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On the Atiyah-Drinfeld-Hitchin-Manin Construction for Self-Dual Gauge Fields

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Abstract. The vector spaces A, B, C, in terms of which the general construction due to Atiyah, Drinfeld, Hitchin and Manin for self-dual gauge fields defined over some region of Euclidean space is phrased, are shown to be expressible in terms of the spaces spanned by the solutions of certain linear covariant differential equations depending on the gauge field. The corresponding linear maps between A and B, B and C are given with the properties required by ADHM and the results then necessary to verify the construction informally proved. The local problems associated with assuming the gauge field to obey the self-duality equations are separated from the global problems of assuring the required boundary conditions for a particular solution. With suitable global conditions C is shown to be the dual of A and a natural scalar product defined on B so as to reconstruct the gauge field in the standard form given by the construction. A discussion is given of the requirements entailed by the condition of a symmetry on the gauge field and the relation to the usual cohomological treatment is outlined in an appendix.

1. Introduction

Given that non-Abelian gauge theories are an essential feature in our theoretical description of particle physics it seems desirable to explore their mathematical structure in some detail. In this context the analysis of possible solutions of the classical equations obeyed by the gauge field is of interest. An important set, although of course in no way is it the general case, are the solutions of the Euclidean self-duality equations. For a gauge field $A_\mu = A_\mu^a t_a$, $\{t_a\}$ being a set of matrix generators forming a basis for the Lie algebra of a gauge group $\mathcal{G}$, these read

$$F_{\mu\nu} = {}^*F_{\mu\nu} \equiv \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta}\,,$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,. \tag{1.1}$$

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The classical field equations in this case are guaranteed to be satisfied by virtue of the Bianchi identity (throughout we only consider self-dual fields, for anti self-dual fields $F_{\mu\nu} = -{}^*F_{\mu\nu}$ trivial modifications are required).

Such self-dual gauge fields have been considered in the following contexts.

1) The original Belavin, Polyakov, Schwartz and Tyupkin (BPST) instanton [1], and for $\mathscr{G} = \mathrm{SU}(2)$, the multi-instanton generalizations due to 't Hooft [2] and with collinear O(3) rotational symmetry to Witten and Peng [3]. The latter case has been extended to a general gauge group $\mathscr{G}$ [4]. For multi-instanton solutions the gauge fields are defined on the one point compactification of Euclidean four-dimensional space, topologically identical to S^4, so that the gauge field behaves as a pure gauge at infinity, i.e.,

$$|x| \to \infty\,, \qquad A_\mu \sim g\partial_\mu g^{-1}\,, \qquad g(\hat{x}) \in \mathscr{G}\,, \qquad \hat{x} = x/|x|\,. \tag{1.2}$$

These solutions are specified by an integer valued topological index

$$k = -\frac{1}{16\pi^2}\int d^4x\, \mathrm{tr}(F_{\mu\nu}{}^*F_{\mu\nu}) \tag{1.3}$$

$(\mathrm{tr}(t_a t_b) = -1/2\delta_{ab})$ and correspond to minima of the Euclidean gauge theory action. For any given k and gauge group $\mathscr{G}$ the general solution depends on parameters ranging over some space, the dimensions of which is well understood [5], but whose other properties are unclear. The known explicit solutions [2–4] correspond only to a subspace of the full parameter space for $k > 2$.

2) The so-called caloron solutions [6, 7] which are multi-instanton gauge fields periodic in one Cartesian co-ordinate, thus being defined on $R^3 \times S^1$. They are appropriate in considering the quantized gauge field system at finite temperature.

3) The Prasad-Sommerfield monopole solution [8] for $\mathscr{G} = \mathrm{SU}(2)$ of the Bogomolny equations [9], to which the self-duality equations (1.1) reduce when the gauge fields are independent of one Cartesian co-ordinate, e.g., x_4, hence being defined as R^3, and the associated component of the gauge field, $A_4 = \Phi$, behaves as a Higgs scalar with the boundary conditions for $r = |\underset{\sim}{x}| \to \infty$

$$\begin{gathered} \Phi(\underset{\sim}{x}) \sim \varphi(\hat{\underset{\sim}{x}})\,, \qquad \underset{\sim}{A}(\underset{\sim}{x}) \sim \frac{1}{r}\underset{\sim}{a}(\hat{\underset{\sim}{x}})\,, \qquad \hat{\underset{\sim}{x}} = \frac{\underset{\sim}{x}}{r}\,, \\ |\varphi| = C\,, \qquad |\varphi|^2 = -2\,\mathrm{tr}(\varphi^2)\,. \end{gathered} \tag{1.4}$$

These solutions of the Bogomolny equations have been extended to describe a single monopole with spherical symmetry for general gauge groups $\mathscr{G}$ [10] and recently multimonopole solutions have been found for $\mathscr{G} = \mathrm{SU}(2)$ both when they are superposed [11] and, albeit less explicitly, when they are separated [12] with the full set of parameters [13].

4) The so-called chromon solutions [14] which are self-dual excitations of a covariantly constant Abelian self-dual background. They are thus defined on R^4 with the Euclidean action density even asymptotically non-zero.

5) The minimum action field configurations on a hypertorus T^4 with twisted boundary conditions satisfy the self-duality equations and can be given explicitly for $\mathscr{G} = \mathrm{SU}(n)$ [15].

Besides the above the self-duality equations may be dimensionally reduced further [16] but interesting solutions are harder to find. In each case described above the global, and hence topological, aspects differ although techniques of solutions are similar. The caloron solutions [6, 7] are expressed in terms of the 't Hooft representation of self-dual gauge fields [2], as can the PS monopole [17]. Collinear instantons [4] and spherically symmetric monopoles [10] are closely related while recent multimonopole solutions [11, 12] have been obtained in terms of a framework originally developed to describe instantons [18]. However, despite these various results, the sets of solutions of the self-duality equations at present available are by no means complete.

As against this the approach of Atiyah, Drinfeld, Hitchin and Manin [19] (ADHM) offers a general construction for multi-instanton gauge fields on S^4 with many desirable features, in particular

a) It is applicable for arbitrary topological index k and gauge group $\mathscr{G}$, with minor modifications [20, 21], and all multi-instanton fields on S^4 are guaranteed in principle to be expressible in the form given by the construction with the full range of parameters. The gauge arbitrariness of the solutions occurs naturally and explicitly.

b) For a self-dual gauge field A_μ given as a tensor product of differing self-dual gauge fields for differing gauge groups, or for a self-dual gauge field A_μ in other than the fundamental representation of the gauge group, an explicit connection of the formalism of the ADHM construction for A_μ can be made to that for the individual gauge fields in the tensor product, or to that for the corresponding gauge field in the fundamental representation [22].

c) The Green's function for the covariant Laplacian on flat space D^2, $D_\mu = \partial_\mu + A_\mu$, is given very simply in terms of natural objects of the construction, and the result can be used as a basis for the Green's function for higher spin operators [20, 21].

d) Solutions of the Dirac equation, and also higher spin operators, in the self-dual gauge field background are straightforwardly given and their normalization integrals on flat space can be computed [20, 23].

e) A substantial body of results exists for the multi-instanton functional determinants that arise in semi-classical approximations to functional integrals about such self-dual background gauge fields [24].

By virtue of (b) the results described in (c), (d) and (e) can be extended to tensor products and other representations of the gauge group. Despite these various advantages, and its inherent simplicity, the ADHM construction has not so far proved very successful in generating new explicit self-dual solutions with an unconstrained parametrization, save for a $k=3$ SU(2) multi-instanton solution [21]. General solutions for arbitrary k would be desirable for computing the functional measure for gauge theories in the semi-classical approximation. In the hope of eventually finding such a wider class of self-dual gauge fields we have investigated the ADHM construction with a view to making it more constructive and also applying it to the monopole solutions described earlier.

The starting point for the construction is the observation of Ward [25], see also Yang [26], that self-duality of $F_{\mu\nu}$, Eq. (1.1), implies that the connection A_μ is integrable on anti self-dual planes in complexified, compactified Euclidean four-dimensional space. Thus for $n_{i\mu}$, $i=1,2$, two independent tangent vectors (for Euclidean space with metric $\delta_{\mu\nu}$ we make no distinction between tangent vectors and their duals) to such an anti self-dual plane $\mathscr{S}$,

$$n_i \cdot n_j = 0\,, \qquad n_{1\mu} \wedge n_{2\nu} = -*(n_{1\mu} \wedge n_{2\nu})\,. \tag{1.5}$$

We have then

$$[n_1 \cdot D, n_2 \cdot D] = 0\,, \tag{1.6}$$

which is the requisite integrability condition for

$$n_i \cdot D\varrho|_{x \in \mathscr{S}} = 0 \tag{1.7}$$

defining a covariantly constant scalar on $\mathscr{S}$.

The anti self-dual planes $\{\mathscr{S}\}$ in four dimensional space can be specified by points in CP^3, given in homogeneous coordinates by complex four-vector ξ^α, arbitrary up to $\xi^\alpha \to \lambda \xi^\alpha$. Decomposing ξ into two-spinors

$$\xi^\alpha = (\eta_A, \chi^{A'})\,, \qquad A, A' = 1, 2\,, \tag{1.8}$$

then the anti self-dual plane $\mathscr{S}_\xi$ corresponding to ξ is defined by the equation

$$\pi^{A'}(x) \equiv \chi^{A'} - x^{A'A}\eta_A = 0\,, \tag{1.9}$$

for $x^{A'A} = (x_\mu e_\mu)^{A'A}$, e_μ being an appropriate basis of 2×2 matrices or quaternions, described in more detail in Appendix A, where the tangent vectors to $\mathscr{S}_\xi$ are shown to be expressible in terms of η so as to satisfy (1.5). In the analysis of self-dual gauge fields Ward [25] was then led to consider the space E_ξ of all covariantly constant scalar fields on $\mathscr{S}_\xi$ which may be defined by

$$\begin{aligned} &E_\xi = \{\varrho\}/\sim\,, \qquad n_i \cdot D\varrho|_{x \in \mathscr{S}_\xi} = 0\,, \\ &\varrho \sim \varrho' \quad \text{if} \quad (\varrho - \varrho')|_{x \in \mathscr{S}_\xi} = 0\,. \end{aligned} \tag{1.10}$$

Assuming the gauge field A_μ is irreducible, i.e.,

$$D_\mu \varrho = 0 \Rightarrow \varrho = 0\,, \tag{1.11}$$

then E_ξ has the dimension d of the representation space of $\mathscr{G}$ on which A_μ acts. The vector bundle E thereby formed over CP^3 with fibres E_ξ then encodes the gauge connection A_μ and from E, A_μ can be reconstructed up to the usual gauge arbitrariness [25].

