with center p_0 and take $\psi \in E(K)H$.

We want to use a modified "Jost-Lehmann-Dyson" representation. Now

$$G_{[A,\phi]}(\sigma, q) := \int \left(\psi, \left[A^{\mu_1 \ldots \mu_n}\left(-\frac{\xi}{2}\right), \phi^{\nu_1 \ldots \nu_n}\left(\frac{\xi}{2}\right)\right]\Omega\right) \cdot e^{iq\xi} \cos\sigma \sqrt{-\xi^2}\, d^4\xi \tag{3.12}$$

and

$$G_{[\phi,A]}(\sigma, q) := \int \left(\psi, \left[\phi^{\mu_1 \ldots \mu_n}\left(-\frac{\xi}{2}\right), A^{\nu_1 \ldots \nu_n}\left(\frac{\xi}{2}\right)\right]\Omega\right) \cdot e^{iq\xi} \cos\sigma \sqrt{-\xi^2}\, d^4\xi \tag{3.13}$$

exist because $[A^{\mu_1 \ldots \mu_n}(x), \phi^{\nu_1 \ldots \nu_n}(y)] = 0$ if $(y-x)^2 > 0$, and fulfill the ultrahyperbolic equation

$$(\partial_{\sigma\sigma} + \partial_{q^0q^0} - \Delta_{\boldsymbol{q}})G(\sigma, q) = 0, \quad G(-\sigma, q) = G(\sigma, q). \tag{3.14}$$

For $\sigma = 0$ we have

$$G_{[A,\phi]}(0, q) = \int \left(\psi, \left[\tilde{A}^{\mu_1 \ldots \mu_n}\left(\frac{p+q}{2}\right), \tilde{\phi}^{\nu_1 \ldots \nu_n}\left(\frac{p-q}{2}\right)\right]\Omega\right) d^4p. \tag{3.15}$$

Momentum conservation requires $p \in K$. The support of $A(Q)$ is contained in $Q^2 = 0$ and therefore we have

$$\begin{aligned} &\operatorname{supp} G_{[A,\phi]}(0, \cdot) \subseteqq \{q | (K+q)^2 = 0\}, \\ &\operatorname{supp} G_{[\phi,A]}(0, \cdot) \subseteqq \{q | (K-q)^2 = 0\}. \end{aligned} \tag{3.16}$$

The assumption

$$[A^{\mu_1 \ldots \mu_n}(x), \phi^{\nu_1 \ldots \nu_n}(y)] + [\phi^{\mu_1 \ldots \mu_n}(x), A^{\nu_1 \ldots \nu_n}(y)] = 0 \quad \text{for} \quad (y-x)^2 \neq 0 \tag{3.17}$$

implies $(G_{[A,\phi]} + G_{[\phi,A]})(\sigma, q)$ to be a polynomial in σ – i.e. there is a N such that

$$(\partial_\sigma)^{2N}(G_{[A,\phi]} + G_{[\phi,A]})(\sigma, q) = 0. \tag{3.18}$$

As a consequence we have

$$\operatorname{supp}(\partial_\sigma)^{2N} G_{[A,\phi]}(0, \cdot) \subseteqq \{q | (K+q)^2 = 0 \quad \text{and} \quad (K-q)^2 = 0\}. \tag{3.19}$$

But $(\partial_\sigma)^{2N} G_{[A,\phi]}$ still fulfills the ultrahyperbolic equation so we can use the mean value theorem by Asgeirsson [7] and conclude

$$(\partial_\sigma)^{2N} G_{[A,\phi]}(\sigma, q) = 0 \tag{3.20}$$

because $(\partial_\sigma)^{2N} G_{[A,\phi]}(0, q)$ vanishes for all $\boldsymbol{q} \in \mathbb{R}^3$ as long as $|q^0|$ is big enough. This in turn implies

$$\left(\psi, \left[A^{\mu_1 \ldots \mu_n}\left(-\frac{\xi}{2}\right), \phi^{\nu_1 \ldots \nu_n}\left(\frac{\xi}{2}\right)\right]\Omega\right) = 0 \quad \text{for} \quad \xi^2 < 0. \tag{3.21}$$

This completes the proof of Lemma 6.

But if all $\phi_d^{\mu_1\cdots\mu_n}$ are relatively local to $A^{\mu_1\cdots\mu_n}$ then $\phi^{\mu_1\cdots\mu_n}$ has the same property. The transitivity of relative locality gives finally that ϕ is relatively local to A – i.e. ϕ is a Wick polynomial in the free field A.

