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Three Body Asymptotic Completeness for $P(\Phi)_2$ Models

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Abstract. We consider weakly coupled even $\lambda P(\Phi)_2$ models that do not have a two-body bound state, and prove asymptotic completeness on the subspace of states with mass between $3m+a(\lambda)$ and $4m-b(\lambda)$, where a and b are positive functions tending to zero with λ. The analytic structure of the six point function, integrated over the three incoming momenta, shows only two Landau singular manifolds (plus normal thresholds) associated to three particle processes.

I. Introduction

The $P(\Phi)_2$ theory has been for almost ten years a mathematically well defined quantum field theory, with energy momentum spectrum such as required for reasonable scattering properties: isolated (cyclic) vacuum and isolated one particle hyperboloid of mass m (Glimm et al. [15]). This guarantees the existence of asymptotic (Fock) spaces $\mathscr{H}^{\text{in}}$ and $\mathscr{H}^{\text{out}}$. A satisfactory interpretation of scattering further requires

$$\mathscr{H}^{\text{in}} = \mathscr{H}^{\text{out}} = \mathscr{H},$$

where $\mathscr{H}$ is the whole Hilbert space of physical states. A complete proof of this property, called "asymptotic completeness," seems at present to be out of reach for any $P(\Phi)_2$ model. The usual approach is to consider subspaces $\mathscr{H}_{(a,b)}$ and $\mathscr{H}_{(a,b)}^{\text{in, out}}$ of states with zero total momentum, and total energy in a given interval (a, b) and to prove

$$\mathscr{H}_{(a,b)}^{\text{in}} = \mathscr{H}_{(a,b)}^{\text{out}} = \mathscr{H}_{(a,b)}.$$

When the interaction polynomial $P(\Phi)$ is even, which we assume, one also distinguishes odd and even subspaces, generated by products of odd and even

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numbers of field operators. The above mentioned results [15] already imply

$$\mathcal{H}^{\text{in, even}}_{[0,2(m-\varepsilon)[} = \mathcal{H}^{\text{out, even}}_{[0,2(m-\varepsilon)[} = \mathcal{H}^{\text{even}}_{[0,2(m-\varepsilon)[}$$

and

$$\mathcal{H}^{\text{in, odd}}_{[0,3(m-\varepsilon)[} = \mathcal{H}^{\text{out, odd}}_{[0,3(m-\varepsilon)[} = \mathcal{H}^{\text{odd}}_{[0,3(m-\varepsilon)[}$$

for any $\varepsilon > 0$ and λ sufficiently small.

The first results involving continuous spectrum of the mass operator, which intuitively corresponds to scattering states, are due to Spencer and Zirilli [21] who proved asymptotic completeness in the two body region:

$$\mathcal{H}^{\text{in, even}}_{[2(m-\varepsilon),4(m-\varepsilon)[} = \mathcal{H}^{\text{out, even}}_{[2(m-\varepsilon),4(m-\varepsilon)[} = \mathcal{H}^{\text{even}}_{[2(m-\varepsilon),4(m-\varepsilon)[}.$$

This means in particular that the mass spectrum below $2m$ is discrete and corresponds to possible additional particles (bound states). The presence of one such two body bound state was then established by Dimock and Eckmann [9, 10] under the necessary and sufficient condition that the coefficient of Φ^4 in $P(\Phi)$ is nonpositive. More recently, Neves da Silva [19] has proved the existence of a three body bound state (just below $3m$ in $\mathcal{H}^{\text{odd}}$) under the condition that the coefficients of Φ^4 and Φ^6 are both strictly negative. The discrete spectrum below $2m$ in $P(\Phi)_2$ models without the $\Phi \to -\Phi$ symmetry has been studied by Koch [18], Glimm and Jaffe [14], and Imbrie [17].

In the present paper we consider the three body continuous spectrum and scattering states for even $\lambda P(\Phi)_2$ models without a two body bound state; i.e. we assume that the coefficient of Φ^4 is strictly positive. As in all the results outlined above, the coupling constant λ will have to be taken sufficiently small. The desired result is the following:

$$\mathcal{H}^{\text{in, odd}}_{[3(m-\varepsilon),5(m-\varepsilon)[} = \mathcal{H}^{\text{out, odd}}_{[3(m-\varepsilon),5(m-\varepsilon)[} = \mathcal{H}^{\text{odd}}_{[3(m-\varepsilon),5(m-\varepsilon)[}$$

but we have only been able to prove

$$\mathcal{H}^{\text{in, odd}}_{]3m+a(\lambda),4m-b(\lambda)[} = \mathcal{H}^{\text{out, odd}}_{]3m+a(\lambda),4m-b(\lambda)[} = \mathcal{H}^{\text{odd}}_{]3m+a(\lambda),4m-b(\lambda)[},$$

where $a(\lambda)$ and $b(\lambda)$ are positive functions tending to zero with λ.

