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Decay of Correlations in the One Dimensional Ising Model With $J_{ij} = |i-j|^{-2}$

John Z. Imbrie*

Department of Physics, Harvard University, Cambridge, MA 02138, USA

Abstract. A low temperature expansion is constructed for the one dimensional Ising model with Hamiltonian $H = \sum_{i<j} |i-j|^{-2}(1-\sigma_i\sigma_j)$. It is shown that the two point function $\langle \sigma_i ; \sigma_j \rangle$ obeys upper and lower bounds of the form $f(\beta)|i-j|^{-2}$ for inverse temperature β sufficiently large.

Introduction

The one dimensional Ising model with Hamiltonian $H = \sum_{i<j} J(i-j)(1-\sigma_i\sigma_j)$ exhibits a phase transition with spontaneous magnetization at low temperatures if the spin–spin coupling $J(i-j)$ is sufficiently long range. If $J(r) = r^{-\alpha}$ with $\alpha > 2$, there is no magnetization at any temperature [7, 15, 18], but if $\alpha \leqq 2$ there is a transition. The case $\alpha < 2$ was treated by Dyson [7] but the borderline case $\alpha = 2$ was not treated rigorously until Fröhlich and Spencer [10] developed a sophisticated Peierls argument for the model. (Anderson, Yuval, and Hamann [1, 2] studied this case in connection with the Kondo problem and predicted a spontaneous magnetization.) In this paper a more detailed analysis of the borderline case $\alpha = 2$ is made. We obtain precise upper and lower bounds on the long distance behavior of correlation functions at low temperatures.

The technique of Fröhlich and Spencer originated in their study of the two dimensional Coulomb gas and plane rotator models [9]. There it provided a tool to study the long distance behavior of correlation functions when truncation was not necessary to obtain decay. When truncation was necessary, their technique did not apply because it is not a full fledged cluster expansion—it is analogous to Peierls expansion half of a mean field expansion [12]. One of the main motivations of this work is a desire to combine the expansion of Fröhlich and Spencer with

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a decoupling expansion, thereby obtaining full control of a number of massless models.

In the two dimensional Coulomb gas, the decoupling expansion would have to be done in various dipole-type gases—this seems too difficult at the moment, although some simplified versions have been studied recently [8, 11]. In the one dimensional Ising model, however, we provide a Mayer or decoupling expansion to go along with the Peierls expansion of [10]. This enables us to handle correlation functions such as $\langle \sigma_i ; \sigma_j \rangle$ for which the truncation is nontrivial (due to the magnetization).

The model involves spins $\sigma_i = \pm 1, i \in \mathbb{Z}$, with interactions given by the Hamiltonian

$$H_L(\{\sigma_i\}) = \sum_{i<j} |i-j|^{-2}(1-\sigma_i\sigma_j). \tag{1.1}$$

We impose the boundary condition that $\sigma_i = 1$ for $|i| \geqq L$, and establish estimates uniform as $L \to \infty$. Expectations are constructed as usual:

$$\langle F \rangle_L = \frac{1}{Z} \sum_{\{\sigma_i\}} F e^{-\beta H_L(\{\sigma_i\})},$$
$$\langle F_1 ; F_2 \rangle_L = \langle F_1 F_2 \rangle_L - \langle F_1 \rangle_L \langle F_2 \rangle_L, \tag{1.2}$$

where Z is the normalization. The following two theorems are our main results about this model.

Theorem 1.1. *Let β be sufficiently large. Then*

$$\left| \left\langle \prod_{i \in A_1} \sigma_i ; \prod_{i \in A_2} \sigma_i \right\rangle_L \right| \leqq e^{c(|A_1| + |A_2|)} (\operatorname{dist}(A_1, A_2))^{-2}. \tag{1.3}$$

Theorem 1.2. *Let $\varepsilon > 0$ be given, and suppose $i, j \in (-L, L)$ with $|i-j| > M_0(\varepsilon)$. Then for $\beta > \beta_0(\varepsilon)$*

$$|\langle \sigma_i ; \sigma_j \rangle_L - 4e^{-8\beta\zeta(2)}(e^{4\beta|i-j|^{-2}} - 1)| \leqq 4|i-j|^{-2} e^{-\beta(1-\varepsilon)(12\zeta(2)-4)}, \tag{1.4}$$

where $\zeta(2) = \sum_{j=1}^{\infty} j^{-2} = \frac{\pi^2}{6}$. *If* $|i-j| \geqq \beta^{1/2}$, then this implies that

$$\langle \sigma_i ; \sigma_j \rangle_L = 16\beta e^{-8\beta\zeta(2)} |i-j|^{-2} [1 + O(\beta|i-j|^{-2}) + O(e^{-\beta(4\zeta(2)-4-\varepsilon')})] \tag{1.5}$$

for $\beta > \beta_0(\varepsilon')$.

It is particularly satisfying that the expansion is sensitive enough to obtain lower bounds on the two point function and thus establish the power law fall-off. Upper bounds are usually much easier to obtain. More detailed bounds on the two and higher point functions could also be obtained from our expansion with a more careful study of the leading terms. General Ursell functions could also be treated. More general systems with $c_1|i-j|^{-2} \leqq J_{ij} \leqq c_2|i-j|^{-2}$ could be treated with minor changes in the estimates.

The thermodynamic limit $L \to \infty$ of this model exists by [13]. (It would also follow from our expansion.) Thus these bounds carry over to the infinite volume correlation functions.

Thouless [17] predicted a discontinuity in the spontaneous magnetization of this model as β decreases through β_c, the point where magnetization disappears. Simon and Sokal [16] prove that this happens, provided there is a spontaneous magnetization for β large and provided there is a power fall-off in $\langle \sigma_i ; \sigma_j \rangle$ uniformly for $\beta > \beta_c$. Our proof of power fall-off is restricted to large β, so the proof of existence of the Thouless effect remains an open question. Bhattacharjee et al. [19] present a different picture of the region near β_c. They argue that the power law decay of $\langle \sigma_i ; \sigma_j \rangle$ varies with β (in contrast to the situation at large β where we find a constant r^{-2} decay) and that the power goes to zero as $\beta \to \beta_c$.

In Sect. 2, we give the Peierls or contour expansion in a form very close to the one in [10]. Since we intend to quote certain estimates from [10], an explanation of the ideas is given here. Each contour is a set of spin flips, and summing over collections of contours is the same as summing over spin configurations. Each contour has an even number of spin flips, so if no contour surrounds a spin then it must be plus. To prove that there is a spontaneous magnetization, one must show that it is unlikely for any contour to surround a given site. When constructing contours it is therefore necessary to insure that the energy of contours in some class is commensurate with the logarithm of the number of such contours (the "entropy"). Then at low temperature the energy will win. This balance is assured by forcing isolated parts of a contour to have an odd number of spin flips, and by keeping contours very far apart. An odd set of spin flips has an energy that behaves like the logarithm of the distance to the nearest other spin flip, so these isolated parts of a contour provide a lot of energy. The entropy also behaves like the logarithm of the distance between spin flips: There are $D-1$ ways of placing a spin flip pair of length D over a given point. Thus the aforementioned energy-entropy balance is possible. Keeping contours far apart insures that interactions between contour do not cause much of their energy to be lost. In fact it is necessary to keep large contours even farther apart than in proportion to their diameter.

In Sect. 3, we give the Mayer expansion. The weak interaction between contours is removed step by step until the partition function is expressed as a sum over collections of clusters (or contours connected into Mayer graphs) which affect each other only by their excluded volume. Interactions between contours in a cluster decay as r^{-2}.

The next section deals with the hard core conditions on clusters. At the same time, disconnected processes in an expectation are cancelled between the numerator and the partition function in the denominator. The result is the expansion for the correlation functions.

Section 5 proves convergence of the expansion and establishes Theorems 1.1–1.2. We proceed through the various levels of structure, taking care to preserve a decay like r^{-2} at each stage.

2. The Contour Expansion

We begin the construction of a cluster expansion by identifying an appropriate set of contours for each configuration of spins. Each contour γ will be a collection of bonds across which the spin changes sign ("spin flips"). Unlike in the two-dimensional Ising model, it is not immediately obvious what the appropriate notion of connectedness of contours should be. If one includes too many spin flips in a connected contour, then there will be too many contours of a given energy surrounding a fixed point and so energy will not dominate entropy. On the other hand, if too few spin flips are included in a connected contour, then the contour can have strong interactions with the rest of the system. The construction of contours we give here is basically the same as in [10], but we enforce certain irreducibility conditions on the contours in an ensemble. This has the effect of removing any trace of the inductive nature of their construction. While it may not be absolutely necessary, this simplification aids the control of ratios of partition functions since all constraints will be very explicit.

We fix the size of our finite system at $2L-1$, and require that $\sigma_i = 1$ for $|i| \geqq L$. Letters γ, ρ, Γ will denote contours in this system, that is, sets of spin flips or bonds $b \in \mathbb{Z}^* \cap [-L, L]$ such that $\sigma_i \sigma_{i+1} = -1$ or i the site just before b. We write $|\gamma|$ for the number of spin flips in γ.

Definition. A contour γ is called *irreducible* if:

A. The number of spin flips in γ is even.

B. There is no decomposition of γ into subsets $\gamma = \gamma_1 \cup \ldots \cup \gamma_n$ such that $|\gamma_i|$ is even for all i and $\operatorname{dist}(\gamma_i, \gamma_j) \geqq M(\min\{d(\gamma_i), d(\gamma_j)\})^{3/2}$ for $i \neq j$. Here $d(\gamma)$ is the diameter of γ.

Proposition 2.1. *There is a unique way of partitioning the spin flips of any spin configuration into irreducible contours γ_α such that*

C. $$\operatorname{dist}(\gamma_\alpha, \gamma_\beta) \geqq M(\min\{d(\gamma_\alpha), d(\gamma_\beta)\})^{3/2}.$$

Proof. To find a partition satisfying A, B, C above, start by considering the set of all spin flips as a single contour. Conditions A, C are certainly satisfied, but the contour may not be irreducible because of condition B. If B is not satisfied, divide up the contour into $\gamma_1 \cup \ldots \cup \gamma_n$ for which A and C will hold. If B does not hold for each of $\gamma_1, \ldots, \gamma_n$, then we split up the reducible γ_i's further. The procedure repeats (at most finitely many times) until all contours are irreducible. Odd contours ($|\gamma|$ odd) are never generated by this procedure. We need only check that the distance rule C is never violated.

Whenever γ is split into $\gamma_1 \cup \ldots \cup \gamma_n$ as per B, we have C for $\gamma_\alpha, \gamma_\beta$ in this splitting. Furthermore

$$\operatorname{dist}(\gamma, \gamma') \geqq M(\min\{d(\gamma), d(\gamma')\})^{3/2} \tag{2.1}$$

for $\gamma' \neq \gamma$ in the ensemble of contours before the splitting. Since $d(\gamma_\alpha) \leqq d(\gamma)$ and $\operatorname{dist}(\gamma_\alpha, \gamma') \geqq \operatorname{dist}(\gamma, \gamma')$ for γ_α in the splitting of γ, the inequality C is preserved.

Thus existence is proven and we concentrate on uniqueness, which is the main content of the proposition. Suppose we have two decompositions of a set of spin flips into irreducible contours, $\{\gamma_\alpha\}$ and $\{\rho_\alpha\}$. Then $\{\rho_\alpha\}$ cannot be obtained from $\{\gamma_\alpha\}$ by joining two or more γ's into a single contour, because by C the resulting contour would violate B and so be reducible. Hence at least one contour in $\{\gamma_\alpha\}$

must be broken up in forming $\{\rho_\alpha\}$. Let γ be a smallest such contour.