In the ADHM construction E_ξ is determined by linear algebra in terms of vectors spaces A, B, C and linear maps, $f(\xi)$, $g(\xi)$, homogeneous of degree one in ξ where

$$A \xrightarrow{f(\xi)} B \xrightarrow{g(\xi)} C\,, \qquad g(\xi)f(\xi) = 0 \tag{1.12}$$

so that

$$E_\xi = \ker g(\xi)/\mathrm{im}\, f(\xi)\,. \tag{1.13}$$

In the original proof of the construction [19] the spaces A, B, C are represented in term of sheaf cohomology groups, with which the theoretical physics community is perhaps not generally familiar (at least the author) although there are discussions designed specifically to introduce these ideas and then show the general existence of the construction [27–29]. With this powerful framework the global problem of defining the gauge field A_μ on S^4 with given topological index k, i.e., obtaining the limiting behaviour (1.2), is resolved simultaneously with the local problem of getting A_μ to be a solution of the self-duality equations.

Nevertheless the ADHM construction is not only limited to this global requirement since Nahm has shown [30] how it can be made to work for the simple PS monopole with all the usual desirable features. However, in this case the form to take for the spaces A, B, C and the associated linear maps f, g is not so clear cut and was not guaranteed to be possible by general results. Still, as stressed by Witten in an article [31] which stimulated this investigation, the ADHM construction may be considered purely locally for self-dual gauge fields defined over some open region of four-dimensional Euclidean space. The spaces A, B, C are then defined by the solutions of certain covariant linear differential equations with the gauge field A_μ as a background.

Thus in Sect. 2 of this paper we set out these equations, describe the associated linear maps f, g and prove the relevant results to ensure (1.13). It should be stressed that we make no pretensions to exact rigour. Notational details are collected in Appendix A. In Appendix B we give a brief sketch of the relationship between the various differential equations and the corresponding sheaf cohomology groups. Also in another Appendix C we briefly describe how the dimensionality of the space B may be determined. In Sect. 3 we show that with suitable extra global requirements, which ensure amongst other aspects,

$$\ker f(\xi)=0 \tag{1.14}$$

and also a certain free Dirac equation, in which there is no gauge field, should have no solutions then a natural positive definite scalar product $\langle\,,\rangle$ can be defined on B, and also C is the dual of A. In these circumstances the standard form of the construction

$$A_\mu(x)=\langle v(x), \partial_\mu v(x)\rangle\,, \qquad \langle v(x), v(x)\rangle = 1_{\mathscr{G}}\,, \tag{1.15}$$

is realised with $v(x)$ satisfying certain linear equations given later.

Finally in Sect. 4 we show how symmetries of self-dual fields are reflected in the ADHM construction, as required for monopole solutions invariant under a translational symmetry in the t direction, and consider how the present investigation may be extended.

2. Spaces and Maps for the ADHM Construction

With the notational details set out in Appendix A the vector spaces A, B, C are defined, following partially Witten's outline [31], in reverse order.

Space C. This is spanned by solutions of the Dirac equation

$$D_{AA'}\psi^{A'}=0\,, \tag{2.1}$$

or, equivalently, using the spinor derivatives introduced in (A.10)

$$d_u^{A'}\psi^{B'} - d_u^{B'}\psi^{A'} = 0\,, \tag{2.2}$$

for $\psi^{A'}(x)$ independent of u_B. A linearly independent complete set of solutions of (2.1), or (2.2), is assumed to be assembled as a row vector $\psi^{A'}$ so, with this basis for C, an arbitrary element is expressible as

$$\psi^{A'} w'\,, \tag{2.3}$$

for some column vector w', taken to be in a space $\tilde{C}$.

Space B. Just as discussed for C, the elements of B are given in terms of a suitable complete basis, with $v \in \tilde{B}$, by

$$\phi v\,, \qquad \Omega_\beta^{A'} v \tag{2.4}$$

with the pair $(\phi, \Omega_\beta^{A'})$ forming row vectors of linearly independent solutions of

$$d_u^{A'}\phi = \Omega_\beta^{A'} z^\beta\,, \qquad z = (u, xu)\,, \tag{2.5a}$$

$$D_{AA'}\Omega_\beta^{A'} = 0\,, \tag{2.5b}$$

for $(\phi, \Omega_\beta^{A'})$ independent of u. Consistency of Eqs. (2.5) requires that the field strength $F_{\mu\nu}$ corresponding to the gauge field A_μ, $D_\mu = \partial_\mu + A_\mu$, is self-dual. Since C is complete we can write

$$\Omega_\beta^{A'} = \psi^{A'} A'_\beta\,, \tag{2.6}$$

where $A'_\beta : \tilde{B} \to \tilde{C}$. Further defining, for z^β as in (2.5a),

$$A'(z) = A'_\beta z^\beta = \Delta'^B(x) u_B\,, \tag{2.7}$$

then Δ'^B is linear in x, explicitly, with $A'_\beta = (a'^B, b'_{B'})$

$$\Delta'^B(x) = a'^B + b'_{B'} x^{B'B}\,. \tag{2.8}$$

With Δ' in the form (2.8), Eqs. (2.5) can be alternatively expressed as

$$D^{A'A}\phi = \psi^{A'}\Delta'^A\,. \tag{2.9}$$

Space A. Similarly the elements of A may be represented with $w \in \tilde{A}$, by

$$\lambda_{B'} w\,, \tag{2.10}$$

with $\lambda_{B'}$ a row vector of a complete set of linearly independent solutions of

$$D_{AA'} D^2 \tilde{\lambda}^{A'} = 0\,. \tag{2.11}$$

In order to discuss the map $f : A \to B$ an alternative characterization of A, involving some redundancy, is necessary. Define

$$\lambda_\beta = (\lambda^B, \lambda_{B'})\,, \qquad \lambda^B(x) = -\lambda_{B'}(x) x^{B'B}\,, \tag{2.12}$$

and then

$$d_u^{A'}\lambda_\beta = P_{\beta\gamma}^{A'} z^\gamma\,, \tag{2.13}$$

with u arbitrary, $P^{A'}_{\beta\gamma}$ independent of u and

$$P^{A'}_{B'C'} = \tfrac{1}{2} D^2 \tilde{\lambda}^{A'} \varepsilon_{B'C'} . \tag{2.14}$$

From (2.12)

$$\lambda_\beta x^{\beta D} = 0 , \qquad z^\beta = x^{\beta D} u_D \tag{2.15}$$

and applying $d^{A'}_u$ to this using (2.13) gives

$$-2\tilde{\lambda}^{A'} \varepsilon^{DE} = P^{A'}_{\beta\gamma} x^{\beta D} x^{\gamma E} , \tag{2.16}$$

so that we require, consistent with (2.14),

$$P^{A'}_{\beta\gamma} = -P^{A'}_{\gamma\beta} . \tag{2.17}$$

Further from (2.12), (2.13), (2.14) and (2.17) we can show

$$D^2 \lambda_\beta = 2P^{A'}_{\beta A'}$$

and hence, applying $D_{AA'}$ to (2.13)

$$(D_{AA'} P^{A'}_{\beta\gamma}) z^\gamma = 0 . \tag{2.18}$$

But (2.11) implies $D_{AA'} P^{A'}_{B'C'} = 0$ and, in conjunction with (2.18), it is now easy to see that

$$D_{AA'} P^{A'}_{\beta\gamma} = 0 . \tag{2.19}$$

The alternative description of A is then in terms of the pair $(\lambda_\beta, P^{A'}_{\beta\gamma})$ obeying (2.12), (2.13), (2.17), and (2.19), which are equivalent to the simple Eq. (2.11).

However, this set of equations is clearly similar in form to Eqs. (2.5) and since B is furthermore complete then, as in (2.6),

$$\lambda_\beta = \phi A_\beta , \qquad P^{A'}_{\beta\gamma} = \Omega^{A'}_\gamma A_\beta = \psi^{A'} A'_\gamma A_\beta , \tag{2.20}$$

where $A_\beta : \tilde{A} \to \tilde{B}$ and from (2.17)

$$A'_\gamma A_\beta = -A'_\beta A_\gamma . \tag{2.21}$$

If Δ^B is introduced in terms of A_β just as in (2.7), and so is also expressible in the form (2.8) for $A_\beta = (a^B, b_{B'})$, then (2.21) becomes equivalently

$$\Delta'^A \Delta^B = \varepsilon^{AB} c , \tag{2.22}$$

with $c(x) : \tilde{A} \to \tilde{C}$ quadratic in x. From (2.12) or (2.15)

$$\phi \Delta^B = 0 . \tag{2.23}$$

Alternatively, with $\Delta^B(x) = a^B + b_{B'} x^{B'B}$, $b_{B'}$ can be determined by requiring it to comprise the maximal linearly independent set of transformations, with range $\tilde{B}$, such that $b'_{A'} b_{B'} \propto \varepsilon_{A'B'}$ (up to the arbitrariness $b_{B'} \sim b_{B'} R$ for R non-singular). This condition suffices, with $\lambda_{B'} = \phi b_{B'}$, to ensure that (2.11) holds, and then Δ^B is uniquely fixed by the requirement (2.23). The condition (2.22) follows automatically, being implied when $D^{A'A}$ is applied to (2.23), as in the connection between (2.15) and (2.16), and furthermore then giving the relation

$$\psi^{A'} c = 2\tilde{\lambda}^{A'} = 2\phi \tilde{b}^{A'} . \tag{2.24}$$

Equation (2.24) does not entail any further constraints; applying $\tilde{D}_{AA'}$ to both sides gives an identity, using (2.1), (2.5) or (2.13), and $\partial_{AA'}c = 2\tilde{\Delta}'_A b_{A'}$ from its definition (2.22).