Conceived in the same spirit as Spencer and Zirilli's method for proving two particle asymptotic completeness, our proof keeps the benefit of various aspects of the program inspired by the work of Symanzik [23] and developed since 1968 by Bros [1–3] in the axiomatic framework of quantum field theory. This program, based on the study of generalized Bethe Salpeter equations, has displayed the general nature of the connection between:

(i) asymptotic completeness in a given energy strip $E < (n+1)m$,

(ii) the n particle irreducibility of corresponding Bethe Salpeter type kernels,

(iii) the analytic and monodromic structure of the relevant connected Green's functions near the physical regions of the corresponding processes.

For the two-body region, steps (i)⇒(ii) and (ii)⇒(iii) were proved in [1], while the equivalence (i)⇔(ii) was completed in [3b] up to the problem of possible poles in the continuum. Spencer and Zirilli's method, developed later but independently for the models $\lambda P(\Phi)_2$, has much in common with the above program in the directions (ii)⇒(iii) and (iii)⇒(i). Concerning three particle asymptotic completeness, the first results were stated by Bros [2], where the main ideas and methods of proofs are given, and where the steps (i)⇔(ii) and (ii)⇒(iii) are treated up to some technical limitations. The present work deals with steps (ii)⇒(iii) and (iii)⇒(i) for weakly coupled $P(\Phi)_2$ models, for which property (ii) was proved in [20, 5].

Let $x_j=(x_j^0,\mathbf{x}_j)\in\mathbb{R}^2$ and let $S_n(x_1,\ldots,x_n)$ be the Schwinger functions of a weakly coupled two dimensional $P(\Phi)$ model. Let

$$R_1(x,y)\equiv S_2(x,y)$$

and let R_1^{-1} be the inverse of R_1 considered as an integral operator. Let

$$\begin{aligned} R_3(x_1,x_2,x_3;y_1,y_2,y_3)=&S_6(x_1,x_2,x_3,y_1,y_2,y_3)\\ &-\int dxdyS_4(x_1,x_2,x_3,x)R_1^{-1}(x,y)S_4(y,y_1,y_2,y_3), \end{aligned} \tag{1.1}$$

$$R_{03}(x_1,x_2,x_3,y_1,y_2,y_3)=6\prod_{j=1}^{3}R_1(x_j,y_j), \tag{1.2}$$

$$K_{1,3}^{(1)}(x;y_1,y_2,y_3)=\int dyR_1^{-1}(x,y)\left[S_4(y,y_1,y_2,y_3)-\sum_{j=1}^{3}S_2(y,y_j)S_2(y_k,y_l)\right], \tag{1.3}$$

and let

$$R_3(p_1,p_2,p_3;p_1',p_2',p_3'),$$
$$K_{1,3}^{(1)}(p;p_1',p_2',p_3'),$$
$$R_1^{-1}(p),$$

be the Fourier transforms, taken at

$$p_1+p_2+p_3=p_1'+p_2'+p_3'=p=(ik,0).$$

These functions are "one particle irreducible" and therefore analytic for $|\mathrm{Re}\,k|<3(m-\varepsilon)$ [20]. The bulk of our work is a study of their analytic structure in the three body region $\mathrm{Re}\,k\in(3(m-\varepsilon),5(m-\varepsilon))$. For suitable analytic test functions f and g, we consider

$$\begin{aligned} \langle fR_3(k)g\rangle=&\int\prod(dp_jdp_j')\delta((ik,0)-\textstyle\sum p_j)f(p_1,p_2,p_3)g(p_1',p_2',p_3')\\ &\cdot R_3(p_1,p_2,p_3;p_1',p_2',p_3'),\\ (K_{1,3}^{(1)}g)(k)=&\int\prod dp_jK_{1,3}^{(1)}((ik,0);p_1,p_2,p_3)g(p_1,p_2,p_3). \end{aligned}$$