Consider the splitting of γ given by $\{\rho_\alpha \cap \gamma\}$. At least one element must be odd, otherwise this splitting would violate B. (If the distance condition holds for $\{\rho_\alpha\}$ then it holds for $\{\rho_\alpha \cap \gamma\}$.) Since γ is even, at least two elements of $\{\rho_\alpha \cap \gamma\}$ are odd; denote them by $\rho_1 \cap \gamma, \rho_2 \cap \gamma$. Each of ρ_1, ρ_2 intersects a $\gamma_\alpha \neq \gamma$ in an odd set, otherwise they could not be even. Call these contours γ_1 for ρ_1, γ_2 for ρ_2; it is possible that $\gamma_1 = \gamma_2$. Since $\gamma_i \cap \rho_i$ is odd, γ_i must be broken up in $\{\rho_\alpha\}$. Hence by definition of γ,

$$d(\gamma_1) \geqq d(\gamma), d(\gamma_2) \geqq d(\gamma). \tag{2.2}$$

However,

$$\begin{aligned} d(\gamma) &\geqq \operatorname{dist}(\rho_1, \rho_2) \\ &\geqq M(\min\{d(\rho_1), d(\rho_2)\})^{3/2} \\ &\geqq M(\min\{\operatorname{dist}(\gamma,\gamma_1), \operatorname{dist}(\gamma,\gamma_2)\})^{3/2}, \end{aligned} \tag{2.3}$$

and since

$$\begin{aligned} \operatorname{dist}(\gamma,\gamma_i) &\geqq M(\min\{d(\gamma), d(\gamma_i)\})^{3/2} \\ &= M d(\gamma)^{3/2}, i = 1, 2, \end{aligned} \tag{2.4}$$

we have

$$d(\gamma) \geqq M(M d(\gamma)^{3/2})^{3/2} > d(\gamma) \tag{2.5}$$

for $M > 1$. We have a contradiction, so our supposition that $\{\gamma_\alpha\} \neq \{\rho_\alpha\}$ cannot be correct, and the proposition is proven. □

Clearly any set of irreducible contours satisfying the distance rule C determines a unique spin configuration. Thus there is a one-to-one correspondence between spin configurations and collections of irreducible contours satisfying C. This means that our contour expansion

$$\sum_{\{\sigma_i\}} e^{-\beta H(\{\sigma_i\})} = \sum_{\substack{\{\gamma_i\} \text{ irreducible} \\ \operatorname{dist}(\gamma_i,\gamma_{i'}) \geqq M(\min\{d(\gamma_i), d(\gamma_{i'})\})^{3/2}}} \exp\left(-\beta H\left(\bigcup_i \gamma_i\right)\right) \tag{2.6}$$

is valid. This formula will be the starting point for the Mayer expansion of the next section. It is particularly useful in that all constraints on the sum over contours are explicitly given as conditions on pairs of contours.

The next proposition proves the charged constituent condition of [10]. It is used to extract self energies of contours.

Proposition 2.2. *If γ is an irreducible contour and $\rho \subseteq \gamma, \rho \neq \gamma$, then*

D. $\operatorname{dist}(\rho, \gamma \backslash \rho) \geqq 2M d(\rho)^{3/2}$ *implies that ρ is odd.*

Proof. If ρ were even, then $\gamma \backslash \rho$ would also be even and

$$\operatorname{dist}(\rho,\gamma\backslash\rho) \geqq M(\min\{d(\rho), d(\gamma\backslash\rho)\})^{3/2}. \tag{2.7}$$

This violates condition B for γ. Thus ρ must be odd. □

3. The Mayer Expansion

The Mayer expansion expresses the partition function as a sum over collections of noninteracting connected graphs (or "clusters"). We know from [10] that for M large a contour interacts only weakly with the other contours of the system. It was shown that

$$0 \leqq H(\gamma_1) + H(\Gamma) - H(\gamma_1 \cup \Gamma) \leqq \frac{c}{M}(\log M)^3 H(\gamma_1), \tag{3.1}$$

for $\Gamma = \gamma_2 \cup \cdots \cup \gamma_n$ and $\{\gamma_\alpha\}_{\alpha=1}^n$ satisfying A, C, D. This explains why a convergent Mayer expansion should be possible. We will also need to control the 3-body and higher body interactions between contours. Actually, their will remain hard core type conditions on the clusters; these will be treated in Sect. 4.

We will use the tree graph formulation; see [5], or [6] for an overview on the method. For this section, we will concentrate on a fixed $\{\gamma_\alpha\}$ satisfying A, B, C, D. Letters Γ will refer to unions $\bigcup_\beta \gamma_{\alpha(\beta)}$ where $\{\gamma_{\alpha(\beta)}\} \subseteq \{\gamma_\alpha\}$.

The Hamiltonian for Γ is

$$H(\Gamma) = 2 \sum_{i<j} |i-j|^{-2} \chi_\Gamma(i,j), \tag{3.2}$$

where

$$\chi_\Gamma(i,j) = \begin{cases} 1 & \text{if an odd number of spin flips in } \Gamma \text{ separate } i \text{ from } j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Interpolating Hamiltonians are defined inductively:

$$H_{s_1}(\Gamma_1; \Gamma_2) = H(\Gamma_1 \cup \Gamma_2)s_1 + (H(\Gamma_1) + H(\Gamma_2))(1-s_1), \tag{3.4}$$

and in general

$$\begin{aligned} &\mathrm{H}_{s_1 s_2 \ldots s_n}(\Gamma_1, \Gamma_2, \ldots, \Gamma_n; \Gamma_{n+1}) \\ &\quad = H_{s_1 s_2 \cdots s_{n-1}}(\Gamma_1, \ldots, \Gamma_{n-1}; \Gamma_n \cup \Gamma_{n+1}) s_n \\ &\quad\quad + (H_{s_1 s_2 \ldots s_{n-1}}(\Gamma_1, \ldots, \Gamma_{n-1}; \Gamma_n) + H(\Gamma_{n+1}))(1-s_n). \end{aligned} \tag{3.5}$$

Here s_i are interpolation parameters, $s_i \in [0,1]$. When $s_n = 0$ we have

$$H_{s_1 \ldots s_{n-1} 0}(\Gamma_1, \ldots, \Gamma_n; \Gamma_{n+1}) = H_{s_1 \ldots s_{n-1}}(\Gamma_1, \ldots, \Gamma_{n-1}; \Gamma_n) + H(\Gamma_{n+1}) \tag{3.6}$$

and the contours in $\Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_n$ are decoupled from the rest of the system (Γ_{n+1}).

This interpolation procedure has the nice property of preserving "stability" as the following proposition shows.

Proposition 3.1. *Let* $\Gamma_i = \bigcup_\alpha \gamma_{\alpha i}$, $i = 1, \ldots, n+1$, *and suppose the contour configuration* $\{\gamma_{\alpha i}\}_{\substack{i=1,\ldots,n+1 \\ \alpha=1,\ldots}}$ *satisfies conditions* A, B, C, D. *Put*

$$\bar{H}(\Gamma_i) = \left(1 - \frac{c}{M}(\log M)^3\right) \sum_\alpha H(\gamma_{\alpha i}), \tag{3.7}$$

where c is the constant that appears in (3.1). Then

$$H_{s_1\ldots s_n}(\Gamma_1,\ldots,\Gamma_n;\Gamma_{n+1}) \geqq \sum_{i=1}^{n} \bar{H}(\Gamma_i) + H(\Gamma_{n+1}) \geqq \sum_{i=1}^{n+1} \bar{H}(\Gamma_i). \tag{3.8}$$

Proof. When $n=0$ the first inequality in (3.8) is an identity; we establish the general case by induction. Since $H_{s_1\ldots s_n}(\Gamma_1,\ldots,\Gamma_n;\Gamma_{n+1})$ is a convex sum of $H_{s_1\ldots s_{n-1}}(\Gamma_1,\ldots,\Gamma_{n-1};\Gamma_n\cup\Gamma_{n+1})$ and $(H_{s_1\ldots s_{n-1}}(\Gamma_1,\ldots,\Gamma_{n-1};\Gamma_n)+H(\Gamma_{n+1}))$, it suffices to prove the lower bound for each term separately.

For the first term we have by induction the lower bound

$$\sum_{i=1}^{n-1} \bar{H}(\Gamma_i) + H(\Gamma_n\cup\Gamma_{n+1}). \tag{3.9}$$

By (3.1),

$$\begin{aligned} H(\Gamma_n\cup\Gamma_{n+1}) &= H(\gamma_{1n}\cup\ldots\cup\gamma_{ln}\cup\Gamma_{n+1}) \\ &\geqq \left(1-\frac{c}{M}(\log M)^3\right)H(\gamma_{1n}) + H(\gamma_{2n}\cup\ldots\cup\gamma_{ln}\cup\Gamma_{n+1}) \\ &\geqq \ldots \geqq \sum_{\alpha=1}^{l}\left(1-\frac{c}{M}(\log M)^3\right)H(\gamma_{\alpha n}) + H(\Gamma_{n+1}) \\ &= \bar{H}(\Gamma_n) + H(\Gamma_{n+1}), \end{aligned} \tag{3.10}$$

and together with (3.9) this proves the lower bound for

$$H_{s_1\ldots s_{n-1}}(\Gamma_1,\ldots,\Gamma_{n-1};\Gamma_n\cup\Gamma_{n+1}).$$

Applying induction to the second term yields the lower bound

$$\sum_{i=1}^{n-1} \bar{H}(\Gamma_i) + H(\Gamma_n) + H(\Gamma_{n+1}) \tag{3.11}$$

and we can apply the steps in (3.10) to show

$$H(\Gamma_n) \geqq \sum_{\alpha=1}^{l-1}\left(1-\frac{c}{M}(\log M)^3\right)H(\gamma_{\alpha n}) + H(\gamma_{ln}) \geqq \bar{H}(\Gamma_n). \tag{3.12}$$

This completes the proof of the first inequality; the second is immediate if we apply (3.12) to Γ_{n+1}. □

The next ingredient for the Mayer expansion is a formula for $\frac{d}{ds_n}H_{s_1\ldots s_n}(\Gamma_1,\ldots,\Gamma_n;\Gamma_{n+1})$. Define

$$\begin{aligned} \theta_\Gamma(i,j) &= \begin{cases} 1 \text{ if an even number of spin flips in } \Gamma \text{ lie} \\ \text{between } i \text{ and } j, \text{ or if } \Gamma=\varnothing, \\ -1 \text{ otherwise} \end{cases} \\ &= 1-2\chi_\Gamma(i,j). \end{aligned} \tag{3.13}$$

Proposition 3.2. *If $H_{s_1\ldots s_n}(\Gamma_1,\ldots,\Gamma_n;\Gamma_{n+1})$ is defined through Eqs. (3.2)–(3.5), then*

$$\frac{d}{ds_n}H_{s_1\ldots s_n}(\Gamma_1,\ldots,\Gamma_n;\Gamma_{n+1})$$

$$= -4\sum_{l=1}^{n} s_l \dots s_{n-1} \sum_{i<j} |i-j|^{-2} \chi_{\Gamma_l}(i,j)\chi_{\Gamma_{n+1}}(i,j)\theta_{\Gamma_{l+1}\cup\dots\cup\Gamma_n}(i,j). \tag{3.14}$$

Proof. The important aspect of this formula is the fact that Γ_l is "connected" to Γ_{n+1} in the term with a factor $s_l \dots s_{n-1}$. We make the interpretation $\Gamma_{l+1}\cup\dots\cup\Gamma_n = \emptyset$ if $l \geqq n$. We derive (3.14) from the following formula for the undifferentiated Hamiltonian:

$$H_{s_1\dots s_n}(\Gamma_1,\dots,\Gamma_n;\Gamma_{n+1}) = \sum_{k=1}^{n+1} H(\Gamma_k) - \sum_{k=1}^{n}\sum_{l=1}^{k} 4 s_l \dots s_k \sum_{i<j} |i-j|^{-2}\chi_{\Gamma_l}(i,j)\chi_{\Gamma_{k+1}}(i,j)\theta_{\Gamma_{l+1}\cup\dots\cup\Gamma_k}(i,j). \tag{3.15}$$

Equation (3.14) follows immediately, since differentiating with respect to s_n picks out the $k=n$ term.