The linear maps f, g in (1.12) required for the construction are now straightforwardly defined, for $A \to B$ by

$$\begin{gathered} \lambda_\beta w,\, P^{A'}_{\beta\gamma} w \xrightarrow{f(\xi)} \lambda_\beta \xi^\beta w\,,\ P^{A'}_{\beta\gamma}\xi^\beta w = \phi v,\, \Omega^{A'}_\gamma v\,, \\ v = A(\xi) w\,, \end{gathered} \tag{2.25}$$

and for $B \to C$ by

$$\begin{gathered} \phi v\,, \qquad \Omega^{A'}_\beta v \xrightarrow{g(\xi)} \Omega^{A'}_\beta \xi^\beta v = \psi^{A'} w'\,, \\ w' = A'(\xi) v\,. \end{gathered} \tag{2.26}$$

Obviously $g(\xi) f(\xi) = 0$ as a result of (2.17) or (2.21), $A'(\xi) A(\xi) = 0$.

The analysis of $\ker g(\xi)$ and $\operatorname{im} f(\xi)$, which appear in (1.13), is simple:

$\ker g(\xi)$. Suppose, for some v,

$$\Omega^{A'}_\beta \xi^\beta v = 0\,, \quad \text{or} \quad A'(\xi) v = 0\,, \tag{2.27}$$

then, if $\Omega^{A'}_\beta = (\Omega^{A'B}, \Omega^{A'}_{B'})$, from (2.5), and the definitions (1.8), (1.9),

$$d^{A'}_\eta \phi v = -\,\Omega^{A'}_{B'} v \pi^{B'}\,. \tag{2.28}$$

As discussed in Appendix A, $d^{A'}_\eta A' = 1, 2$ form a basis for tangential derivatives to the anti-self-dual plane $\mathscr{S}_\xi$, on which $\pi^{B'}(x) = 0$. Hence (2.28) shows that ϕv is covariantly constant on $\mathscr{S}_\xi$.

$\operatorname{im} f(\xi)$. Suppose $(\phi v, \Omega_\alpha v) \in \operatorname{im} f(\xi)$, or $v = A(\xi) w$ for some w, then from (2.12) and (1.8), (1.9)

$$\phi v = \lambda_{B'} w \pi^{B'}\,, \tag{2.29}$$

so ϕv vanishes on $\mathscr{S}_\xi$.

Thus by (1.10) $\ker g(\xi)$ and $\operatorname{im} f(\xi)$ have the properties required for (1.13). To prove (1.13) it is necessary further to show that B is large enough to include all covariantly constant scalars on $\mathscr{S}_\xi$, and A large enough relative to B to ensure that E_ξ, as defined in (1.13), depends only on ξ. These requirements are the consequence respectively of the corollary to Lemma 1 and Lemma 2, which are now demonstrated,

Lemma 1. *$g(\xi)$ is onto.*

Proof. For any $\psi^{A'} \in C$ it can be expressed in the form

$$\psi^{A'} - d^{A'}_\eta \varrho = \Lambda^{A'}_{B'} \pi^{B'}\,. \tag{2.30}$$

The content of (2.30) is that, since the Dirac equation in the form (2.2) in terms of $d^{A'}_\eta$ is the requisite integrability condition, on the plane $\mathscr{S}_\xi \psi^{A'} = d^{A'}_\eta \varrho$, by virtue of (A.11) using a version of the Poincaré lemma. Requiring the Dirac equation (2.1) to hold in general gives

$$(D_{AA'} \Lambda^{A'}_{B'}) \pi^{B'} = (2\Lambda^{A'}_{A'} - D^2 \varrho) \eta_A\,, \tag{2.31}$$

so that

$$2\Lambda^{A'}_{A'} - D^2\varrho = \mu_{B'}\pi^{B'}. \tag{2.32}$$

Now let

$$\hat{\varrho} = \varrho + \nu_{C'}\pi^{C'}, \qquad \hat{\Lambda}^{A'}_{B'} = \Lambda^{A'}_{B'} - d^{A'}_{\eta}\nu_{B'}, \tag{2.33}$$

where

$$D^2\nu_{C'} = \mu_{C'}, \tag{2.34}$$

supposing only that D^2 can be inverted, although not necessarily uniquely. With the redefinition (2.33), (2.30) becomes

$$\psi^{A'} = d^{A'}_{\eta}\hat{\varrho} + \hat{\Lambda}^{A'}_{B'}\pi^{B'}, \tag{2.35}$$

but in this case

$$D_{AA'}\hat{\Lambda}^{A'}_{B'} = 0 \tag{2.36}$$

since, by the definition of $\nu_{C'}$ in (2.34), $2\hat{\Lambda}^{A'}_{A'} = D^2\hat{\varrho}$. Assuming

$$d^{A'}_{u}\hat{\varrho} = \hat{\Lambda}^{A'}_{\beta}z^{\beta} \tag{2.37}$$

for arbitrary u then defines $\hat{\Lambda}^{A'B}$, with the decomposition $\hat{\Lambda}^{A'}_{\beta} = (\hat{\Lambda}^{A'B}, \hat{\Lambda}^{A'}_{B'})$. Applying $D_{AA'}$ to (2.37) finally shows, with (2.36), that $D_{AA'}\hat{\Lambda}^{A'}_{\beta} = 0$. Hence $(\hat{\varrho}, \hat{\Lambda}^{A'}_{\beta}) \in B$ and from (2.35) and (2.37)

$$\hat{\Lambda}^{A'}_{\beta}\xi^{B} = \psi^{A'}. \tag{2.38}$$

Thus for any w' there is a v, $w' = A'(\xi)v$.

Corollary. *For ϱ covariantly constant on $\mathscr{S}_{\xi}$ there is a v so that $(\varrho - \phi v)|_{x \in \mathscr{S}_{\xi}} = 0$, $A'(\xi)v = 0$.*

If $\psi^{A'} = 0$ in (2.30) then it becomes the equation defining a general covariantly constant scalar on $\mathscr{S}_{\xi}$. The preceding argument then shows that there is a $(\hat{\varrho}, \hat{\Lambda}^{A'}_{\beta}) \in \mathrm{B}$ with $(\varrho - \hat{\varrho})|_{x \in \mathscr{S}_{\xi}} = 0$, by (2.33), and further, by (2.38)

$$\hat{\Lambda}^{A'}_{\beta}\xi^{\beta} = 0. \quad \square \tag{2.39}$$

Lemma 2. *If $(\phi v, \Omega^{A'}_{\beta}v) \in \ker g(\xi)$ and $\phi v|_{x \in \mathscr{S}_{\xi}} = 0$ then $(\phi v, \Omega^{A'}_{\beta}v) \in \operatorname{im} f(\xi)$.*

Proof. If ϕv vanishes on $\mathscr{S}_{\xi}$ then it can be assumed to be of the form

$$\phi v = \nu_{A'}\pi^{A'} \tag{2.40}$$

for some $\nu_{A'}$ and (2.28) becomes

$$(d^{A'}_{\eta}\nu_{B'} + \Omega^{A'}_{B'}v)\pi^{B'} = 0. \tag{2.41}$$

Hence

$$d^{A'}_{\eta}\nu_{B'} = -\Omega^{A'}_{B'}v + X^{A'}\tilde{\pi}_{B'} \tag{2.42}$$

defines $X^{A'}$ with the property $\tilde{d}_{\eta A'}X^{A'}=0$ so we can solve

$$D^2 r\eta_A = D_{AA'}X^{A'}. \tag{2.43}$$

Now $v_{B'}$ may be redefined

$$v_{B'} = \lambda_{B'} + r\tilde{\pi}_{B'}, \tag{2.44}$$

so that (2.42) becomes

$$d_\eta^{A'}\lambda_{B'} = -\Omega_{B'}^{A'}v - T^{A'}\tilde{\pi}_{B'}, \tag{2.45}$$

where

$$T^{A'} = -X^{A'} + d_\eta^{A'}r, \qquad D_{AA'}T^{A'}=0. \tag{2.46}$$

Thus, since from (2.45),

$$D^2\tilde{\lambda}^{B'} = 2T^{B'}, \tag{2.47}$$

we see that $\lambda_{B'}$ obeys (2.11) and may be used to construct the corresponding λ_β, $P_{\beta\gamma}^{A'}$. Equation (2.45) then implies

$$\Omega_{B'}^{A'}v = -P_{B'\gamma}^{A'}\xi^\gamma$$

and, in conjunction with (2.5a), we finally get

$$\Omega_\beta^{A'}v = P_{\gamma\beta}^{A'}\xi^\gamma, \qquad \phi v = \lambda_{B'}\pi^{B'} = \lambda_B\xi^B \tag{2.48}$$

with $(\lambda_\beta, P_{\beta\gamma}^{A'}) \in A$. □

Hence the spaces A, B, C as defined at the start of this section, with the consequential maps f, g, are sufficient to give (1.13). For the subsequent discussion two further results are important.

Lemma 3. $\ker f(\xi)=$ *space of solutions of* $D^2r=0$.