Remark. One could try to adapt the above proof to the case where the asymptotic fields carry spin n. But to avoid too many technical complications one should try to formulate a proof within the algebraic framework of quantum field theory.

Appendix A

We need the explicit form of the 2-point-function for a symmetric, traceless tensorfield $\phi^{\mu_1\cdots\mu_n}$ of rank n given by Oksak and Todorov (see [6] and [10], Appendix F). From $\phi^{\mu_1\cdots\mu_n}$ we go over to

$$\phi(x,z):=\phi^{\mu_1\cdots\mu_n}(x)(z\sigma_{\mu_1}\bar{z})\dots(z\sigma_{\mu_n}\bar{z}),\ z\in\mathbb{C}^2\backslash\{0\}. \tag{A2}$$

The 2-point-function

$$(\Omega,\phi(x,w)\phi(y,z)\Omega)=:F(y-x;w,z) \tag{A2}$$

is a homogeneous function in $w,\bar{w},z,\bar{z}$ of degree n. The Fourier transform of $F(\xi;w,z)$ is given by

$$\tilde{F}(p;w,z)=(z\underset{\sim}{p}\bar{z})^n(w\underset{\sim}{p}\bar{w})^n\sum_{k=0}^{n}f_k(p^2)P_k(v), \tag{A3}$$

with

$$v:=\frac{|z\underset{\sim}{p}\bar{w}|^2-p^2|z\varepsilon w|^2}{(z\underset{\sim}{p}\bar{z})(w\underset{\sim}{p}\bar{w})},$$

$$P_k(v):=2^{-k}\sum_{\ell=0}^{k}\binom{k}{\ell}^2(v-1)^{k-\ell}(v+1)^{\ell},$$

"Legendre polynomials," and positive distributions $f_k(p^2)$ with support in $[0,\infty)$. Now we assume in addition $\phi(x,z)$ to have dimension d – i.e.

$$F(\lambda\xi;w,z)=\lambda^{-2d}F(\xi;w,z),\quad \lambda>0. \tag{A4}$$

This implies that the f_k's are homogeneous distributions of degree $d-n-2$

$$f_k((\lambda p)^2)=\lambda^{2(d-n-2)}f_k(p^2),\quad \lambda>0. \tag{A5}$$

But the f_k's are positive distributions and therefore $d-n-2$ must be greater than or equal to -1. We get

$$f_k(p^2)=c_k\begin{cases}(p^2)^{d-n-2}, & d>n+1,\\ \delta(p^2), & d=n+1.\end{cases} \tag{A6}$$

This proves the following

Lemma. *For a symmetric, traceless tensorfield $\phi^{\mu_1\cdots\mu_n}$ of rank n and dimension d we have*

i) $d\geqq n+1$,
ii) $d=n+1$ *if and only if* $\Box\phi^{\mu_1\cdots\mu_n}(x)=0$.

Remark. There might be a problem if $f_k(p^2)$ contains a $\delta(p^2)$-contribution because of the peculiarities of mass zero fields.

Appendix B

We want to characterize solutions of the wave equation in 3 space and 1 time dimensions, that vanish for timelike respectively spacelike arguments.

Any weak solution $f \in \mathscr{S}'(\mathbb{R}^4)$ of the wave equation $\Box f(x)=0$ can be decomposed into plane waves – i.e.

$$f(x)=\int_{\mathbb{R}^3} \{e^{i(\boldsymbol{p}\boldsymbol{x}-|\boldsymbol{p}|x^0)}a(\boldsymbol{p})+e^{i(\boldsymbol{p}\boldsymbol{x}+|\boldsymbol{p}|x^0)}b(\boldsymbol{p})\}\,d^3p \tag{B1}$$

with $a, b \in \mathscr{S}'(\mathbb{R}^3)$. This decomposition is unique up to solutions, which have support only in the point $p=0$

1. Solutions which Vanish for Timelike Arguments. Now we require in addition to the wave equation that $f(x)=0$ for $x^2>0$. By the mean value theorem of Asgeirsson [7] this is equivalent to

$$\operatorname{supp}(P(\partial)f)(x^0,\mathbf{0}) \subseteqq \{x^0=0\} \tag{B2}$$

for all polynomials P in $\partial=(\partial_1,\partial_2,\partial_3)$.