We prove (3.15) by induction on n; it clearly holds for $n=0$. By (3.5) and the induction hypothesis we have

$$\begin{aligned}
&H_{s_1\dots s_n}(\Gamma_1,\dots,\Gamma_n;\Gamma_{n+1}) \\
&= \left(\sum_{k=1}^{n-1} \mathrm{H}(\Gamma_k) + \mathrm{H}(\Gamma_n\cup\Gamma_{n+1})\right)s_n + \left(\sum_{k=1}^{n+1} H(\Gamma_k)\right)(1-s_n) \\
&\quad - \sum_{k=1}^{n-2}\sum_{l=1}^{k} 4 s_l\dots s_k \sum_{i<j}|i-j|^{-2}\chi_{\Gamma_l}(i,j)\chi_{\Gamma_{k+1}}(i,j)\theta_{\Gamma_{l+1}\cup\dots\cup\Gamma_k}(i,j). \\
&\quad - \sum_{l=1}^{n-1} 4 s_l\dots s_{n-1}\sum_{i<j}|i-j|^{-2}\chi_{\Gamma_l}(i,j)\theta_{\Gamma_{l+1}\cup\dots\cup\Gamma_{n-1}}(i,j)\,(\chi_{\Gamma_n\cup\Gamma_{n+1}}(i,j)s_n \\
&\quad + \chi_{\Gamma_n}(i,j)(1-s_n)).
\end{aligned} \tag{3.16}$$

The first two terms equal the first term in (3.15) plus

$$\begin{aligned}
&s_n(H(\Gamma_n\cup\Gamma_{n+1}) - H(\Gamma_n) - H(\Gamma_{n+1})) \\
&\quad = 2s_n\sum_{i<j}|i-j|^{-2}(\chi_{\Gamma_n\cup\Gamma_{n+1}}(i,j) - \chi_{\Gamma_n}(i,j) - \chi_{\Gamma_{n+1}}(i,j)) \\
&\quad = -4s_n\sum_{i<j}|i-j|^{-2}\chi_{\Gamma_n}(i,j)\chi_{\Gamma_{n+1}}(i,j),
\end{aligned} \tag{3.17}$$

which is the $k=l=n$ term in (3.15). The third term in (3.16) was unaffected by the interpolation since it was present in both terms of the convex sum. It provides the $k \leqq n-2$ terms in (3.15). We use the identities

$$\begin{aligned}
\chi_{\Gamma_n\cup\Gamma_{n+1}}(i,j)s_n + \chi_{\Gamma_n}(i,j)(1-s_n) &= \chi_{\Gamma_n}(i,j) + \theta_{\Gamma_n}(i,j)\chi_{\Gamma_{n+1}}(i,j)s_n, \\
\theta_{\Gamma_1\cup\Gamma_2} &= \theta_{\Gamma_1}\theta_{\Gamma_2}
\end{aligned} \tag{3.18}$$

to write the last term in (3.16) as

$$\begin{aligned}
&\sum_{l=1}^{n-1} -4s_l\dots s_{n-1}\sum_{i<j}|i-j|^{-2}\chi_{\Gamma_l}(i,j)\chi_{\Gamma_n}(i,j)\theta_{\Gamma_{l+1}\cup\dots\cup\Gamma_{n-1}}(i,j) \\
&\quad - \sum_{l=1}^{n-1} 4s_l\dots s_n\sum_{i<j}|i-j|^{-2}\chi_{\Gamma_l}(i,j)\chi_{\Gamma_{n+1}}(i,j)\theta_{\Gamma_{l+1}\cup\dots\cup\Gamma_n}(i,j).
\end{aligned} \tag{3.19}$$

The first term is the $k=n-1$ term in (3.15); the second provides the $k=n$, $l<n$ terms. Thus (3.15) and the proposition are established. □

Remark. Any time $\theta_{\Gamma_{l_0+1}\cup\ldots\cup\Gamma_n}(i,j)=-1$, the term in (3.14) with l the next integer after l_0 such that $\chi_{\Gamma_l}(i,j)=1$ will have $\theta=1$. Since that term has fewer factors of s_i, it dominates the earlier term. Thus $\frac{d}{ds_n}(-\beta H_{s_1\ldots s_n}(\Gamma_1,\ldots,\Gamma_n;\Gamma_{n+1}))$ is always non-negative.

We now give the expansion for $e^{-\beta H(\gamma_1\cup\ldots\cup\gamma_n)}$, using (3.14).

$$e^{-\beta H(\gamma_1\cup\ldots\cup\gamma_n)}=e^{-\beta H(\gamma_1)}e^{-\beta H(\gamma_2\cup\ldots\cup\gamma_n)}$$
$$+\int_0^1 ds_1 4\beta\sum_{i_1<j_1}|i_1-j_1|^{-2}\chi_{\gamma_1}(i_1,j_1)\chi_{\gamma_2\cup\ldots\cup\gamma_n}(i_1,j_1)e^{-\beta H_{s_1}(\gamma_1;\gamma_2\cup\ldots\cup\gamma_n)}. \tag{3.20}$$

We put $\Gamma_1=\gamma_1$ and choose Γ_2 depending on which term (i_1,j_1) we are expanding further. Let $\Gamma=\gamma_1\cup\ldots\cup\gamma_n$. Γ_l will be defined as the union of all contours $\gamma_m\subseteq\Gamma\backslash\Gamma_1\backslash\ldots\backslash\Gamma_{l-1}$ such that $\chi_{\gamma_m}(i_{l-1},j_{l-1})=1$. This choice insures that

$$\chi_{\gamma_2\cup\ldots\cup\gamma_n}(i_1,j_1)=\chi_{\Gamma_2}(i_1,j_1). \tag{3.21}$$

If $\Gamma_2=\varnothing$ or if $\chi_{\Gamma_2}(i_1,j_1)=0$ then the (i_1,j_1) term in (3.20) vanishes, and we disregard it. The second term in (3.20) is expanded into

$$\sum_{i_1<j_1}4\beta|i_1-j_1|^{-2}\chi_{\Gamma_1}(i_1,j_1)\chi_{\Gamma_2}(i_1,j_1)\int_0^1 ds_1 e^{-\beta H_{s_1}(\Gamma_1;\Gamma_2)}e^{-\beta H(\Gamma\backslash\Gamma_1\backslash\Gamma_2)}$$
$$+\sum_{i_1<j_1}\sum_{i_2<j_2}(4\beta)^2|i_1-j_1|^{-2}|i_2-j_2|^{-2}\chi_{\Gamma_1}(i_1,j_1)\chi_{\Gamma_2}(i_1,j_1)\sum_{\eta(3)=1}^{2}\int ds_1 ds_2$$
$$s_{\eta(3)}\cdots s_1\chi_{\Gamma_{n(3)}}(i_2,j_2)\chi_{\Gamma\backslash\Gamma_1\backslash\Gamma_2}(i_2,j_2)\theta_{\Gamma_{\eta(3)+1}\cup\ldots\cup\Gamma_2}(i_2,j_2)e^{-\beta H_{s_1s_2}(\Gamma_1,\Gamma_2;\Gamma\backslash\Gamma_1\backslash\Gamma_2)} \tag{3.22}$$

Continuing the interpolations until Γ is exhausted, we obtain

$$e^{-\beta H(\Gamma)}=\sum_{k=1}^{\infty}\sum_{i_1<j_1}\cdots\sum_{i_{k-1}<j_{k-1}}\sum_{\eta}\int_0^1 ds_1\ldots ds_{k-1}$$
$$\Big\{(4\beta)^{k-1}\prod_{l=2}^{k}[|i_{l-1}-j_{l-1}|^{-2}\chi_{\Gamma_{\eta(l)}}(i_{l-1},j_{l-1})\chi_{\Gamma_l}(i_{l-1},j_{l-1})\theta_{\Gamma_{\eta(l)+1}\cup\ldots\cup\Gamma_{l-1}}(i_{l-1},j_{l-1})$$
$$s_{\eta(l)}\cdots s_{l-2}]e^{-\beta H_{s_1\cdots s_{k-1}}(\Gamma_1,\ldots,\Gamma_{k-1};\Gamma_k)}\Big\}e^{-\beta H(\Gamma\backslash\Gamma_1\backslash\ldots\backslash\Gamma_k)}. \tag{3.23}$$

Here η is summed over all maps from $\{1,\ldots,k\}$ to itself such that $\eta(l)<l$. It specifies which term in the sum over l in (3.14) is taken at each interpolation. If $k=1$ we replace $\sum_\eta$ with 1. Our choice of Γ_l allows us to eliminate the dependence of the χ- and θ-factors on $\Gamma\backslash\Gamma_1\backslash\ldots\backslash\Gamma_l$, as in (3.21).

Given Γ, the above construction was completely deterministic with the exception of the choice of $\gamma_1=\Gamma_1$. To eliminate this freedom, we order once and for all the set of all possible irreducible contours (not just the ones in a particular collection $\{\gamma_\alpha\}$). When applying (3.23), we always choose Γ_1 to be the first contour of Γ. We write $\gamma_1 \llcorner \gamma_2$ when γ_1 is before γ_2.

Equation (3.23) defines the notion of cluster (or connected graph) that we will use in the expansion. A cluster consists of the following data:

An integer $k > 1$;
an irreducible contour Γ_1;
a tree η (that is, integers $0 < \eta(l) < l$ for $l = 2, \ldots, k$);
for each $l = 2, \ldots, k$, a pair of integers $i_{l-1} < j_{l-1}$,

a union of irreducible contours $\Gamma_l = \bigcup_\alpha \gamma_{\alpha l}$,
and an interpolation parameter $s_{l-1} \in [0, 1]$.

These data satisfy certain compatibility conditions:

(*i*) The irreducible contours forming $\Gamma_1, \ldots, \Gamma_k$ satisfy $\operatorname{dist}(\gamma, \gamma') \geqq M(\min\{d(\gamma), d(\gamma')\})^{3/2}$,
(*ii*) Γ_1 is before all $\gamma_{\alpha l}$ in the ordering fixed above,
(*iii*) For $l = 2, \ldots, k$ and all α, $\chi_{\gamma_{\alpha l}}(i_{l-1}, j_{l-1}) = 1$ and $\chi_{\gamma_{\alpha l}}(i_m, j_m) = 0$ for $m < l - 1$,
(*iv*) For $l = 2, \ldots, k$, $\chi_{\Gamma_{\eta(l)}}(i_{l-1}, j_{l-1}) = \chi_{\Gamma_l}(i_{l-1}, j_{l-1}) = 1$.