Proof. Suppose $A(\xi)w=0$ or equivalently

$$\lambda_\beta\xi^\beta w=0, \qquad P_{\beta\gamma}^{A'}\xi^\beta w=0 \tag{2.49}$$

for some w. The condition on $\lambda_\beta w$ in (2.49), in conjunction with (2.12), requires

$$\lambda_{B'}w = r\tilde{\pi}_{B'} \tag{2.50}$$

for a scalar r. Applying $d_\eta^{A'}$ to (2.50) then gives, from (2.13) and (2.49)

$$P_{B'C'}^{A'}w = -d_\eta^{A'}r\varepsilon_{B'C'}, \tag{2.51}$$

and since this must satisfy the Dirac equation as in (2.19)

$$D^2r=0. \tag{2.52}$$

Conversely solutions of (2.52) may be used, via (2.50) and (2.51), to generate elements of $\ker f(\xi)$. □

Lemma 4. $g(\xi)f(\bar{\xi})$ *is onto*, $\bar{\xi}^\alpha$ *is defined in* (A.8), *and, if* ξ^α *is decomposed as in* (1.8), *we may then write*

$$\xi=(\eta, q\eta), \qquad \bar{\xi}=(\bar{\eta}, q\bar{\eta}), \tag{2.53}$$

with $q=q_\mu e_\mu$, q_μ *a real four-vector.*

Proof. Suppose $\eta \neq 0$ (for $\eta = 0$ a similar proof can easily be given), then for $\psi^{A'} \in C$ define $\lambda_{B'}$ through

$$\begin{aligned}&-\bar{\eta}\eta|x-q|^2(x-q)^{A'A}D^2\frac{(\overline{x-q})_{AB'}}{|x-q|^2}\tilde{\lambda}^{B'}(x)\\&=\bar{\eta}\eta(-|x-q|^2D^2\delta^{A'}_{B'}+2(\overline{x-q})_{AB'}D^{A'A}+4\delta^{A'}_{B'})\tilde{\lambda}^{B'}(x)=2\psi^{A'}(x),\end{aligned}\tag{2.54}$$

where the solution is required to be regular at $x=q$. Since $\psi^{A'}$ obeys the Dirac equation, $\lambda_{B'}$ satisfies (2.11) so that it is possible to construct $(\lambda_\beta, P^{A'}_{\beta\gamma}) \in A$. Inserting (2.13), (2.14) and (2.16) into (2.54) then gives the desired result

$$P^{A'}_{\beta\gamma}\bar{\xi}^\beta\xi^\gamma = \psi^{A'}. \tag{2.55}$$

Corollary. $\ker g(\xi)f(\bar{\xi}) =$ *space of solutions of* $D^2 s_B = 0$.

If $P^{A'}_{\beta\gamma}\bar{\xi}^\beta\xi^\gamma w = 0$ then (2.54) holds for $\lambda_{B'}w$ with $\psi^{A'}=0$. The solution of the resulting homogeneous equation can obviously be written

$$\tilde{\lambda}^{B'}(x)w = (x-q)^{B'B}s_B(x), \quad D^2 s_B = 0. \quad \square \tag{2.56}$$

By (2.7), (2.22), and (2.53) we have

$$A'(\xi)A(\bar{\xi}) = \bar{\eta}\eta c(q), \tag{2.57}$$

so Lemma 4 ensures that $w' = c(q)w$ for any w', and if $c(q)w = 0$, $\lambda_\beta w$ is of the form entailed by (2.56).

The spaces A, B, C have so far been defined in this section in terms of general solutions of four-dimensional partial differential equations which will involve arbitrary functions of three variables. With suitable global conditions on the gauge field A_μ they may be considerably restricted while still ensuring the results that are necessary for the validity of the construction in the form (1.13). For self-dual fields on S^4, represented by $A_\mu(x)$ obeying the boundary condition (1.2), then solutions of the equations defining A, B, C are strongly constrained by requiring them to be globally defined, as sections, on this compact manifold. The conditions, corresponding to (1.2), are equivalent to imposing, on flat space, the asymptotic behaviour

$$\begin{aligned}&\psi^{A'}(x) \sim \frac{x^{A'A}\gamma_A}{|x|^4}g(\hat{x}), \qquad \phi(x) \sim \beta g(\hat{x}),\\&\tilde{\lambda}^{A'}(x) \sim \frac{x^{A'A}\alpha_A}{|x|^2}g(\hat{x}), \qquad r(x) \sim \frac{\delta}{|x|^2}g(\hat{x}),\\&s_B(x) \sim \frac{\varepsilon_B}{|x|^2}g(\hat{x}).\end{aligned}\tag{2.58}$$

Simple positivity arguments, with Eqs. (2.52) or (2.56), then show that r or s_B must vanish, $\ker f(\xi) = \ker g(\xi)f(\bar{\xi}) = 0$, so $c(q)$ is invertible. Further the spaces A, B, C are finite dimensional and by application of index theorems, or in Appendix C for B, then for $\mathcal{G} = \mathrm{SU}(n)$,

$$\dim A = \dim C = k, \qquad \dim B = 2k+n. \tag{2.59}$$

Thus in this case A'_α, A_β are just finite matrices.

For $\mathscr{G}=\mathrm{Sp}(n)$, or $\mathrm{O}(n)$, there are extra reality conditions that can be imposed. For $\mathrm{Sp}(n)$ [for $\mathrm{O}(n)$ the modifications are obvious] the group elements obey

$$g^\dagger g=1_{\mathscr{G}}, \quad g^T\sigma g=\sigma, \quad \sigma=\sigma^*=-\sigma^T, \quad \sigma^2=-1, \tag{2.60}$$

so $g^*\sigma=\sigma g$. The reality conditions are then

$$\begin{gathered}\sigma\bar{\psi}^{A'}=\psi^{A'}\sigma_C, \quad \sigma_C^2=1,\\ \sigma\phi^*=\phi\sigma_B, \quad \sigma\bar{\Omega}^{A'}\cdot\xi^*=\Omega^{A'}\cdot\bar{\xi}\sigma_B, \quad \sigma_B^2=-1,\\ \sigma\lambda^*\cdot\xi^*=\lambda\cdot\bar{\xi}\sigma_A, \quad \sigma\bar{P}^{A'}\cdot\xi^*\wedge z^*=P^{A'}\cdot\bar{\xi}\wedge\bar{z}\sigma_A, \quad \sigma_A^2=1,\end{gathered} \tag{2.61}$$

for some $\sigma_{A,B,C}=\sigma^*_{A,B,C}$ and with the notations $\Omega\cdot\xi=\Omega_\alpha\xi^\alpha$, $\lambda\cdot\xi=\lambda_\alpha\xi^\alpha$, $P^{A'}\cdot\xi\wedge z=P^{A'}_{\alpha\beta}\xi^\alpha z^\beta$. From (2.61) we get

$$\begin{gathered}\sigma_C A'(\xi)^*=A'(\bar{\xi})\sigma_B, \quad \sigma_B A(\xi)^*=A(\bar{\xi})\sigma_A,\\ \sigma_C\bar{\Delta}'=-\tilde{\Delta}'\sigma_B, \quad \sigma_B\bar{\Delta}=-\tilde{\Delta}\sigma_A.\end{gathered} \tag{2.62}$$

3. Reconstruction of the Gauge Fields

As mentioned in the introduction the ADHM construction provides, besides the vector bundle E, a succinct prescription for the gauge field, (1.15), in terms of the vector spaces and their maps, once suitable scalar products have been defined. To do this it is necessary to require that the spaces A, B, C are so restricted that, as for S^4, $\ker f(\xi)=\ker g(\xi)f(\bar{\xi})=0$. By virtue of Lemma 4, $c(x)$ is then invertible for all x in the region over which the self-dual gauge field is defined, as is now assumed henceforth.

An explicit formula for A_μ, in terms of ϕ and Δ' as defined in the previous section, may be derived by also introducing $F(x):\tilde{C}\to\tilde{C}^*$ with the properties

$$F^\dagger=F, \quad F \text{ positive definite}. \tag{3.1}$$

We may then construct $I:\tilde{B}\to\tilde{B}^*$ and $L:\tilde{A}\to\tilde{C}^*$ by

$$I=I^\dagger=\phi^\dagger\phi+\tilde{\Delta}'^{\dagger A}F\tilde{\Delta}'_A, \quad L=Fc, \tag{3.2}$$

(here $\tilde{\Delta}'^{\dagger A}=(\bar{\Delta}'^A)^T$, also subsequently $\psi^\dagger_{A'}=\bar{\psi}^T_{A'}$) so that

$$I\Delta=\tilde{\Delta}'^\dagger L, \tag{3.3}$$

using (2.22) in the form

$$\tilde{\Delta}'_A\Delta^B=\delta^B_A c. \tag{3.4}$$

Further in terms of the projection operator P, acting on $\tilde{B}$ and given by

$$P=1-\Delta^A c^{-1}\tilde{\Delta}'_A, \quad P\Delta=0, \quad \tilde{\Delta}'P=0, \tag{3.5}$$

it is straightforward to show that

$$P^\dagger\partial_\mu IP=0, \tag{3.6}$$

directly from the definition of I in (3.2), using (2.9) and its conjugate

$$\phi^\dagger\overleftarrow{D}_{AA'}=\Delta'^\dagger_A\psi^\dagger_{A'}, \quad \overleftarrow{D}_\mu=\overleftarrow{\partial}_\mu-A_\mu. \tag{3.7}$$