Therefore the Fourier transform with respect to x^0

$$\begin{aligned} q_p(\omega) &:= (2\pi)^{-1}\int_{-\infty}^{+\infty} (P(\partial)f)(x^0,\mathbf{0})e^{i\omega x^0}dx^0 \\ &= \int_{\mathbb{R}^3} \{\delta(\omega-|\boldsymbol{p}|)P(i\boldsymbol{p})a(\boldsymbol{p})+\delta(\omega+|\boldsymbol{p}|)P(i\boldsymbol{p})b(\boldsymbol{p})\}\,d^3p \end{aligned} \tag{B3}$$

is a polynomial in ω. It is sufficient to take only the special polynomials

$$|\boldsymbol{p}|^{\ell}\, Y_{\ell m}(\Omega),\ Y_{\ell m}\text{: spherical harmonics}. \tag{B4}$$

By introducing polar coordinates we get

$$\begin{aligned} q_{\ell m}(\omega) &= \Theta(\omega)\omega^{2+\ell}\int_{|\boldsymbol{p}|=\omega} a(\boldsymbol{p})\,Y_{\ell m}(\Omega)\,d\Omega \\ &\quad + \Theta(-\omega)(-\omega)^{2+\ell}\int_{|\boldsymbol{p}|=-\omega} b(\boldsymbol{p})\,Y_{\ell m}(\Omega)\,d\Omega \\ &= \Theta(\omega)\omega^{2+\ell}a_{\ell m}(\omega)+\Theta(-\omega)(-\omega)^{2+\ell}b_{\ell m}(-\omega)\,. \end{aligned} \tag{B5}$$

Therefore we have proved

Lemma 1. *For a solution f of the wave equation to vanish for $x^2>0$ it is necessary and sufficient that*

$$\begin{aligned} &q_{\ell m}(\omega)=\Theta(\omega)\omega^{2+\ell}a_{\ell m}(\omega)+\Theta(-\omega)(-\omega)^{2+\ell}b_{\ell m}(-\omega) \\ &\ell=0,1,2,\ldots,m=-\ell,\ldots,\ell \end{aligned}$$

are polynomials in ω.

2. Solutions which Vanish for Spacelike Arguments. Now we require $f(x)=0$ for $x^2<0$. Because Huyghens' principle is valid in $3+1$ dimensions the solution is determined by the Cauchy data for $x^0=0$:

$$\begin{aligned} \operatorname{supp} f(0,\boldsymbol{x}) &\subseteqq \{\boldsymbol{x}=0\}\,, \\ \operatorname{supp}(\partial_0 f)(0,\boldsymbol{x}) &\subseteqq \{\boldsymbol{x}=0\}\,, \end{aligned} \tag{B6}$$

or expressed in $a(\boldsymbol{p})$ and $b(\boldsymbol{p})$

$$\begin{aligned} a(\boldsymbol{p}) &= \tfrac{1}{2}\left(P(\boldsymbol{p}) + \frac{i}{|\boldsymbol{p}|}Q(\boldsymbol{p})\right), \\ b(\boldsymbol{p}) &= \tfrac{1}{2}\left(P(\boldsymbol{p}) - \frac{i}{|\boldsymbol{p}|}Q(\boldsymbol{p})\right) \end{aligned} \tag{B7}$$

with $P(\boldsymbol{p})$ and $Q(\boldsymbol{p})$ polynomials.

Expanding P and Q in terms of $|\boldsymbol{p}|^{\ell} Y_{\ell m}$ we get

$$q_{\ell m}(\omega) = \omega^{2+2\ell}\tfrac{1}{2}\left(P_{\ell m}(\omega^2) + \frac{i}{\omega}Q_{\ell m}(\omega^2)\right), \tag{B8}$$

and there is a $L \in \mathbb{N}$ such that $q_{\ell m} \equiv 0$ for $\ell > L$. Therefore we have

Lemma 2. *For a solution f of the wave equation to vanish for $x^2<0$ it is necessary and sufficient that*

i) *there is a $L\in\mathbb{N}$ such that $q_{\ell m}\equiv 0$ for $\ell > L$,*

ii) $q_{\ell m}(\omega) = \Theta(\omega)\omega^{2+\ell}a_{\ell m}(\omega) + \Theta(-\omega)(-\omega)^{2+\ell}b_{\ell m}(-\omega)$,

$\ell = 0, \ldots, L,\ m = -\ell, \ldots, \ell$

are polynomials with a zero at $\omega=0$ of the order greater than or equal to $2\ell+1$.

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