Condition (*ii*) arises from our choise of Γ_1; (*iii*) from our choice of $\Gamma_2, \ldots, \Gamma_k$. Denote by Y a set of data satisfying these conditions, and let $\Gamma(Y) = \Gamma_1 \cup \ldots \cup \Gamma_k$, $\Gamma_1(Y) = \Gamma_1$, etc. We associate to the cluster Y the quantity in braces in (3.23), that is,

$$\rho(Y) = (4\beta)^{k-1} \prod_{l=2}^{k} [|i_{l-1} - j_{l-1}|^{-2} \theta_{\Gamma_{\eta(l)+1} \cup \ldots \cup \Gamma_{l-1}}(i_{l-1}, j_{l-1}) s_{\eta(l)} \ldots s_{l-2}]$$
$$\cdot e^{-\beta H_{s_1 \ldots s_{k-1}}(\Gamma_1, \ldots, \Gamma_{k-1}; \Gamma_k)}. \tag{3.24}$$

The χ-factors have been dropped by condition (*iv*); we can take $\rho(Y) = 0$ when Y does not satisfy (*i*)–(*iv*).

For two unions of irreducible contours Γ, Γ', each satisfying the distance rule C, write $\Gamma \leqq \Gamma'$ when every contour in Γ is a contour of Γ'. (This has nothing to do with the ordering of irreducible contours chosen above). We also denote by $\gamma_1(\Gamma)$ the first contour in Γ.

With these notations we can write

$$\sum_{Y : \Gamma(Y) \leqq \Gamma, \Gamma_1(Y) = \gamma_1(\Gamma)} \quad \text{for} \quad \sum_{k=1}^{\infty} \sum_{i_1 < j_1} \ldots \sum_{i_{k-1} < j_{k-1}} \sum_{\eta} \int_0^1 ds_1 \ldots ds_{k-1}$$

in (3.23). Thus (3.23) becomes

$$e^{-\beta H(\Gamma)} = \sum_{Y : \Gamma(Y) \leqq \Gamma, \Gamma_1(Y) = \gamma_1(\Gamma)} \rho(Y) e^{-\beta H(\Gamma \setminus \Gamma(Y))}. \tag{3.25}$$

Next, we apply (3.25) to $e^{-\beta H(\Gamma \setminus \Gamma(Y))}$ and continue expanding until Γ is exhausted. We say a collection of clusters $\{Y_1, \ldots, Y_k\}$ is compatible if

(1) The contours of each cluster satisfy the distance rule with respect to the contours in the other clusters.

(2) Whenever $\Gamma_1(Y_r) \mathrel{\llcorner} \Gamma_1(Y_{r'})$, the contours in $Y_{r'}$ do not span the vertices of Y_r. That is, if γ is a contour of $Y_{r'}, \chi_\gamma(i,j)=0$ for each of the $k-1$ pairs of sites $i_{l-1}<j_{l-1}$ of Y_r.

Furthermore, $\{Y_1,\dots,Y_k\}$ is compatible with Γ if in addition

(3) $$\Gamma(Y_1)\cup\dots\cup\Gamma(Y_k)=\Gamma.$$

The expansion now takes the form

$$e^{-\beta H(\Gamma)}=\sum_{\{Y_1,\dots,Y_k\}\text{compatible with }\Gamma}\ \prod_{s=1}^{k}\rho(Y_s). \tag{3.26}$$

Condition (2) takes care of the constraints on contours in $\Gamma\backslash\Gamma(Y)$ arising from the choice of Y. Any compatible collection of clusters is automatically ordered according to the order of their Γ_1's. Thus condition (*ii*) insures that $\Gamma_1(Y)$ is the first contour in Γ each time (3.25) is applied.

4. The Correlation Functions

In this section we combine the contour and Mayer expansions, deal with the remaining constraints on clusters, and cancel disconnected processes in the ratio

$$\langle F(\{\sigma_i\})\rangle_L=\frac{\sum_\Gamma F(\{\sigma_i(\Gamma)\})e^{-\beta H(\Gamma)}}{\sum_\Gamma e^{-\beta H(\Gamma)}}\equiv\frac{Z_F}{Z}. \tag{4.1}$$

We take $F=F_1(\sigma_{A_1})F_2(\sigma_{A_2})$ where $\sigma_A=\{\sigma_i\}_{i\in A}$, A a finite set of lattice sites. For example $F=\sigma_i\sigma_j$. Γ runs over all unions of irreducible contours in $\mathbb{Z}^*\cap[-L,L]$ and satisfying the distance rule. In Sect. 5, we prove convergence of the resulting expansion for correlation functions and derive decay estimates uniform as $L\to\infty$.

Using (3.26) we can write the numerator in (4.1) as

$$Z_F=\sum_{\{Y_1,\dots,Y_k\}\text{ compatible}} F(\{Y_s\})\prod_{s=1}^{k}\rho(Y_s), \tag{4.2}$$

where by abuse of notation we write $F(\{Y_s\})$ for the value F takes on the spin configuration determined by the contour configuration $\bigcup_{s=1}^{k}\Gamma(Y_s)$.

We say a cluster Y is odd with respect to i if there is an odd number of spin flips in $\Gamma(Y)$ left of i. Consider two cases for each $\{Y_s\}$. If no cluster is odd with respect to at least one site in A_1 and at least one in A_2, we define $X_j=(A_j,\{Y_s\}_{s\in\mathcal{O}_j})$, where $\{Y_s\}_{s\in\mathcal{O}_j}$ are the clusters that are odd with respect to at least one site in A_j. Otherwise, we define $X_{12}=(A_1\cup A_2,\{Y_s\}_{s\in\mathcal{O}_{12}})$ where $\{Y_s\}_{s\in\mathcal{O}_{12}}$ are the clusters odd with respect to any site in A_1 or A_2. Write $\{X_\alpha\}$ for $\{X_1,X_2\}$ or $\{X_{12}\}$. F depends only on $\{X_\alpha\}$. Summing separately over $\{X_\alpha\}$ and the other clusters, we have

$$Z_F = \sum_{\{X_\alpha\}} \sum_{\substack{(Y_1,\ldots,Y_j) \\ \text{compatible with } \{X_\alpha\}}} \frac{1}{j!} F(\{X_\alpha\}) \prod_\alpha \rho(X_\alpha) \prod_{r=1}^{j} \rho(Y_r), \tag{4.3}$$

where

$$\rho(X_\alpha) = \prod_{s \in \mathcal{O}_\alpha} \rho(Y_s). \tag{4.4}$$

We have summed over ordered families of Y_r's not part of an X_α, and compensated with a $\frac{1}{j!}$. The $\llcorner$-ordering of Y_r's does not necessarily agree with their indices r.

We next remove the compatibility conditions between X_1 and X_2, between X_α and Y_r, and between the Y_r's using functions U taking values 0 and 1 (this is a standard trick, see [3, 4, 11, 14]):

$$U(Y_1, Y_2) = \begin{cases} 0 & \text{if any pair of irreducible contours forming } Y_1, Y_2 \text{ satisfy} \\ & \operatorname{dist}(\gamma_1, \gamma_2) < M(\min\{d(\gamma_1), d(\gamma_2)\})^{3/2}, \\ 0 & \text{if } \Gamma_1(Y_1) \llcorner \Gamma_1(Y_2) \text{ and any irreducible contour } \gamma \text{ of } Y_2 \\ & \text{satisfies } \chi_\gamma(i,j) = 1 \text{ for } (i,j) \text{ a vertex pair of } Y_1\text{; likewise} \\ & \text{when } \Gamma_1(Y_2) \llcorner \Gamma_1(Y_1), \\ 1 & \text{otherwise;} \end{cases}$$

$$U(X_\alpha, Y) = \begin{cases} 0 & \text{if } Y \text{ is odd with respect to a site in } A_\alpha \text{ (denoting } A_1 \cup A_2 \text{ by} \\ & A_{12}), \\ 0 & \text{if } U(Y_s, Y) = 0 \text{ for } Y_s \text{ a cluster from } X_\alpha, \\ 1 & \text{otherwise;} \end{cases}$$

$$U(X_1, X_2) = \begin{cases} 0 & \text{if any cluster in } X_1 \text{ is odd with respect to a site in } A_2, \\ & \text{likewise for } X_2 \text{ and } A_1, \\ 0 & \text{if } U(Y_{s_1}, Y_{s_2}) = 0 \text{ for } Y_{s_1} \text{ a cluster from } X_1,\ Y_{s_2} \text{ from } X_2, \\ 1 & \text{otherwise.} \end{cases} \tag{4.5}$$

Equation (4.3) becomes

$$Z_F = \sum_{\{X_\alpha\}} \prod_\alpha F_\alpha(X_\alpha) \sum_{(Y_1,\ldots,Y_j)} \frac{1}{j!} \prod_{\mathscr{L}} U(\mathscr{L}) \prod_\alpha \rho(X_\alpha) \prod_{r=1}^{j} \rho(Y_r), \tag{4.6}$$

where $F_{12} = F$ and the product over $\mathscr{L}$ is over all pairs $\{X_1, X_2\}$, $\{X_\alpha, Y_r\}$, or $\{Y_{r_1}, Y_{r_2}\}$. $\{X_\alpha\}$ is still of the form $\{X_1, X_2\}$ or $\{X_{12}\}$ but now the clusters in X_1 can overlap with A_2, etc.

Standard manipulations allow us to extract the connected part of this expression. We put $U = 1 + A$ and expand the product over $\mathscr{L}$. This produces a sum of graphs of vertices X_α, Y_r connected with lines $\mathscr{L}$ and factors $A(\mathscr{L})$:

$$Z_F = \left(\sum_{\{X_\alpha\}} \prod_\alpha F_\alpha(X_\alpha) \sum_{(Y'_1,\ldots,Y'_{j_c})} \frac{1}{j_c!} \sum_{G_c} \prod_{\mathscr{L} \in G_c} A(\mathscr{L}) \prod_\alpha \rho(X_\alpha) \prod_{r=1}^{j_c} \rho(Y'_r) \right)$$

$$\cdot\left(\sum_{(Y''_1,\ldots,Y''_{j_0})}\frac{1}{j_0!}\sum_{G_0}\prod_{\mathscr{L}\in G_0}A(\mathscr{L})\prod_{r=1}^{j_0}\rho(Y''_r)\right). \tag{4.7}$$

Here G_c is a graph involving all of $(Y'_1,\ldots,Y'_{j_c})$ such that each Y'_r is connected directly or indirectly to some X_α; G_0 is an arbitrary graph on $(Y''_1,\ldots,Y''_{j0})$. The second factor is what we would have obtained if $A=\emptyset$; it is the partition function Z. Thus, we obtain the final form of the expansion:

$$\langle F_1F_2\rangle_L=\sum_{\{X_\alpha\}}\prod_\alpha F_\alpha(X_\alpha)\sum_{(Y_1,\ldots,Y_j)}\frac{1}{j!}\sum_{G_c}\prod_{\mathscr{L}\in G_c}A(\mathscr{L})\prod_\alpha\rho(X_\alpha)\prod_{r=1}^{j}\rho(Y_r). \tag{4.8}$$

Notice that the sum of all terms with $\{X_\alpha\}=\{X_1,X_2\}$ and G_c not connecting X_1 to X_2 factorizes into independent sums associated with each X_α. Thus these terms sum to $\langle F_1\rangle_L\langle F_2\rangle_L$ and so

$$\langle F_1;F_2\rangle_L=\langle F_1F_2\rangle_L-\langle F_1\rangle_L\langle F_2\rangle_L$$

is given by the same formula (4.8) except that G_c is connected with respect to all Y's and X's.