For F satisfying (3.1) I is positive definite, and hence invertible. Manifestly $v^\dagger I v \geqq 0$ for all v and if $v^\dagger I v = 0$ necessarily $\phi v = 0$, $\Delta' v = 0$ but from an extension of Lemma 2 $\phi v = 0$ requires $v = \Delta w$, thus $cw = 0$ or, with $\ker c = 0$, $w = 0$. Similarly L is invertible. Hence from (3.3), (2.23) can be written

$$\Delta' I^{-1} \phi^\dagger = 0 \tag{3.8}$$

so that $P I^{-1} \phi^\dagger = I^{-1} \phi^\dagger$ and (3.6) can be re-expressed as

$$\phi \partial_\mu I^{-1} \phi^\dagger = 0 . \tag{3.9}$$

Using (2.9) and (3.8) we obtain

$$(D_\mu \phi) I^{-1} \phi^\dagger = 0 , \tag{3.10}$$

and moreover with (3.7) and (3.9)

$$\mathscr{D}_\mu (\phi^\dagger I^{-1} \phi) = 0 , \qquad \mathscr{D}_\mu X = \partial_\mu X + [A_\mu, X] . \tag{3.11}$$

Since also, from (3.2) and (3.8),

$$\phi = \phi I^{-1} I = \phi I^{-1} \phi^\dagger \phi \tag{3.12}$$

it follows that, with A_μ irreducible as in (1.11) so $\mathscr{D}_\mu X = 0 \Rightarrow X \propto 1_{\mathscr{G}}$, (3.11) and (3.12) give

$$\phi I^{-1} \phi^\dagger = 1_{\mathscr{G}} . \tag{3.13}$$

Hence (3.10) implies the simple expression

$$A_\mu = -\partial_\mu \phi I^{-1} \phi^\dagger = \phi I^{-1} \partial_\mu \phi^\dagger , \tag{3.14}$$

which allows reconstruction of the gauge field.

The conventional form of the ADHM construction for A_μ is obtained if F can be chosen so that I, L, given by (3.2), are independent of x. The appropriate choice may be represented in terms of the complete set of solutions of the Dirac equation ψ by requiring

$$\psi^\dagger_{A'} \psi^{A'} = -\partial^2 F \tag{3.15}$$

for $F(x)$ regular and obeying appropriate boundary conditions on the region over which the gauge fields are defined. In suitable contexts constancy of I, L is then a consequence of the following lemma.

Lemma 5. *If for $H^{A'} : \tilde{B} \to \tilde{C}^*$ satisfying $\partial_{AA'} H^{A'} = 0$ ensures $H^{A'} = 0$, then I, L are constant.*

Proof. (i) For L let

$$X^{A'}_\gamma = \psi^\dagger_{B'} \varepsilon^{B'A'} \phi A_\gamma , \tag{3.16}$$

so that

$$\tilde{\partial}_{uA'} X^{A'}_\gamma = -\psi^\dagger \psi A'(z) A_\gamma , \qquad X^{A'}_\gamma z^\gamma = 0 . \tag{3.17}$$

Also if

$$q_{\gamma\delta} = -F A'_\delta A_\gamma \tag{3.18}$$

and

$$\hat{X}_\gamma^{A'} = \partial^{A'B}(q_{\gamma\delta}\tilde{x}_B^\delta) - 2q_{\gamma D'}\varepsilon^{D'A'}, \tag{3.19}$$

where $\tilde{x}_B^\delta = x^{\delta C}\varepsilon_{CB}$ and otherwise as in (2.15), then from (3.15) and (3.17)

$$\tilde{\partial}_{uA'}\hat{X}_\gamma^{A'} = \tilde{\partial}_{uA'}X_\gamma^{A'}. \tag{3.20}$$

Hence

$$X_\gamma^{A'} = \hat{X}_\gamma^{A'} + H^{A'}A_\gamma, \qquad \partial_{AA'}H^{A'} = 0. \tag{3.21}$$

However, since $q_{\gamma\delta}x^{\gamma A}x^{\delta B} = L\varepsilon^{AB}$, (3.19) gives

$$\hat{X}_\gamma^{A'}z^\gamma = -\partial_u^{A'}L, \tag{3.22}$$

so that if $H^{A'}=0$ (3.17), (3.21), and (3.22) require $\partial_\mu L = 0$.

(ii) For I a similar proof is obtained by letting

$$X_\gamma^{A'} = \phi^\dagger\psi^{A'}A'_\gamma + \bar{A}'_\gamma\psi^\dagger_{B'}\varepsilon^{B'A'}\phi, \tag{3.23}$$

where $\bar{A}'(z) = \tilde{u}^A\Delta'(x)^\dagger_A = -A'(\bar{z})^\dagger$, and

$$\begin{gathered}\tilde{\partial}_{uA'}X_\gamma^{A'} = -s_{\gamma\delta}z^\delta, \qquad s_{\gamma\delta} = \bar{A}'_\gamma\psi^\dagger\psi A_\delta - \bar{A}'_\delta\psi^\dagger\psi A_\gamma, \\ X_\gamma^{A'}z^\gamma = \partial_u^{A'}(\phi^\dagger\phi).\end{gathered} \tag{3.24}$$

In this case $s_{\gamma\delta} = \partial^2 q_{\gamma\delta}$, or replacing (3.18)

$$q_{\gamma\delta} = -\bar{A}'_\gamma F A_\delta + \bar{A}'_\delta F A_\gamma, \tag{3.25}$$

which with (3.19) ensures (3.20) still holds and

$$X_\gamma^{A'} = \hat{X}_\gamma^{A'} + \bar{A}'_\gamma H^{A'} + \bar{H}^{A'}A'_\gamma, \qquad \bar{H}^{A'} = \tilde{H}^{\dagger A'}. \tag{3.26}$$

With $q_{\gamma\delta}$ given by (3.25) we get

$$\hat{X}_\gamma^{A'}z^\gamma = -\partial_u^{A'}(\tilde{\Delta}'^\dagger F\tilde{\Delta}'), \tag{3.27}$$

so if $H^{A'}=0$ (3.24) and (3.27), from the definition (3.2), imply $\partial_\mu I = 0$. □

An explicit expression for $H^{A'}$ can be obtained by inserting into the above, from (2.24).

$$\psi^{A'} = 2\phi\tilde{b}^{A'}c^{-1}, \qquad \phi^\dagger\phi = IP, \tag{3.28}$$

together with the derivative of (3.3)

$$Ib_{B'} + \tfrac{1}{4}\partial_{BB'}I\Delta^B = \tilde{b}'^\dagger_{B'}L + \tfrac{1}{4}\tilde{\Delta}'^{\dagger B}\partial_{BB'}L \tag{3.29}$$

to achieve

$$H^{A'} = \partial^{A'B}Lc^{-1}\tilde{\Delta}'_B + \tfrac{1}{2}c^{\dagger -1}\Delta^\dagger_B\partial^{A'B}IP. \tag{3.30}$$

Manifestly if L, I are constant $H^{A'}=0$ and from (3.21), (3.26)

$$\partial^2 L = 2H^{A'}b_{A'}, \qquad \partial^2 I = 2(\tilde{b}'^\dagger_{A'}H^{A'} + \bar{H}^{A'}b'_{A'}), \tag{3.31}$$

which may be directly verified.

For self-dual gauge fields on S^4 then with the asymptotic behaviour in (2.58) so that $\psi^\dagger\psi = O(|x|^{-6})$, then for $|x|\to\infty$

$$F(x) = O(|x|^{-2}), \qquad I, L = O(1) \tag{3.32}$$

and

$$H^{A'}(x) \sim \frac{x^{A'A} a_A}{|x|^4}. \tag{3.33}$$

Hence $H^{A'}=0$, since there are no regular solutions of the Dirac equation everywhere on S^4 (also (3.15) is solved uniquely for F with (3.32)), and I, L are clearly nonzero and constant.

In this situation it is possible to define a natural, positive definite, scalar product on $\tilde{B}$ and make $\tilde{C}$ the dual of $\tilde{A}$

$$\begin{aligned} \langle v, u \rangle &= v^\dagger I u, \\ \langle w', w \rangle &= w'^\dagger L w. \end{aligned} \tag{3.34}$$

By virtue of (3.3)

$$\begin{aligned} \langle v, \Delta w \rangle &= \langle \tilde{\Delta}' v, w \rangle, \\ \langle v, A(\xi) w \rangle &= -\langle A'(\bar{\xi}) v, w \rangle. \end{aligned} \tag{3.35}$$

With this structure the ADHM construction requires the finding of $v(x) \in \tilde{B}$ satisfying

$$\begin{aligned} \langle v(x), \Delta(x) w \rangle &= 0 \quad \text{for all} \quad w \in \tilde{A}, \\ \langle v(x), v(x) \rangle &= 1_{\mathscr{G}}, \end{aligned} \tag{3.36}$$

and then, as in (1.15),

$$A'_\mu = \langle v, \partial_\mu v \rangle \tag{3.37}$$

defines, as easily verified [20, 21], a self-dual gauge field. From (3.3) and (3.13) the general solution of (3.36) is given by

$$v(x)^\dagger = g(x)\phi(x) I^{-1}, \qquad g(x) g(x)^\dagger = 1_{\mathscr{G}}, \tag{3.38}$$

and thus from (3.14)

$$A'_\mu = A^g_\mu = g A_\mu g^{-1} + g \partial_\mu g^{-1}, \tag{3.39}$$

thereby demonstrating the reconstruction of the original gauge field up to the gauge transformation $g(x) \in \mathscr{G}$ [for Sp(n), or O(n), the additional reality requirements below ensure that the defining conditions such as (2.60) are met].

The bases chosen for $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are of course arbitrary and the results are invariant under

$$\begin{aligned} &w' \to P w', \qquad v \to Q v, \qquad w \to R w, \\ &\Delta \to Q \Delta R^{-1}, \qquad \Delta' \to P \Delta' Q^{-1}, \end{aligned} \tag{3.40}$$

for P, Q, R members of the general set of linear transformations on $\tilde{C}$, $\tilde{B}$, $\tilde{A}$. In consequence of (3.40)

$$I \to Q^{-1\dagger} I Q^{-1}, \qquad L \to P^{-1\dagger} L R^{-1}. \tag{3.41}$$

The freedom in (3.41) can be used to set

$$I = 1, \qquad L = 1 \Rightarrow \tilde{\Delta}' = \Delta^\dagger, \tag{3.42}$$

and then, in (3.40), the remaining arbitrariness is restricted to $Q^\dagger Q = 1$, $P^\dagger R = 1$.