We show that $\langle fR_3(k)g\rangle$, $(K_{1,3}^{(1)}g)(k)$ and $R_1^{-1}((ik,0))$ have analytic continuations through the three-body cut from above and from below, at least across the interval $(3m+a(\lambda),4(m-\varepsilon))$ with $a(\lambda)\to 0$ as $\lambda\to 0$. In particular there are no resonances in a neighborhood $|\mathrm{Im}\,k|<c\lambda(\mathrm{Re}\,k-3m)^{1/2}$ of this interval. The discontinuities (differences of the two determinations) along this interval satisfy the

formulae

$$\langle fR_3(k)g\rangle_+ - \langle fR_3(k)g\rangle_- = 3!(2\pi Z)^3 \int \prod_{j=1}^{3} (dp_j\delta(p_j^2+m^2)\theta(\mathrm{Im}\,p_j^0))\,\delta\left((ik,0)-\sum_1^3 p_j\right) \cdot (R_{03}^{-1}R_3f)_\mp(p_1,p_2,p_3)(R_{03}^{-1}R_3g)_\pm(p_1,p_2,p_3), \tag{1.4}$$

$$(K_{1,3}^{(1)}g)_+(k)-(K_{1,3}^{(1)}g)_-(k) = 3!(2\pi Z)^3 \int dp \int \prod_{j=1}^{3} (dp_j\delta(p_j^2+m^2)\theta(\mathrm{Im}\,p_j^0))\,\delta\left((ik,0)-\sum_1^3 p_j\right) \cdot (K_{1,3}^{(1)}R_{03}^{-1})_\mp(p;p_1,p_2,p_3)(R_{03}^{-1}R_3g)_\pm(p_1,p_2,p_3), \tag{1.5}$$

$$R_1^{-1}((ik,0))_+ - R_1^{-1}((ik,0))_- = -3!(2\pi Z)^3 \int dp\,dp' \prod_1^3 (dp_j\delta(p_j^2+m^2)\theta(\mathrm{Im}\,p_j^0))\,\delta\left((ik,0)-\sum_1^3 p_j\right) \cdot (K_{1,3}^{(1)}R_{03}^{-1})_\mp(p;p_1,p_2,p_3)(K_{1,3}^{(1)}R_{03}^{-1})_\pm(p_1,p_2,p_3;p'), \tag{1.6}$$

where $Z=Z(\lambda)$ is the field strength renormalization [coefficient of the pole term $(p^2+m^2)^{-1}$ in $R_1(p)$], and the subscripts + or − label the two determinations with respect to the variable k. These formulae imply, via methods and results of the axiomatic field theory [1–3, 12, 22], the announced result:

$$\mathscr{H}^{\mathrm{in,odd}}_{]3m+a(\lambda),4(m-\varepsilon)[} = \mathscr{H}^{\mathrm{out,odd}}_{]3m+a(\lambda),4(m-\varepsilon)[} = \mathscr{H}^{\mathrm{odd}}_{]3m+a(\lambda),4(m-\varepsilon)[}.$$

In the remainder of this introduction, we shall give an idea of the technique and explain the unresolved difficulties around $3m$ and above $4(m-\varepsilon)$. The basic objects will be the n-particle irreducible functions ($n\leqq 3$) for which we proved "Spencer irreducibility" in our previous paper [5]. It follows, via Spencer's cluster expansion [20], that these functions are analytic in tubes in momentum space approximately as suggested by perturbation theory. Precisely let $K_3(p_1,p_2,p_3;\,p_1',p_2',p_3')$ be the three body Bethe Salpeter kernel, defined as the connected part of $-R_3^{-1}$, the inverse of $-R_3$ considered as an integral operator. K_3 is analytic in

$$|\mathrm{Re}\,k|<5(m-\varepsilon), \tag{1.7}$$

$$|\mathrm{Im}(p_i^0+p_j^0)|<4(m-\varepsilon), \tag{1.8}$$

$$|\mathrm{Im}\,p_i^0|<3(m-\varepsilon), \tag{1.9}$$

$$|\mathrm{Im}\,\mathbf{p}_j|<\delta, \tag{1.10}$$

$$|\mathrm{Im}(p_i'^0+p_j'^0)|<4(m-\varepsilon), \tag{1.11}$$

$$|\mathrm{Im}\,p_i'^0|<3(m-\varepsilon), \tag{1.12}$$

$$|\mathrm{Im}\,\mathbf{p}_j'|<\delta, \tag{1.13}$$

$$|\mathrm{Im}(p_i^0+p_j^0-p_k'^0)|<3(m-\varepsilon), \tag{1.14}$$

$$|\mathrm{Im}(p_i^0-p_j'^0)|<2(m-\varepsilon), \tag{1.15}$$

where δ and ε, with $0<\delta<\sqrt{\varepsilon m}$, can be taken arbitrarily small for sufficiently small λ. We note that the analyticity domain defined by (1.7)–(1.15) contains

$$\bigcup_{3(m-\varepsilon)<\mathrm{Re}k<5(m-\varepsilon)} \{k, \mathscr{D}(k)\times\mathscr{D}_0(k)\},$$