5. Decay of Correlations

We now estimate the expansions (4.8) and prove upper and lower bounds of the type $c|i-j|^{-2}$ on decay of correlations. We proceed in two stages: First, the sum over clusters containing a pair of given points is estimated, and then the full expansion is controlled using the first estimate. The second part is essentially a ratio of partition function estimate, and we use the algebraic formalism. We must also take care to preserve the $|i-j|^{-2}$ decay at each state of the estimation, and never to let a contour γ generate combinatorics worse than $\varepsilon\beta H(\gamma)$. We wish to use only a small part of the energy of contours to control sums so that we can see that the leading contribution to the two point function dominates all other contributions. Finitely many factors $e^{-\varepsilon\beta H(\gamma)}$ for each γ will control the summations. Thus with ε small, most of the energy is still available. We take M large depending on $\varepsilon>0$, and β large depending on M. Let $I(\gamma)$ denote the set of sites between the extremal spin flips of γ. For $\Gamma=\bigcup_\alpha\gamma_\alpha$ we put $I(\Gamma)=\bigcup_\alpha I(\gamma_\alpha)$. Define

$$C(i,j)=\min\{1,|i-j|^{-2}\}. \tag{5.1}$$

Lemma 5.1. *For $M>M_0(\varepsilon),\beta>\beta_0(M)$, and γ an irreducible contour,*

$$\sum_{\gamma:i\in I(\gamma),j\in I(\gamma)}e^{-\varepsilon\beta H(\gamma)}\leqq C(i,j)^2e^{-\varepsilon\beta}. \tag{5.2}$$

Proof. This is not the optimal estimate, but it is all we will need here. As in [10], we define $N_n(\gamma)$ to be the minimum number of open intervals of length 2^n needed to cover γ, and put

$$N(\gamma)=\sum_{n=0}^{[\log_2 d(\gamma)]+1}N_n(\gamma). \tag{5.3}$$

It was proven in [10] that the number of γ with $N(\gamma) \leqq R$ and $0 \in I(\gamma)$ is less than e^{cR}. (Constants independent of ε, M, β will be denoted by c.) Furthermore, $N(\gamma)$ was estimated in terms of $H(\gamma)$ by the inequality

$$N(\gamma) \leqq c(\log M)^2 H(\gamma). \tag{5.4}$$

Hence the number of contours spanning 0 with $H \leqq E$ is less than $e^{c(\log M)^2 E}$. Thus

$$\sum_{\gamma: i \in I(\gamma), j \in I(\gamma)} e^{-\varepsilon\beta H(\gamma)} \leqq \sup_{\gamma: i \in I(\gamma), j \in I(\gamma)} e^{-\varepsilon\beta H(\gamma)/2}. \tag{5.5}$$

We need at least one interval of length 2^n to cover γ for $0 \leqq n \leqq [\log_2(1+|i-j|)]+1$. By (5.3) and (5.4) we have

$$H(\gamma) \geqq \frac{[\log_2(1+|i-j|)]+1}{c(\log M)^2}, \tag{5.6}$$

and thus

$$\begin{aligned} e^{-\varepsilon\beta H(\gamma)/2} &\leqq (1+|i-j|)^{-\varepsilon\beta/(4c(\log M)^2)} e^{-\varepsilon\beta} \\ &\leqq C(i,j)^2 e^{-\varepsilon\beta}. \end{aligned} \tag{5.7}$$

We have taken β sufficiently large, and used $H(\gamma) > 4$ for non-empty γ. This completes the proof. □

In the next estimate, we use a modified function of clusters

$$\rho_1(Y) = \prod_{l=2}^{k} [|i_{l-1} - j_{l-1}|^{-2} s_{\eta(l)} \dots s_{l-2}] \prod_{l=1}^{k} \prod_{\alpha} e^{-3\varepsilon\beta H(\gamma_{\alpha l})} \tag{5.8}$$

related to $\rho(Y)$ by taking absolute value and throwing out some factors we wish to discuss later. Define $L(\Gamma) = \sum_\alpha L(\gamma_\alpha)$ for $\Gamma = \bigcup_\alpha \gamma_\alpha$, where $L(\gamma)$ is the logarithmic length of γ defined in [10] as

$$L(\gamma) = \sum_k \{[\log_2(i_{k-1} - i_k)] + 1\}, \tag{5.9}$$

where $i_1, i_2, \dots$ are the coordinates in $\mathbb{Z}^*$ of the spin flips in γ.

Lemma 5.2. *Let $M > M_0(\varepsilon)$ and $\beta > \beta_0(M)$, and consider the following restricted sum over clusters. Let k be fixed and let three sites i_0, i, j and two integers $p_1, p_2 \in \{1, \dots, k\}$ be given. Consider only clusters Y with k contours such that $i_0 \in I(\Gamma_1(Y))$, $i \in I(\Gamma_{p_1}(Y)) \cup \{i_{p_1-1}, j_{p_1-1}\}$, and $j \in I(\Gamma_{p_2}(Y)) \cup \{i_{p_2-1}, j_{p_2-1}\}$. Fix in addition the backbone of $\eta(Y)$: This is the sequence B of integers $1 = b_1 < b_2 < \dots < b_r = \max\{p_1, p_2\}$, $p_1, p_2 \in B$, obtained by repeated application of η to p_1 or p_2. A tree η_B is obtained from η by restriction. It contains only the lines $(l, \eta(l))$ that are essential in connecting p_1 or p_2 to 1. Define for $x \leqq y \leqq z$,*

$$C(x,y,z) = C(x,y)C(y,z) \tag{5.10}$$

and let $C(x,y,z)$ be invariant under permutations of x, y, z. With the above restrictions,

$$\sum_Y \rho_1(Y) \leqq C(i_0, i, j). \tag{5.11}$$

Proof. Let $c_1 < c_2 < \ldots < c_t$ be the integers in $\{1,\ldots,k\}\backslash B$. We represent the sum over Y as

$$\sum_{\Gamma_1} \sum_{i_{b_2-1}<j_{b_2-1}} \sum_{\Gamma_{b_2}} \int ds_{b_2-1} \sum_{i_{b_3-1}<j_{b_3-1}} \sum_{\Gamma_{b_3}} \int s_{b_3-1} \cdots \sum_{i_{b_r-1}<j_{b_r-1}} \sum_{\Gamma_{b_r}} \int ds_{b_r-1}$$
$$\cdot \sum_{\eta(c_1)} \sum_{i_{c_1-1}<j_{c_1-1}} \sum_{\Gamma_{c_1}} \int ds_{c_1-1} \cdots \sum_{\eta(c_t)} \sum_{i_{c_t-1}<j_{c_t-1}} \sum_{\Gamma_{c_t}} \int ds_{c_t-1}, \tag{5.12}$$

with each summation variable compatible with the ones to its left and with the restrictions in the lemma. Sums are converted into supremums from left to right using appropriately chosen combinatoric coefficients. We use the identity

$$\sum_T f(T) \leqq \sup_T C_T f(T), \tag{5.13}$$

valid for $f(T)$, $C_T \geqq 0$ when $\sum_T C_T^{-1} \leqq 1$.

For controlling the sum over $\Gamma_1 = \gamma_1$ we put

$$C_{\Gamma_1}^{-1} = e^{-\varepsilon\beta H(\gamma_1)}. \tag{5.14}$$

Since $i_0 \in I(\Gamma_1)$, Lemma 5.1 bounds $\sum_{\Gamma_1} C_{\Gamma_1}^{-1}$ by 1.

Before going to the other sums in (5.12) we derive some estimates on sums involving $C(x,y)$ and $C(x,y,z)$. We need the bound

$$\sum_y C(x,y)C(y,z,w) \leqq cC(x,z,w). \tag{5.15}$$

Let us prove first that

$$\sum_x C(0,x)C(x,y) \leqq cC(0,y). \tag{5.16}$$

Assume $y \geqq 0$ and split the sum over x into two parts, $x > y/2$ and $x \leqq y/2$. On the second part $C(x,y) \leqq \min\{1, 4/y^2\}$, so

$$\begin{aligned}\sum_x C(0,x)C(x,y) &\leqq 2\sum_x C(0,x)\min\{1,4/y^2\}\\ &\leqq cC(0,y).\end{aligned} \tag{5.17}$$

By an easy induction, we obtain

$$\sum_{x_1,\ldots,x_k} C(0,x_1)C(x_1,x_2)\ldots C(x_k,y) \leqq c^k C(0,y). \tag{5.18}$$

Returning to (5.15), suppose $x < z < w$. We have

$$\begin{aligned}\sum_{y \leqq z} C(x,y)C(y,z,w) &\leqq \sum_{y} C(x,y)C(y,z)C(z,w) \\ &\leqq cC(x,z)C(z,w) \\ &\leqq cC(x,z,w). \end{aligned} \tag{5.19}$$

The same argument works for the regions $z < y < w$ and $w \leqq y$, and also for other orderings of x, z, w. Thus (5.15) is proven.

We control each group of summations $\sum\limits_{\eta(l)} \sum\limits_{i_{l-1} < j_{l-1}} \sum\limits_{\Gamma_l} \int ds_{l-1}$ together, writing $T_l = (\eta(l), i_{l-1}, j_{l-1}, \Gamma_l, s_{l-1})$. The combinatoric coefficient is given by

$$C_{T_l}^{-1} = e^{(s_{l-1}-1)L_{l-1}} s_{\eta(l)} \dots s_{l-2} \prod_{\alpha} e^{-\varepsilon\beta H(\gamma_{\alpha l})} C(l_{l-1}, j_{l-1}) \tag{5.20}$$

for $l = c_1, \dots, c_t$, where

$$L_{l-1} = \sum_{\alpha=1}^{l-1} s_\alpha \dots s_{l-2} L(\Gamma_\alpha).$$

Just one of i_{l-1}, j_{l-1} is odd with respect to $\Gamma_{\eta(l)}$; call it i_{l-1}^- and call the other one i_{l-1}^+. At least one of $i_{\eta(l)-1}, j_{\eta(l)-1}$ must be odd with respect to an irreducible contour $\gamma_{\alpha\eta(l)}$ that is odd with respect to i_{l-1}^-. (See compatibility conditions (iii) and (iv), Sect. 3.) Choose the smaller one, say, and call it $\hat{\imath}(i_{l-1}^-)$. When $\eta(l) = 1$ we define $i_{\eta(l)-1}^- = i_{\eta(l)-1}^+ = \hat{\imath}(i_{l-1}^-) = i_0$. Let b_p be the element of B at which η_B branches—b_p is the only element of B which is the image of two b's. If no such element exists, put $b_p = \min\{p_1, p_2\}$. Larger b's connecting b_p to p_1 are denoted $b_\alpha^{(1)}$ and those connecting b_p to p_2 are denoted $b_\alpha^{(2)}$. For the other groups T_l we put