For $\mathscr{G}=\mathrm{Sp}(n)$, [or $\mathrm{O}(n)$] then with (2.61) and (2.62) further conditions can be imposed. Define

$$w'_\sigma=\sigma_C w'^*, \qquad v_\sigma=\sigma_B v^*, \qquad w_\sigma=\sigma_A w^*, \tag{3.43}$$

and then the bilinear forms

$$\begin{aligned}[v,u]&=-\langle v_\sigma,u\rangle=v^TJu,\\ [w',w]&=\langle w'_\sigma,w\rangle=w'^TKw,\end{aligned} \tag{3.44}$$

where from (2.61) and (2.62), with (3.2) and (3.15),

$$\begin{aligned}J=-\sigma_B^T I&=\phi^T\sigma\phi+\Delta'^{AT}F\tilde{\Delta}'_A=-J^T,\\ K=\sigma_C^T L=fc,\qquad &\tilde{\psi}_{A'}^T\sigma\psi^{A'}=-\partial^2 f,\qquad f^T=f.\end{aligned} \tag{3.45}$$

Thus (3.44) gives a skew symmetric form on $\tilde{B}$, $[v,u]=-[u,v]$, and in addition to (3.36) we have

$$[v(x),v(x)]=\sigma, \tag{3.46}$$

Also from (3.35)

$$[v,A(\xi)w]=[A'(\xi)v,w] \tag{3.47}$$

or

$$J\Delta(x)=\Delta'(x)^TK. \tag{3.48}$$

With a suitable basis as with (3.42) we can choose $\sigma_C=\sigma_B=K=1$ and $J=-\sigma_B^T=\sigma_B$. The spaces $\tilde{A}$, $\tilde{C}$ are then real while $\tilde{B}$ can be taken to be quaternionic.

4. Concluding Remarks

For the ADHM construction to hold in its complete form, so that, as shown in the previous section, $\ker c=0$ and the conditions of Lemma 5 are met, some global conditions have to be imposed on the class of self-dual gauge fields considered. These may be realised, as perhaps for the different examples mentioned in the introduction, when various spatial symmetries are imposed. The symmetry transformations are then incorporated in the function spaces A, B, C. Suppose for $x\to x_R$ there is a symmetry

$$A_\mu(x_R)dx_{R\mu}=A_\mu^{g_R}(x)dx_\mu, \tag{4.1}$$

with $g_R(x)$ an appropriate gauge transformation, as in (3.39). Then the solutions of the equations defining the spaces A, B, C transform as

$$\psi\to g_R S\psi P_R, \qquad \phi\to g_R\phi Q_R, \qquad \tilde{\lambda}\to g_R S\tilde{\lambda}R_R, \tag{4.2}$$

where S and T, supposed to be constant, are given by $D^{A'A}\to S^{A'}_{B'}g_R D^{B'B}g_R^{-1}T^A_B$. The maps Δ', Δ correspondingly satisfy

$$\Delta'(x_R)=P_R^{-1}\Delta'(x)Q_RT, \qquad \Delta(x_R)=Q_R^{-1}\Delta(x)R_RT, \tag{4.3}$$

and if I, L are constant

$$I=Q_R^\dagger IQ_R, \qquad L=P_R^\dagger LR_R. \tag{4.4}$$

The monopole solutions [8, 10–12], which are independent of $t=x_4$, correspond to a realisation of the translational symmetry $t\to t+s$, with the associated gauge transformation $g_s=1$, and also $S, T=1$. In this case $\{P_s\}$, $\{Q_s\}$, $\{R_s\}$ define abelian groups so that, in a suitable basis, the representations formed by the spaces C, B, A are one-dimensional and can be parameterized by a single variable γ. The elements can then be written as

$$e^{i\gamma s} \tag{4.5}$$

and the dependence on t becomes explicit, for instance

$$\psi(x)=\psi_0(\underset{\sim}{x})e^{i\gamma t}, \qquad \phi(x)=\phi_0(\underset{\sim}{x})e^{i\gamma t}. \tag{4.6}$$

Since $x=t1_2+i\underset{\sim}{x}\cdot\underset{\sim}{\sigma}$ the map Δ', also analogously Δ, is required from (4.3) to be

$$\Delta'(x)=\left(i\frac{\overleftarrow{\partial}}{\partial\gamma}B'(\gamma)1_2+B'(\gamma)x\right)+A'(\gamma), \tag{4.7}$$

with $B'_{A'}$, A'^{A} acting on the space of solutions for fixed γ.

Nahm [32] has shown how the known monopole solutions can be recovered and extended with this form for Δ, Δ'. In these cases γ is real and is restricted in range by

$$\int d^3x\psi^\dagger\psi<\infty\,. \tag{4.8}$$

It should presumably be possible to construct a general proof of the applicability of the ADHM construction to monopole solutions of the Bogomolny [9] equations by analysing the required restrictions on the spaces A, B, C to ensure the desiderata of Sect. 3.

In the simple case of the 't Hooft [2] solution, and also the collinear Witten-Peng [3] solutions, it is possible [33] to locally explicitly solve the equations defining the spaces A, B, C and to restrict the spaces of solutions to ensure the ADHM construction is realised, both for the known multi-instanton case and the PS [8] one monopole solution. It would also be natural to consider self-dual gauge fields described by the higher ansatz of Atiyah and Ward [18], of which the 't Hooft solution is the first. It was in this framework that multi-monopole solutions were first found [11, 12]. The spherically symmetric monopole solutions for large groups [10] were also first obtained by an extension of the SU(2) Witten ansatz.

More generally it is possible that self-dual gauge fields are described by a completely integrable system [34], which involves a linearization of the full non-linear equations. The inverse problem of constructing the ADHM framework which has been discussed here is also basically a linear programme. On the other hand the ADHM framework may possibly be extendable, in some part, to describe the complete equations of motion of gauge fields, even when coupled to sources [35].

Appendix A

The basis for 2×2 matrices, or quaternions, provided by e_μ, or its conjugate $\bar{e}_\mu$, used here has the properties

$$e_\mu \bar{e}_\nu = \delta_{\mu\nu} 1_2 + \eta^+_{\mu\nu}, \qquad \bar{e}_\mu e_\nu = \delta_{\mu\nu} 1_2 + \eta^-_{\mu\nu}, \qquad \eta^\pm_{\mu\nu} = \pm {}^*\eta^\pm_{\mu\nu}. \tag{A.1}$$

They may be represented explicitly by $e_\mu = (i\underset{\sim}{\sigma}, 1_2)$, $\bar{e}_\mu = (-i\underset{\sim}{\sigma}, 1_2)$, and in terms of primed and unprimed spinorial indices are written

$$(e_\mu)^{A'A}, \qquad (\bar{e}_\mu)_{AA'}, \qquad (e_\mu)^{A'A}(\bar{e}_\mu)_{BB'} = 2\delta^A_B \delta^{A'}_{B'}. \tag{A.2}$$

Thus any four-vector q_μ may be encoded as a quaternion, $q = e_\mu q_\mu$ or $\bar{q} = \bar{e}_\mu q_\mu$. The spinorial indices may be raised and lowered with the two-dimensional anti-symmetric symbol ε, $\varepsilon_{12} = \varepsilon^{12} = 1$, so

$$\tilde{u}^A = \varepsilon^{AB} u_B, \qquad \tilde{s}_A = s^B \varepsilon_{BA}, \qquad \tilde{v}_{A'} = \varepsilon_{A'B'} v^{B'}, \qquad \tilde{t}^{A'} = t_{B'} \varepsilon^{B'A'}. \tag{A.3}$$

For the spinors u, v we may also define their conjugates

$$\bar{u}^A = u^*_A, \qquad \bar{v}_{A'} = v^{A'*}, \tag{A.4}$$

where from (A.3) and (A.4)

$$\tilde{\tilde{u}} = u, \qquad \bar{\bar{u}} = u, \qquad \tilde{\bar{u}} = -\bar{\tilde{u}}, \qquad \tilde{u}u' = -\tilde{u}'u, \tag{A.5}$$

and similarly for v. Also useful is the relation

$$\bar{q}_{AA'} = -\varepsilon_{A'B'} q^{B'B} \varepsilon_{BA}, \tag{A.6}$$

so, for instance, $\widetilde{qu} = \tilde{u}\bar{q}$.

The tangent vectors to the anti self-dual plane $\mathscr{S}_\xi$, may, from (1.8) and (1.9), be represented by

$$n_i^{A'A} = \lambda_i^{A'} \tilde{\eta}^A, \tag{A.7}$$

using $\tilde{\eta}\eta = 0$ and where λ_i, $i = 1, 2$, are two independent primed spinors. Since $n_i \bar{n}_j = 0$ then, from (A.1), this is easily seen to imply (1.5).