$$C_{T_l}^{-1} = \begin{cases} \dfrac{C(\hat{\imath}(i_{l-1}^-), i_{l-1}^-)C(i_{l-1}^-, i_{l-1}^+)(C(i_{l-1}^-, i, j) + C(i_{l-1}^+, i, j))}{C(i_{\eta(l)-1}^-, i, j) + C(i_{\eta(l)-1}^+, i, j)} \prod_\alpha e^{-\varepsilon\beta H(\gamma_{\alpha l})}, \\ \hfill l = b_2, b_3, \dots, b_{p-1}, \\ \dfrac{C(\hat{\imath}(i_{l-1}^-), i_{l-1}^-)C(i_{l-1}^-, i_{l-1}^+)(C(i_{l-1}^-, i) + C(i_{l-1}^+, i))}{C(i_{\eta(l)-1}^-, i) + C(i_{\eta(l)-1}^+, i)} \prod_\alpha e^{-\varepsilon\beta H(\gamma_{\alpha l})}, \\ \hfill l = p_1 \neq b_p \text{ or } l = b_\alpha^{(1)} \text{ for some } \alpha, \\ \dfrac{C(\hat{\imath}(i_{l-1}^-), i_{l-1}^-)C(i_{l-1}^-, i_{l-1}^+)(C(i_{l-1}^-, j) + C(i_{l-1}^+, j))}{C(i_{\eta(l)-1}^-, j) + C(i_{\eta(l)-1}^+, j)} \prod_\alpha e^{-\varepsilon\beta H(\gamma_{\alpha l})} \\ \hfill l = p_2 \neq b_p \text{ or } l = b_\alpha^{(2)} \text{ for some } \alpha, \\ \dfrac{C(\hat{\imath}(i_{l-1}^-), i_{l-1}^-)C(i_{l-1}^-, i_{l-1}^+)(C(i_{l-1}^-, i) + C(i_{l-1}^+, i))(C(i_{l-1}^-, j) + C(i_{l-1}^+, j))}{C(i_{\eta(l)-1}^-, i, j) + C(i_{\eta(l)-1}^+, i, j)} \\ \quad \cdot \prod_\alpha e^{-\varepsilon\beta H(\gamma_{\alpha l})}, \hfill l = b_p. \end{cases} \tag{5.21}$$

Each C_{T_l} depends only on variables from earlier sums, which are fixed at the point of converting $\sum\limits_{T_l}$ to $\sup\limits_{T_l}$.

We bound $\sum\limits_{T_l} C_{T_l}^{-1}$ for $l=c_1,\ldots,c_t$. With $\eta(l)$ fixed, sum over $i_{l-1}<j_{l-1}$. One of these (i_{l-1}^-) lies in some elementary interval I of length l_I of some contour $\gamma_{\alpha\eta(l)}$, while the other (i_{l-1}^+) lies outside this interval. Changing the coordinate on $\mathbb{Z}$ appropriately, we have

$$\begin{aligned}\sum_{i_{l-1}^+\leqq 0<i_{l-1}^-\leqq l_I} |i_{l-1}^+-i_{l-1}^-|^{-2} &\leqq c \sum_{0<i_{l-1}^-\leq l_I} |i_{l-1}^-|^{-1}\\ &\leqq c\{[\log_2 l_I]+1\}\end{aligned}$$

and so using (5.9),

$$\begin{aligned}\sum_{i_{l-1}<j_{l-1}} C(i_{l-1},j_{l-1}) &\leqq \sum_{\alpha,\text{ intervals } I} c\{[\log_2 l_I]+1\}\\ &\leqq cL(\Gamma_{\eta(l)}).\end{aligned} \tag{5.22}$$

The sum over Γ_l surrounding i_{l-1} or j_{l-1} is bounded using Lemma 5.1. If there are m irreducible contours in Γ_l then

$$\sum_{\Gamma_l}\prod_{\alpha=1}^{m} e^{-\varepsilon\beta H(\gamma_{\alpha l})}\leqq \sum_{m=1}^{\infty}(2e^{-\varepsilon\beta})^m\leqq e^{-\varepsilon\beta/2}. \tag{5.23}$$

Combining (5.22) and (5.23) yields the estimate

$$\begin{aligned}\sum_{T_l} C_{T_l}^{-1} &\leqq \sum_{\eta(l)=1}^{l-1}\int ds_{l-1}e^{(s_{l-1}-1)L_{l-1}}s_{\eta(l)}\ldots s_{l-2}L(\Gamma_{\eta(l)})\\ &=\int ds_{l-1}e^{(s_{l-1}-1)L_{l-1}}L_{l-1}\\ &=1-e^{-L_{l-1}}\leqq 1.\end{aligned}$$

Consider now $l=b_2,\ldots,b_{p-1}$. We use (5.15) and (5.16) to obtain

$$\begin{aligned}\sum_{i_{l-1}^-,\,i_{l-1}^+} & C(\hat{\imath}(i_{l-1}^-),i_{l-1}^-)C(i_{l-1}^-,i_{l-1}^+)\,(C(i_{l-1}^-,i,j)+C(i_{l-1}^+,i,j))\\ &\leqq cC(\hat{\imath}(i_{l-1}^-),i,j).\end{aligned}$$

This is one of the terms in the denominator of $C_{T_l}^{-1}$. The sum over Γ_l is controlled as before, and the integral over s_{l-1} is bounded by the supremum over s_{l-1}. Thus we obtain $\sum\limits_{T_l} C_{T_l}^{-1}\leqq 1$ for these l. The estimate is similar for the next two classes of l's in (5.21). The case $l=b_p$ is also similar if we use inequalities like $C(i_{l-1}^-,i)\,C(i_{l-1}^-,j)\leqq C(i_{l-1}^-,i,j)$.

To bound the full sum we need only collect the coefficients C_{T_l}:

$$\sum_Y \rho_1(Y)\leqq \sup_Y C_{\Gamma_1}C_{T_2}\ldots C_{T_k}\rho_1(Y).$$

The factors $e^{\varepsilon\beta H(\gamma_{\alpha l})}$ in the C's cancel against factors in $\rho_1(Y)$, as do factors $(s_{\eta(l)}\dots s_{l-2})^{-1}$ and $C(i_{l-1}^-, i_{l-1}^+)^{-1}$. Note that $C(\hat{\imath}(i_{l-1}^-), i_{l-1}^-)^{-1}\prod_\alpha e^{-\varepsilon\beta H(\gamma_{\alpha(l-1)})/2} \leqq 1$ by (5.2), since some $\gamma_{\alpha(l-1)}$ contains both $\hat{\imath}(i_{l-1}^-)$ and i_{l-1}^-. The factor $(C(i_{\eta(l)-1}^-, i, j) + C(i_{\eta(l)-1}^+, i, j)$ in $C_{T_{b_\alpha}}$ cancels against the same term in the denominator of $C_{T_{b_{\alpha-1}}}$; likewise for terms with $l = b_\alpha^{(1)}$ or $b_\alpha^{(2)}$. There are two factors in the denominator of $C_{T_{b_p}}$ to cancel against the factors coming from any legs running into b_p. There remains the term $C(i_0, i, j)$ from $l = b_2$ and uncanceled terms at $l = p_1$ or p_2. The bound now takes the form

$$\begin{aligned}\sum_Y \rho_1(Y) \leqq \sup_Y \prod_{l=2}^{k} e^{(1-s_{l-1})L_{l-1}} \prod_{l=1}^{k}\prod_\alpha e^{-\varepsilon\beta H(\gamma_{\alpha l})} C(i_0, i, j) \\ \cdot (C(i_{p_1-1}^-, i) + C(i_{p_1-1}^+, i))^{-1}(C(i_{p_2-1}^-, j) + C(i_{p_2-1}^+, j))^{-1} \\ \cdot \prod_{\mu=1,2}\prod_\alpha e^{-\varepsilon\beta H(\gamma_{\alpha p_\mu})1/2}\end{aligned}$$

Adding the extra factors $e^{(1-s_{l-1})L_{l-1}}$ only makes the inequality stronger. If $i \in \{i_{p_1-1}, j_{p_1-1}\}$ then $(C(i_{p_1-1}^-, i) + C(i_{p_1-1}^+, i))^{-1} \leqq 1$. Otherwise some $\gamma_{\alpha p_1}$ contains i and at least one of i_{p_1-1}, j_{p_1-1} and we can use $(C(i_{p_1-1}^-, i) + C(i_{p_1-1}^+, i))^{-1} e^{-\varepsilon\beta H(\gamma_{\alpha p_1})/2} \leqq 1$. Similar remarks hold for the factors at p_2.

The s-dependent factor is handled by noting the cancellations between terms:

$$\begin{aligned}\prod_{l=2}^{k} e^{(1-s_{l-1})L_{l-1}} &= \exp\left(\sum_{l=2}^{k}\sum_{\alpha=1}^{l-1}(s_\alpha\dots s_{l-2} - s_\alpha\dots s_{l-1})L(\Gamma_\alpha)\right) \\ &= \exp\left(\sum_{\alpha=1}^{k-1}(1 - s_\alpha\dots s_{k-1})L(\Gamma_\alpha)\right) \\ &\leqq \prod_{l=1}^{k}\prod_\alpha e^{\varepsilon\beta H(\gamma_{\alpha l})}.\end{aligned}$$

The last inequality follows because $L(\gamma)$ is less than $N(\gamma)$ (see [10]) and hence by (5.4),

$$L(\Gamma_l) \leqq \sum_\alpha c(\log M)^2 H(\gamma_{\alpha l}). \tag{5.24}$$

This completes the proof of (5.11). □

The next proposition controls the remaining parts of the sum over all clusters containing three given points. Define

$$\rho_2(Y) = \rho_1(Y)(4\beta)^{k-1} \prod_{l=1}^{k}\prod_\alpha e^{-\varepsilon\beta H(\gamma_{\alpha l})}. \tag{5.25}$$

Proposition 5.3. *For $M > M_0(\varepsilon)$, $\beta > \beta_0(M)$, let Y vary over all clusters such that $i_0 \in I(\Gamma_1(Y))$ and such that i and j are in $\bigcup_l (I(\Gamma_l(Y)) \cup \{i_{l-1}, j_{l-1}\})$. Then*

$$\sum_Y \rho_2(Y) \leqq C(i_0, i, j)e^{-2\varepsilon\beta}. \tag{5.26}$$

Proof. There are k^2 choices of p_1, p_2 such that $i \in \Gamma_{p_1} \cup \{i_{p_1-1}, j_{p_1-1}\}, j \in \Gamma_{p_2} \cup \{i_{p_2-1}, j_{p_2-1}\}$. There are less than 2^k possible backbone subsets B of $\{1, \ldots, k\}$ and less than k choices for b_p. There are less than 2^k assignments of the $b > b_p$ to the two classes $\{b_\alpha^{(1)}\}, \{b_\alpha^{(2)}\}$, which then completely specifies the backbone of η as described in Lemma 5.2. The lemma now yields

$$\sum_Y \rho_2(Y) \leqq \sum_{k=1}^{\infty} k^3 2^{2k} (4\beta)^{k-1} e^{-4\varepsilon\beta k} C(i_0, i, j)$$

to complete the proof. □

An immediate corollary is obtained by summing over i_0. Since $\sum_{i_0} C(i_0, i, j) \leqq cC(i, j)$ we have

$$\sum_{Y: i \in Y, j \in Y} \rho_2(Y) \leqq C(i, j) e^{-\varepsilon\beta}, \tag{5.27}$$

$$\sum_{Y: i \in Y} \rho_2(Y) \leqq e^{-\varepsilon\beta}, \tag{5.28}$$

where $i \in Y$ means $i \in \Gamma_1(Y) \cup \bigcup_{l=2}^{k} (\Gamma_l(Y) \cup \{i_{l-1}, j_{l-1}\})$.

We now proceed with the second stage of the estimation of the expansion (4.8), namely the control of the "external sum" over collections of clusters.

Define for $R = 3, 4, \ldots$

$$\rho_R(Y) = \rho_2(Y) \prod_{l=1}^{k} \prod_{\alpha} e^{-(R-2)\varepsilon\beta H(\gamma_{\alpha l})}. \tag{5.29}$$

Lemma 5.4. *For $M > M_0(\varepsilon), \beta > \beta_0(M)$,*

$$\sum_{Y_2} |A(Y_1, Y_2)| \rho_3(Y_2) \leqq e^{-\varepsilon\beta} L(Y_1), \tag{5.30}$$

where $L(Y) = \sum_{l=1}^{k} L(\Gamma_l)$, and $A = U - 1$ with U given in (4.5).

Proof. There are two ways for A to be nonzero. A contour γ of Y_2 could come within $Md(\gamma)^{3/2}$ of a spin flip or a vertex i_{l-1} or j_{l-1} of Y_1. Secondly, a vertex pair (i^-, i^+) of Y_2 could satisfy $\chi_\gamma(i^-, i^+) = 1$ for γ a contour in Y_1.

In the first case, summation over which spin flip or vertex is bounded by $3L(Y_1)$. Suppose $2^m \leqq d(\gamma) < 2^{m+1}$. Then there are less than $4M2^{3m/2}$ choices of $i \in I(\gamma)$ that we need consider. Furthermore, by (5.7)

$$e^{-\varepsilon\beta H(\gamma)} \leqq (2^m)^{-\varepsilon\beta/(c(\log M)^2)} e^{-\varepsilon\beta} \leqq 2^{-2m} e^{-\varepsilon\beta}. \tag{5.31}$$

Thus using (5.28) we obtain

$$\sum_{Y_2 \text{ in case } 1} |A(Y_1, Y_2)| \rho_3(Y_2) \leqq 3L(Y_1) \sum_{m=0}^{\infty} \sum_i 2^{-2m} e^{-\varepsilon\beta} \sum_{Y_2 \text{ containing } i} \rho_2(Y_2)$$

$$\leq 3L(Y_1) \sum_{m=0}^{\infty} 4M 2^{-m/2} 2^{-2\varepsilon\beta}$$
$$\leq \tfrac{1}{2} e^{-\varepsilon\beta} L(Y_1). \tag{5.32}$$

In the second case we know that the vertex pair (i^-, i^+), and hence Y_2, has one point in an elementary interval $I = [1, l_I]$ of γ and one point outside I. If we sum over all Y_2's compatible with a choice of points (i^-, i^+) using (5.27), we obtain

$$\sum_{Y_2 \text{ in case } 2} |A(Y_1, Y_2)| \rho_3(Y_2) \leq \sum_{\alpha, l} \sum_{\text{intervals} I} \sum_{i^+ < 0 \leq i^- \leq l_I} 2C(i^-, i^+) e^{-2\varepsilon\beta}$$
$$\leq \tfrac{1}{2} e^{-\varepsilon\beta} L(Y_1). \tag{5.33}$$

The i^-, i^+ sums have been estimated as in (5.23), (5.24). The lemma follows from (5.32) and (5.33). □

Define the following function of clusters Z_r, Y_s:

$$\Phi(Z_1, \ldots, Z_j; Y_1, \ldots, Y_k) = \sum_{G_c} \sum_{\mathscr{L} \in G_c} A(\mathscr{L}) \prod_{s=1}^{k} \rho_4(Y_s), \tag{5.34}$$

which appears as part of expressions such as (4.8). Here Z_r is either an X or a Y. G_c is summed over graphs composed of lines $\{Z_r, Y_s\}$ or $\{Y_{s_1}, Y_{s_2}\}$ connected with respect to $\{Z_r\}$ (that is, each Y_s is connected directly or indirectly to some Z_r). We put $L(X) = \sum_{Y \text{ in } X} L(Y) + |A|$, where $|A|$ is the number of sites in A, $X = (A, \{Y_s\})$. We say $x \in X$ if $x \in Y$ for some Y in X or if $x \in A$.

Proposition 5.5. *For $M > M_0(\varepsilon), \beta > \beta_0(M)$,*

$$\sum_{(Y_1, \ldots, Y_k)\ x \in \bigcup_{s=1}^{k} Y_s} |\Phi(Z_1, \ldots, Z_j; Y_1, \ldots, Y_k)|$$
$$\leq k! \exp\left(\sum_{r=1}^{j} L(Z_r) \right) e^{-\varepsilon\beta k} \sum_{r=1}^{j} \sum_{y \in Z_r} C(y, x), \tag{5.35}$$

$$\sum_{(Y_1, \ldots, Y_k)} |\Phi(Z_1, \ldots, Z_j; Y_1, \ldots, Y_k)| \leq k! \exp\left(\sum_{r=1}^{j} L(Z_r) \right) e^{-\varepsilon\beta k}. \tag{5.36}$$

Proof. When $k = 0$ we define $\Phi(Z_1, \ldots, Z_j; \emptyset) = 1$ and the proposition holds. For general k we proceed by induction. The following Kirkwood-Salzburg type equation will be used:

$$\Phi(Z_1, \ldots, Z_j; Y_1, \ldots, Y_k) = \sum_{\Omega} \prod_{s \in \Omega} A(Z_1, Y_s) \prod_{r=2}^{j} \prod_{s \in \Omega} U(Z_r, Y_s)$$
$$\prod_{s_1 < s_2, s_1, s_2 \in \Omega} U(Y_{s_1}, Y_{s_2}) \prod_{s \in \Omega} \rho_4(Y_s) \Phi(Z_2, \ldots, Z_j, (Y_s)_{s \in \Omega}; (Y_s)_{s \notin \Omega}).$$

Here Ω is summed over subsets of $\{1, \ldots, k\}$. See [14] or [3] for a derivation. Inserting this into (5.35) and taking absolute values we can drop the U- and A-factors (which are bounded by 1) and just enforce the condition that $A(Z_1, Y_s) \neq 0$

for $s \in \Omega$. This yields

$$\sum_{(Y_1,\ldots,Y_k):x\in \bigcup_{s=1}^{k} Y_s} |\Phi(Z_1,\ldots,Z_j;Y_1,\ldots,Y_k)|$$

$$\leqq \sum_{\Omega} \sum_{\substack{(Y_1,\ldots,Y_k):x\in \bigcup_{s=1}^{k} Y_s \\ A(Y_s,Z_1)=0 \text{ for } s\in\Omega}} \prod_{s\in\Omega} \rho_4(Y_s)|\Phi(Z_2,\ldots,Z_j,(Y_s)_{s\in\Omega};(Y_s)_{s\notin\Omega})|$$

$$\leqq \sum_{|\Omega|=0}^{k} \binom{k}{|\Omega|} \sum_{\substack{(Y_1',\ldots,Y_{|\Omega|}') \\ A(Y_s',Z_1)\neq 0}} \prod_{s=1}^{|\Omega|} \rho_4(Y_s')\exp\left(\sum_{r=2}^{j} L(Z_r) + \sum_{s=1}^{|\Omega|} L(Y_s')\right)(k-|\Omega|)!$$

$$\cdot e^{-\varepsilon\beta(k-|\Omega|)}\left(\sum_{r=2}^{j}\sum_{y\in Z_r} C(y,x) + \sum_{s=1}^{|\Omega|}\sum_{y\in Y_s'} C(y,x)\right)$$

$$+\sum_{|\Omega|=1}^{k} \binom{k}{|\Omega|}^{|\Omega|} \sum_{\substack{Y_1':x\in Y_1' \\ A(Y_1',Z_1)\neq 0}} \sum_{\substack{(Y_2',\ldots,Y_{|\Omega|}') \\ A(Y_s',Z_1)\neq 0}} \prod_{s=1}^{|\Omega|} \rho_4(Y_s')(k-|\Omega|)!$$

$$\cdot\exp\left(\sum_{r=2}^{j} L(Z_r) + \sum_{s=1}^{|\Omega|} L(Y_s')\right)e^{-\varepsilon\beta(k-|\Omega|)}. \tag{5.37}$$

We have applied the induction hypothesis, and considered the cases $x\in Y_s$, $s\in\Omega$ and $x\in Y_s$, $s\notin\Omega$ separately. Replacing ρ_4 with ρ_3, we pick up factors $\prod_l \prod_\alpha e^{-\varepsilon\beta H(\gamma_{\alpha l})}$ for each Y_s'. By (5.24), these dominate the $\exp\left(\sum_{s=1}^{|\Omega|} L(Y_s')\right)$ factors and the $e^{\varepsilon\beta|\Omega|}$ factors. We can also insert a factor $e^{-\varepsilon\beta|\Omega|}C(w,y_0)$ for some $w(Y_s')\in Z_1$, and some $y_0(Y_s')\in Y_s'$. To see this, consider three cases: A vertex of Y_s' lies within a contour of Z_1; some $\gamma_{\alpha l}$ of Y_s' lies within $Md(\gamma_{\alpha l})^{3/2}$ of a contour of Z_1; or a point of the set A of Z_1 lies in Y_s'. In the first and third cases we can take $w=y_0$, in the second we use (5.7) to show

$$e^{-\varepsilon\beta H(\gamma_{\alpha l})/2} \leqq (Md(\gamma_{\alpha l})^{3/2})^{-2} \leqq C(w,y_0). \tag{5.38}$$

Now (5.37) is bounded by

$$k!\sum_{|\Omega|=0}^{k} \frac{1}{|\Omega|!} L(Z_1)^{|\Omega|}\exp\left(\sum_{r=2}^{j} L(Z_r)\right)e^{-\varepsilon\beta k}\left(\sum_{r=2}^{j}\sum_{y\in Z_r} C(y,x)\right.$$

$$\left.+\, e^{-\varepsilon\beta|\Omega|}\sum_{s=1}^{|\Omega|}\sum_{w\in Z_1,y_0,y} c\, C(w,y_0)C(y_0,y)C(y,x)\right)$$

$$+\,k!\sum_{|\Omega|=1}^{k} \frac{1}{(|\Omega|-1)!} L(Z_1)^{|\Omega|-1}\exp\left(\sum_{r=2}^{j} L(Z_r)\right)e^{-\varepsilon\beta(k+|\Omega|)}$$

$$\cdot \sum_{w\in Z_1,y_0} cC(w,y_0)C(y_0,x)$$

$$\leqq k!\exp\left(\sum_{r=1}^{j} L(Z_r)\right)e^{-\varepsilon\beta k}\sum_{r=1}^{j}\sum_{y\in Z_r} C(y,x). \tag{5.39}$$

We have applied Lemma 5.4 to some sums over Y'_s and (5.27) to others. When $Z_1=(A,\{Y_u\})$ the lemma yields a $L(Y_u)$ for each case $A(Y'_s,Y_u)\neq 0$ and (5.28) yields an additional $|A|$ for the remaining cases, or $L(Z_1)$ in all. In each term $s=1,\dots,|\Omega|$ we sum over Y'_s compatible with w, y_0, y (yielding the factor $C(y_0,y)$) and then sum over y_0, y. By (5.18) we obtain the desired decay $cC(w,x)$. This proves (5.35). To obtain (5.36) we can drop the second term in (5.37), where Y'_1 is required to contain x. The factor $\sum C(y,x)$ is also dropped. The sums over Y'_s are handled as in (5.39). □