For each point in CP^3 represented by a four vector ξ^α, decomposed into unprimed and primed spinors according to (1.8), its conjugate may be defined by

$$\bar{\xi}^\alpha = (\tilde{\bar{\eta}}_A, \tilde{\bar{\chi}}^{A'}), \qquad \bar{\bar{\xi}}^\alpha = -\xi^\alpha. \tag{A.8}$$

By using e_μ, or $\bar{e}_\mu$, the covariant derivative $D_\mu = \partial_\mu + A_\mu$ may be reexpressed so as to act from unprimed to primed spinors $D^{A'A}$, or vice versa $D_{AA'}$ (dropping the bar on D for convenience in this case). For self-dual gauge fields satisfying (1.1), then from (A.1)

$$D_{AA'} D^{A'B} = D^2 \delta_A^B, \tag{A.9}$$

as often used in the text. With an unprimed spinor u we may define a spinorial derivative

$$d_u^{A'} = D^{A'A} u_A, \tag{A.10}$$

which for $A' = 1, 2$ provides two independent tangential derivatives for the anti-self-dual planes whose tangent vectors are given in terms of u as in (A.7). The self-duality condition then takes the form

$$d_u^{A'} d_u^{B'} - d_u^{B'} d_u^{A'} = 0, \tag{A.11}$$

which expresses the integrability on the anti-self-dual planes just as (1.6). It is useful to note

$$\partial_u^{A'} v^{B'} = 0, \qquad v^{B'} = x^{B'B} u_B. \tag{A.12}$$

Appendix B

In the analysis of the ADHM construction in terms of sheaf cohomology [27–29] for self-dual gauge fields on S^4 it is possible to identify the spaces A, B, C with certain H^1 sheaf cohomology groups for sheaves over CP^3. Thus

$$C = H^1(E(-1)), \qquad B = H^1(E \times \Omega^1), \qquad A = H^1(E \times \Omega_1^2), \tag{B.1}$$

with notation for which a partial explanation is contained below.

Homogeneous co-ordinates for CP^3 are provided by a complex four-vector

$$z = (u, v), \qquad z \sim \lambda z, \tag{B.2}$$

and the projection $CP^3 \to S^4$ is given by

$$u \neq 0, \qquad z = (u, xu), \tag{B.3a}$$

$$v \neq 0, \qquad z = (\xi v, v), \tag{B.3b}$$

where $x = x_\mu e_\mu$, $\xi = x_\mu \bar{e}_\mu / |x|^2$ are co-ordinates for the regions $V_0(\xi \neq 0)$, $V_\infty(x \neq 0)$ covering S^4. Sections of vector bundles with connection A_μ over S^4 are joined by an appropriate group transformation on the intersection $V_0 \cap V_\infty$, for instance for the scalar ϕ this is the gauge transformation $g_{0\infty}$ so that

$$\phi(x) = g_{0\infty} \phi_\infty(\xi). \tag{B.4}$$

For a covering of CP^3 provided by $\{U_i\}$ ($i = 1, \ldots, 4$ is sufficient) the Čech H^1 cohomology groups listed in (B.1) are defined as the additive group of holomorphic vector functions in the vector bundle E over CP^3, within certain classes differing in each particular case, which are defined on each intersection $U_i \cap U_j$ and are cocycles

$$\begin{gathered} c_{ij} = -g_{ij} c_{ji}, \\ (\delta c)_{ijk} \equiv c_{ij} + g_{ij} c_{jk} + g_{ik} c_{ki} = 0 \quad \text{on} \quad U_i \cap U_j \cap U_k, \end{gathered} \tag{B.5}$$

modulo coboundaries

$$c_{ij} \sim c'_{ij} \quad \text{if} \quad c_{ij} - c'_{ij} = (\delta c)_{ij} \equiv c_i - g_{ij} c_j, \tag{B.6}$$

for c_i correspondingly defined on U_i. In (B.5) and (B.6) $g_{ij}(z)=g_{ij}(\lambda z)$, defined on $U_i \cap U_j$, is the transition matrix for the vector bundle E, associated with the fundamental representation of the gauge group $\mathcal{G}$, transforming vectors defined as functions of coordinates on U_j to the corresponding vectors as functions on U_i, $g_{ji} = g_{ij}^{-1}$, $g_{ij}g_{jk}=g_{ik}$. More precisely the cocycles for the cohomology groups in (B.1) are respectively scalars, one and two forms over CP^3 which can be written as

$$\hat{h}_{ij}, \quad \hat{\omega}_{ij}=\hat{\Omega}_{\alpha, ij}dz^\alpha, \quad \hat{\varrho}_{ij}=\tfrac{1}{2}\hat{P}_{\alpha\beta, ij}dz^\alpha \wedge dz^\beta, \tag{B.7}$$

with the following properties

$$\begin{aligned} &\hat{h}_{ij}(z), \quad \hat{\Omega}_{\alpha, ij}(z), \quad \hat{P}_{\alpha\beta, ij}(z) \\ &\text{analytic for} \quad z\in U_i \cap U_j, \quad \text{homogeneous of degree } -1 \text{ in } z, &\text{(B.8a)}\\ &\hat{\Omega}_{\alpha, ij}(z)z^\alpha=0, \quad \hat{P}_{\alpha\beta, ij}(z)z^\beta=0, &\text{(B.8b)}\\ &\hat{P}_{\alpha\beta, ij}(z)=-\hat{P}_{\beta\alpha, ij}(z). &\text{(B.8c)}\end{aligned}$$

[(B.8b) is necessary to ensure that the one and two forms in (B.7) are over CP^3].

For the subset of CP^3 described in (B3a) the transition matrices can be split [25] as

$$\begin{aligned} g_{ij}(z)&=g_i(x,u)g_j(x,u)^{-1}, \\ d_u^{A'}g_i(x,u)^{-1}&=0, \end{aligned} \tag{B.9}$$

for $x\in\mathcal{S}_z$ and $g_i(x,u)$ analytic as a function of $u\sim\lambda u\in CP^1$ on the region corresponding to $z\in U_i$. If we then define

$$h_{ij}(x,u)=g_i(x,u)^{-1}\hat{h}_{ij}(z),$$

and analogously $\Omega_{\alpha, ij}$, $P_{\alpha\beta, ij}$, the requirement (B.8a) becomes, using (A.12) and (B.9)

$$\begin{aligned} &d_u^{A'}h_{ij}=0, \quad d_u^{A'}\Omega_{\alpha, ij}=0, \quad d_u^{A'}P_{\alpha\beta, ij}=0, \\ &h_{ij}, \quad \Omega_{\alpha, ij}, \quad P_{\alpha\beta, ij} \\ &\text{analytic in } u \text{ for} \quad z\in U_i\cap U_j, \quad \text{homogeneous of degree } -1 \text{ in } u. \end{aligned} \tag{B.10}$$

which now involve the gauge connection A_μ. For the region described in (B.3b) it is necessary to use the gauge transformation $g_{0\infty}$ so that $g_i(x,u)g_{0\infty}=g_{\infty i}(\xi, v)$ and $h_{\infty ij}=g_{0\infty}^{-1}h_{ij}$, etc. are regular at $\xi=0$. The equations corresponding to (B.11) on $h_{\infty ij}, \ldots$, are hence in terms of $d_{vA}=D_{\infty AA'}v^{A'}=-\xi_{AA'}/|\xi|^4 g_{0\infty}^{-1}d_u^{A'}g_{0\infty}$.

The twistor programme [36] provides a machinery for passing from H^1 cohomology groups on CP^3 to solutions of corresponding four dimensional field equations. This can be done either locally in the neighbourhood of a line in CP^3 corresponding to a point in S^4 (two regions are then sufficient to provide a covering and the cocycle condition in (B.5) is redundant) or globally over CP^3 when solutions defined as sections over the whole of S^4 are obtained. In the present case, as argued by Madore et al. [27], then, if for $z\in U_i$, U_j the whole of $CP^1=\{u\sim\lambda u\}$ is covered, due to the triviality of $H^1(CP^1, O(-1))$ or explicitly by contour integrals, it is possible to uniquely split the cocycles

$$\begin{gathered} h_{ij}=h_i-h_j, \qquad \Omega_{\alpha,ij}=\Omega_{\alpha,i}-\Omega_{\alpha,j}, \\ P_{\alpha\beta,ij}=P_{\alpha\beta,i}-P_{\alpha\beta,j}, \end{gathered} \tag{B.11}$$

so that $h_i(x,u)$, $\Omega_{\alpha,i}(x,u)$, $P_{\alpha\beta,i}(x,u)$ are analytic in u for all $z\in U_i$. In the global case the cocycle condition (B.5) ensures that (B.11) is valid for all i, j. Following ref [27] then from (B.10)

$$d_u^{A'}h_i=\psi^{A'}, \qquad d_u^{A'}\Omega_{\beta,i}=\Omega_\beta^{A'}, \qquad d_u^{A'}P_{\beta\gamma,i}=P_{\beta\gamma}^{A'}, \tag{B.12}$$

where $\psi^{A'}(x)$, $\Omega_\beta^{A'}(x)$, $P_{\beta\gamma}^{A'}(x)=-P_{\gamma\beta}^{A'}(x)$, of degree 0 in u, are in fact independent of u since they are holomorphic in u over the whole of CP^1. Furthermore, since from (A.11) $d_{uA'}d_u^{A'}=0$, they obey the Dirac equation as in (2.1), (2.5b) and (2.19). Similarly, from (B.8b),

$$\Omega_{\alpha,i}z^\alpha=\phi, \qquad P_{\alpha\beta,i}z^\beta=\lambda_\alpha, \tag{B.13}$$

giving $\phi(x)$, $\lambda_\alpha(x)$, also independent of u, such that $\lambda_\alpha z^\alpha=0$. Applying $d_u^{A'}$ to (B.13), with (B.12), just gives Eqs. (2.5b) and (2.14). Thus the equations whose solutions describe the spaces A, B, C in Sect. 2 are just those corresponding to the cohomology groups in (B.1). It is trivial to see that the arbitrariness due to coboundaries on CP^3 disappears for the solutions of the field equations on regions of S^4. When the H^1 groups in (B.1) are over CP^3 it is straightforward to obtain the transition functions linking the associated solutions defined over the two regions V_0, V_∞ covering S^4, thus for ϕ (B.4) holds and for ψ

$$\psi^{A'}(x)=-\frac{x^{A'A}}{|x|^4}g_{0\infty}\psi_{\infty A}(\xi). \tag{B.14}$$

The asymptotic conditions (2.58) are obtained by shrinking V_∞ to just the point $\xi=0$ with $g_{0\infty}\to g(\hat{x})$.