We now proceed to control the full expansion

$$\langle F_1;F_2\rangle_L=\sum_{\{X_\alpha\}}\prod_{\alpha}F_\alpha(X_\alpha)\sum_{(Y_1,\dots,Y_j)}\frac{1}{j!}\sum_{G_c}\prod_{\mathscr{L}\in G_c}A(\mathscr{L})\prod_{\alpha}\rho(X_\alpha)\prod_{r=1}^{j}\rho(Y_r), \tag{5.40}$$

where G_c is connected with respect to all Y's and X's. Consider first terms where $\{X_\alpha\}=\{X_1,X_2\}$ and G_c does not contain the line $\{X_1,X_2\}$. G_c breaks into three parts, $G_c=G_{c_1}\cup G_i\cup G_{c_2}$. G_{c_1} contains lines $\{X_1,Y_r\}$ and $\{Y_{r_1},Y_{r_2}\}$ that are connected to X_1 without passing through X_2. G_{c_2} contains lines $\{X_2,Y_r\}$ and $\{Y_{r_1},Y_{r_2}\}$ with Y_r, Y_{r_1}, Y_{r_2} not part of G_{c_1}. G_i contains the remaining lines $\{X_2,Y_r\}$ with Y_r part of G_{c_1}. G_{c_1}, G_{c_2} are arbitrary graphs connected with respect to X_1, X_2, respectively. G_i is arbitrary, expect that it cannot be empty because then G_c would be disconnected. Let us fix $\{X_\alpha\}$ in (5.40) and replace ρ with ρ_4. Put $\bar{X}_2=\{x{:}x\in X_2$ or $\mathrm{dist}(x,\gamma)\leqq Md(\gamma)^{3/2}$ for some γ in $X_2\}$. Then

$$\begin{aligned}
&\sum_{(Y_1,\dots,Y_j)}\frac{1}{j!}\left|\sum_{G_c}\prod_{\mathscr{L}\in G_c}A(\mathscr{L})\prod_{r=1}^{j}\rho_4(Y_r)\right|\\
&\leqq\left(\sum_{(Y'_1,\dots,Y'_{j_1})}\frac{1}{j_1!}\left|\sum_{G_{c_1}}\prod_{\mathscr{L}\in G_{c_1}}A(\mathscr{L})\prod_{r=1}^{j_1}\rho_4(Y'_r)\sum_{G_i}\prod_{\mathscr{L}\in G_i}A(\mathscr{L})\right|\right)\\
&\quad\cdot\left(\sum_{(Y''_1,\dots,Y''_{j_2})}\frac{1}{j_2!}\left|\sum_{G_{c_2}}\prod_{\mathscr{L}\in G_{c_2}}A(\mathscr{L})\prod_{r=1}^{j_2}\rho_4(Y''_r)\right|\right)\\
&\leqq\left(\sum_{x\in\bar{X}_2}\sum_{(Y'_1,\dots,Y'_{j_1}):x\in\bigcup_{r=1}^{j_1}Y'_r}\frac{1}{j_1!}2^{j_1}|\Phi(X_1;Y'_1,\dots,Y'_{j_1})|\right)\\
&\quad\cdot\left(\sum_{(Y''_1,\dots,Y''_{j_2})}\frac{1}{j_2!}|\Phi(X_2;Y''_1,\dots,Y''_{j_2})|\right).
\end{aligned} \tag{5.41}$$

The restriction on $(Y'_1,\dots,Y'_{j_1})$ comes from the nonemptiness of G_i. There are less than 2^{j_1} choices of G_i. Proposition 5.5 now bounds (5.41) by

$$2e^{L(X_1)}e^{L(X_2)}\sum_{x\in\bar{X}_2}\sum_{y\in X_1}C(y,x). \tag{5.42}$$

When G_c contains a line $\{X_1,X_2\}$ we obtain a bound $2e^{L(X_1)}e^{L(X_2)}A(X_1,X_2)$ and when $\{X_\alpha\}=\{X_{12}\}$ we obtain a bound $2e^{L(X_{12})}$.

Finally we sum over $\{X_\alpha\}$. Defining $\rho_4(X_\alpha)$ as in (4.4), replacing ρ with ρ_4, we have

$$\sum_{\{X_\alpha\}} \sum_{(Y_1,\dots,Y_j)} \frac{1}{j!} \left| \sum_{G_c} \prod_{\mathscr{L}\in G_c} A(\mathscr{L}) \prod_{r=1}^{j} \rho_4(Y_r) \right| \prod_\alpha \rho_4(X_\alpha)$$
$$\leqq \sum_{x,y} \sum_{X_1 \ni y} \sum_{X_2 : x \in \bar{X}_2} 2e^{L(X_1)} e^{L(X_2)} C(y,x) \rho_4(X_1) \rho_4(X_2)$$
$$+ \sum_x \sum_{X_1 \ni x} \sum_{X_2 : x \in \bar{X}_2} 2e^{L(X_1)} e^{L(X_2)} \rho_4(X_1) \rho_4(X_2)$$
$$+ \sum_{X_{12}} 2e^{L(X_{12})} \rho_4(X_{12}). \tag{5.43}$$

We can replace $L(X_\alpha)$ with $|A_\alpha|$ and 2 with $e^{-\varepsilon\beta}$ if ρ_4 is replaced with ρ_3. A factor $C(x,z)$, $z \in X_2$ can be procured at the same time. The summation over a cluster in X_α containing a point w is bounded by $\sum_{a\in A_\alpha} C(a,w)$ by (5.27). The summation over the others is bounded by $\prod_{\alpha\in A_\alpha} \left(\sum_{n=0}^{\infty} e^{-\varepsilon\beta n} \right) \leqq e^{|A_\alpha|}$ by (5.28). Here n is the number of clusters in X_α containing a. Thus (5.43) is bounded by

$$\sum_{a_1\in A_1} \sum_{a_2 \in A_2} e^{-\varepsilon\beta} e^{2(|A_1|+|A_2|)} \Bigg(\sum_{x,y,z} C(a_1,y) C(y,x) C(x,z) C(z,a_2)$$
$$+ \sum_{x,z} C(a_1,x) C(x,z) C(z,a_2) + C(a_1,a_2) \Bigg)$$
$$\leqq e^{c(|A_1|+|A_2|-2)} (\operatorname{dist}(A_1,A_2))^{-2} e^{-\varepsilon\beta/2}. \tag{5.44}$$

Similar but easier estimates prove the convergence of the comparison series for untruncated functions. In that case one should allow for the possibility of no clusters in $\{X_\alpha\}$, and we obtain an overall bound $e^{c(|A_1|+|A_2|)}$.

Comparison of (5.43)–(5.44) with (5.40) now yields

$$|\langle F_1; F_2 \rangle_L| \leqq e^{c(|A_1|+|A_2|-2)} (\operatorname{dist}(A_1,A_2))^{-2} e^{-\varepsilon\beta/2} \sup_{\substack{\{X_\alpha\},(Y_1,\dots,Y_j) \\ \text{compatible with some } G_c}}$$
$$\cdot \prod_\alpha \left| F_\alpha(X_\alpha) \frac{\rho(X_\alpha)}{\rho_4(X_\alpha)} \right| \prod_{r=1}^{j} \left| \frac{\rho(Y_r)}{\rho_4(Y_r)} \right|. \tag{5.45}$$

Checking back through (3.24), (5.8), (5.25) and (5.29) yields

$$|\rho(Y)/\rho_4(Y)| \leqq e^{-\beta H_{s_1 \dots s_{k-1}}(\Gamma_1,\dots,\Gamma_{k-1};\Gamma_k)} \prod_{l=1}^{k} \prod_\alpha e^{6\varepsilon\beta H(\gamma_{\alpha l})}$$
$$\leqq \prod_{l=1}^{k} \prod_a e^{-\beta(1-7\varepsilon) H(\gamma_{al})} \leqq 1. \tag{5.46}$$

In the second step Proposition 3.1 has been applied, and M has been chosen sufficiently large, depending on ε.

Taking F_i to be a product of spins σ, (5.45) proves the bound of Theorem 1.2:

$$\left|\left\langle \prod_{i\in A_1} \sigma_1 ; \prod_{i\in A_2} \sigma_i \right\rangle_L \right| \leqq e^{c(|A_1|+|A_2|)}(\mathrm{dist}(A_1,A_2))^{-2} e^{-\varepsilon\beta/2}. \tag{5.47}$$

Next we analyze the spin–spin correlation function

$$\langle \sigma_i;\sigma_j \rangle_L = \langle (1-\sigma_i);(1-\sigma_j)\rangle_L \tag{5.48}$$

for $i<j\in(-L,L)$. (Of course $\langle\sigma_i;\sigma_j\rangle_L=0$ if either i or j is outside this interval.) For simplicity we assume $|i-j|\geqq M+2$. The largest terms in the expansion (4.8) are obtained when there are four spin flips, one on each side of i, j. Terms where X_1 or X_2 have no clusters vanish because then $1-\sigma=0$. The spin flips must be part of one cluster in X_{12}, because $|i-j|\geqq M+2$ implies that a connected graph between the two contours is impossible. They must form two contours γ_1,γ_2 because of the charged constituent rule. Thus we have a $k=2$ cluster, with one interpolation. Performing the integral over s yields

$$e^{-\beta H(\gamma_1\cup\gamma_2)} - e^{-\beta(H(\gamma_1)+H(\gamma_2))} = e^{-8\beta\zeta(2)}(e^{4\beta|i-j|-2}-1), \tag{5.49}$$

where we have evaluated

$$H(\gamma_1) = 2\sum_{i<j}|i-j|^{-2}\chi_{\gamma_1}(i,j) = 4\left(1+\frac{1}{4}+\frac{1}{9}+\dots\right) \equiv 4\zeta(2) = \frac{2\pi^2}{3},$$

$$H(\gamma_1\cup\gamma_2) = H(\gamma_1)+H(\gamma_2)-4|i-j|^{-2}. \tag{5.50}$$

There is an additional factor $4=(1-\sigma_i)(1-\sigma_j)$ when we consider the contribution to (4.8).

The sum of all other terms in (4.8) can be estimated as in (5.45). The supremum runs over $\{X_\alpha\}$, $(Y_1,\dots,Y_j)$ whose contours are not of the type already considered. It is easy to see that the next lowest energy configuration has a block of minus spins of size 2 around i or j, but not both. The minimum energy is $H(\gamma_1)+H(\gamma_2)=12\zeta(2)-4$. Hence the error terms are bounded by $4|i-j|^{-2}e^{-\beta(1-7\varepsilon)(12\zeta(2)-4)}$, and we have

$$|\langle\sigma_i;\sigma_j\rangle_L - 4e^{-8\beta\zeta(2)}(e^{4\beta|i-j|^{-2}}-1)| \leqq 4|i-j|^{-2}e^{-\beta(1-7\varepsilon)(12\zeta(2)-4)} \tag{5.51}$$

for $|i-j|\geqq M+2$, $M>M_0(\varepsilon)$, $\beta>\beta_0(M)$.

This completes the proof of Theorem 1.1.

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