By analogous arguments to the above the absence of solutions of $D^2r=0$ corresponds to $H^1(E(-2))=0$ which plays a crucial role in the cohomological analysis.

Representative cocycles for the cohomological groups in (B.1) can be easily found with the results of Sect. 2. Define, with $B_i:\tilde{A}\to\tilde{B}$,

$$Y_i(z)=B_i(A'(z)B_i)^{-1}, \qquad Z_i(z)=\frac{1}{z^i}(1-Y_i(z)A'(z)), \tag{B.15}$$

which are of degree -1 in z, then if U_i is defined by $z^i\neq 0$, $\det(A'(z)B_i)\neq 0$, we can take

$$\begin{gathered} h_i=\phi Y_i, \qquad \Omega_{\alpha,i}=\delta_{i\alpha}\phi Z_i+\phi Y_iA'_\alpha, \\ P_{\alpha\beta,i}=\delta_{i\beta}\phi Z_iA_\alpha-\delta_{i\alpha}\phi Z_iA_\beta+\phi Y_iA'_\beta A_\alpha, \end{gathered} \tag{B.16}$$

which satisfy (B.12) and (B.13). Choosing $B_i \to A_\gamma$, since $A'(z)A(\bar{z})$ is non-singular, it can be seen that these U_i provide a covering of CP^3. The maps between cocycles corresponding to (2.25) and (2.26) are given by

$$\begin{aligned} \hat{P}_{\alpha\beta,ij}\xi^\alpha &= \hat{\Omega}_{\beta,ij}A(\xi) + (\delta\Lambda_\beta)_{ij}, \\ \hat{\Omega}_{\alpha,ij}\xi^\alpha &= \hat{h}_{ij}A'(\xi) + (\delta k)_{ij}, \\ \Lambda_{\beta,i} &= -k_i A_\beta, \quad k_i = \xi^i g_i \phi Z_i. \end{aligned} \tag{B.17}$$

The scalar products of Sect. 3 are given cohomologically by

$$H^1(E(-1)) \times H^1(E \times \Omega_1^2) \cong H^2(\Omega^2) \cong C, \tag{B.18a}$$

$$H^1(E \times \Omega^1) \times H^1(E \times \Omega^1) \cong H^2(\Omega^2) \cong C. \tag{B.18b}$$

The construction of cocycles for $H^2(\Omega^2)$ in (B.18) is given in terms of a product of H^1 cocycles on $U_i \cap U_j \cap U_k$ by

$$c_{ijk} = \tfrac{1}{6}(a_{ij}\cdot b_{ik} - a_{ji}\cdot b_{jk} - a_{ik}\cdot b_{ij} + a_{ki}\cdot b_{kj} + a_{jk}\cdot b_{ji} - a_{kj}\cdot b_{ki}),$$

where $\cdot$ denotes the group invariant scalar product for vectors in E, and for (B.18b) the wedge product for one forms, so that the cocycle is of the form

$$c_{ijk} = \tfrac{1}{2} f_{\alpha\beta,ijk} dz^\alpha \wedge dz^\beta, \quad f_{\alpha\beta,ijk} z^\beta = 0,$$

$$f_{\alpha\beta,ijk} \text{ analytic in } z \text{ on } U_i \cap U_j \cap U_k, \quad \text{homogeneous of degree } -2. \tag{B.19}$$

To analyse $H^2(\Omega^2)$ we assume that it is generally possible to decompose $f_{\alpha\beta,ijk}$ (nonuniquely) according to

$$f_{\alpha\beta,ijk} = t_{\alpha\beta,ij} + t_{\alpha\beta,jk} + t_{\alpha\beta,ki}, \quad t_{\alpha\beta,ij} = -t_{\alpha\beta,ji}, \tag{B.20}$$

for $t_{\alpha\beta,ij}$ suitably analytic functions of, in the appropriate regions described in (B.3), x, u or ξ, v. For $\mathscr{G} = \mathrm{Sp}(n)$, when the transition matrices g_{ij} and g_i in (B.9) are required to satisfy $g^T\sigma g = \sigma$, then (B.20) is obtained for the products in (B.18) by taking respectively

$$\begin{aligned} t^1_{\alpha\beta,ij} &= \tfrac{1}{2}(h_i^T \sigma P_{\alpha\beta,j} - h_j^T \sigma P_{\alpha\beta,i}), \\ t^2_{\alpha\beta,ij} &= \tfrac{1}{2}(\Omega^T_{\alpha,i}\sigma\Omega_{\beta,j} - \Omega^T_{\alpha,j}\sigma\Omega_{\beta,i} - \alpha \leftrightarrow \beta). \end{aligned} \tag{B.21}$$

To ensure that (B.20) realises the conditions in (B.19) it is necessary, for the region described by (B.3a), that

$$\begin{aligned} t_{\alpha\beta,ij} z^\beta &= r_{\alpha,i} - r_{\alpha,j}, \quad r_{\alpha,i} z^\alpha = r, \\ \partial_u^{A'} t_{\beta\gamma,ij} &= t^{A'}_{\beta\gamma,i} - t^{A'}_{\beta\gamma,j}, \quad \tilde{\partial}_{uA'} t^{A'}_{\beta\gamma,i} = s_{\beta\gamma}, \\ \partial_u^{A'} r_{\beta,i} &= t^{A'}_{\beta\gamma,i} z^\gamma + X^{A'}_\beta, \end{aligned} \tag{B.22}$$

where

$$r(x) = r_\infty(\xi), \quad s_{\alpha\beta}(x) = |x|^{-6} s_{\infty\alpha\beta}(\xi), \quad X^{A'}_\beta(x) = -x^{A'A}|x|^{-4} X_{\infty\beta A}(\xi)$$

are independent of u, v. For consistency from (B.22)

$$\tilde{\partial}_{uA'} X^{A'}_B = -s_{\beta\gamma} z^\gamma, \quad X^{A'}_\beta z^\beta = \partial_u^{A'} r. \tag{B.23}$$

On S^4 it is possible to uniquely solve $\partial^2 q_{\alpha\beta}=s_{\alpha\beta}$, with

$$q_{\alpha\beta}(x)=|x|^{-2}q_{\infty\alpha\beta}(\xi) \quad (\partial^2=|x|^{-6}\partial_\xi^2|x|^2),$$

and then, with definition (3.19), (3.20) holds and hence

$$X_\beta^{A'}=\hat{X}_\beta^{A'}+Z_\beta^{A'}, \quad \partial_{AA'}Z_\beta^{A'}=0. \tag{B.24}$$

Since the Dirac equation has no global solutions on S^4, $Z_\beta^{A'}=0$. However from its definition

$$\hat{X}_\beta^{A'}z^\beta=-\partial_u^{A'}q, \quad q_{\gamma\delta}x^{\gamma A}x^{\delta B}=q\varepsilon^{AB}, \tag{B.25}$$

for $q(x)=q_\infty(\xi)$. Combining (B.23) and (B.25) with $Z_\beta^{A'}=0$ gives finally

$$r+q=C, \quad \text{independent of } x, \tag{B.26}$$

providing a unique, independent of the arbitrariness in the original cocycle, representative for $H^2(\Omega^2)$, as suggested in (B.18).

With the explicit construction in (B.21) it is straightforward to follow (B.22), (B.25), using (B.12) and (B.13), to derive appropriate forms for r, q in each case, thus

$$\begin{aligned} r^1=0, \quad & X_\beta^{1A'}=\psi^{A'T}\sigma\lambda_\beta, \quad q^1=fc, \\ r^2=\phi^T\sigma\phi, \quad & X_\beta^{2A'}=\Omega_\beta^{A'T}\sigma\phi+\phi^T\sigma\Omega_\beta^{A'}, \quad q^2=\Delta'^T f\hat{\Delta}'. \end{aligned} \tag{B.27}$$

Hence (B.26) ensures that K, J, as defined in (3.45) are constant and provide the unique realisation of cohomological product of H^1's in (B.18).

Appendix C

The Eqs. (2.5) defining the space B are not all independent. If $\Omega_\beta^{A'}=(\Omega^{A'B},\Omega_{B'}^{A'})$ it is sufficient to solve

$$D^2\phi=2\Omega_{A'}^{A'}, \quad D_{AA'}\Omega_{B'}^{A'}=0, \tag{C.1}$$

since, given ϕ, Eq. (2.5a) determines $\Omega^{A'B}$ so that (2.5b) holds, assuming (A.9) or that the gauge field is self-dual. Counting the solutions of (C.1) can be achieved by recognizing that for every solution $\psi^{A'}$ of the Dirac equation (2.1) there are two possible $\Omega_{B'}^{A'}$. The corresponding ϕ may be found by inverting D^2. In addition it is necessary to include all solutions of $D^2\phi=0$ with, for fields defined on S^4, ϕ having the asymptotic behaviour of (2.58). If d is the dimension of the representation there are d such solutions. This may be seen by considering the Green's function for D^2 which satisfies

$$-D^2G(x,y)=\delta^4(x-y). \tag{C.2}$$

With (1.2) this can be written in the form

$$G(x,y)=\frac{h(x,y)}{4\pi^2|x-y|^2}, \quad h(x,y)\underset{|y|\to\infty}{\sim} u(x)g(\hat{y})^{-1}. \tag{C.3}$$

From (C.2) and (C.3)

$$D^2u=0, \quad u(x)\underset{|x|\to\infty}{\sim} g(\hat{x}), \tag{C.4}$$

giving d solutions. Thus for topological charge k and $\mathscr{G}=\mathrm{SU}(n)$ we get the counting in (2.59).

More systematically Madore et al. [27] have shown that Eqs. (C.1) can be expressed in a homogeneous covariant form on S^4, with the standard metric, so that normal index theorems can be applied.

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