where

$$\mathscr{D}_0(k)=\left\{(p_1,p_2,p_3)\in\mathbb{C}^6, \sum_1^3 p_j=(ik,0), \mathrm{Im}\,p_j^0>m-\varepsilon, |\mathrm{Im}\,\mathbf{p}_j|<\delta\right\}, \qquad (1.16)$$

$$\mathscr{D}(k)=\left\{(p_1,p_2,p_3)\in\mathbb{C}^6, \sum_1^3 p_j=(ik,0), \mathrm{Re}\,k-4(m-\varepsilon)\right.$$

$$\left.<\mathrm{Im}\,p_j^0<3(m-\varepsilon), |\mathrm{Im}\,\mathbf{p}_j|<\delta\right\}. \qquad (1.17)$$

The other three particle irreducible functions are defined as follows

$$K_{1,3}^{(3)}((ik,0));p_1',p_2',p_3')=(K_{1,3}^{(1)}R_3^{-1})((ik,0);p_1',p_2',p_3'), \qquad (1.18)$$

$$K_{1,1}^{(3)}((ik,0))=-R_1^{-1}((ik,0))-(K_{1,3}^{(3)}R_3K_{3,1}^{(3)})((ik,0)), \qquad (1.19)$$

and are analytic in tubes defined by obvious subsets of (1.7)–(1.15).

The task now is to use this analyticity in the three body region to relate the analytic structure of R_3, $K_{1,3}^{(1)}$, and R_1^{-1} to that already known of R_{03} and $R_1\otimes R_2$, where

$$R_2(x_2,x_3;y_2,y_3)=S_4(x_2,x_3,y_2,y_3)-S_2(x_2,x_3)\,S_2(y_2,y_3).$$

The central tool for this purpose is the three body Bethe Salpeter equation,

$$R_3^{-1}=R_{03}^{-1}-\tfrac{1}{3}\sum_\alpha K_2^\alpha\otimes R_1^{-1}-K_3 \qquad (1.20)$$

or

$$R_3=R_{03}+R_3\sum_\alpha K_2^\alpha R_{02}^\alpha+R_3K_3R_{03}, \qquad (1.21)$$

where $R_{02}=2R_1\otimes R_1$ and K_2 is the two body Bethe Salpeter kernel defined as the connected part of $-R_2^{-1}$, and the sum over α has three terms (two body subchannels).

Without the second term this equation could be rather easily solved up to $\mathrm{Re}\,k=5(m-\varepsilon)$, outside a neighborhood of the threshold $k=3m$, where $\langle fR_{03}(k)g\rangle$ has a logarithmic singularity in two dimensions.

The problem is more difficult with the two-body terms in (1.21) due to the two-body threshold at [e.g. $\alpha=(2,3)$]:

$$p_2^0+p_3^0=i\mu(\mathbf{p}_2+\mathbf{p}_3)\equiv i(4m^2+(\mathbf{p}_2+\mathbf{p}_3)^2)^{1/2}. \qquad (1.22)$$

Suppose that we iterate the equation and look at

$$K_2^{12}R_{02}^{12}K_2^{23}R_{02}^{23}f. \qquad (1.23)$$

Suitable analyticity domains, including two-body cuts, are yet to be defined, but (1.23) should be analytic for at least some points below the two-body threshold, i.e. for some points satisfying

$$\mathrm{Im}(p_1^0+p_2^0)<2m\,.$$

The definition of (1.23) involves an integral over the intermediate $p_2^0+p_3^0$ variable. The corresponding contour, originally euclidean, should cross the Minkowski manifold below the two-body threshold, i.e. for

$$\mathrm{Im}(p_2^0+p_3^0)<2m\,.$$

But the analyticity domain of $K_2^{23}R_{02}^{23}f$ will typically be limited by

$$\mathrm{Im}(p_1^0+p_3^0)<4(m-\varepsilon)\,. \tag{1.8'}$$

Summing up the above three inequalities yields the following necessary condition:

$$\mathrm{Re}\,k<4m-2\varepsilon\,. \tag{1.24}$$

That this condition is sufficient for our purposes will be seen in the course of the paper. A similar limitation is also present in the work of Bros [2] who proved, in the axiomatic framework with some additional hypotheses, that asymptotic completeness between $3m$ and $\frac{11}{3}m$ (now raised to $4m$) is equivalent to analyticity of the three particle irreducible kernels in domains limited by Eqs. (1.7)–(1.15) without the ε.

The troublesome inequality is (1.8′) which corresponds to a four body threshold in a subchannel. The only way out is to study the four body threshold (in $\mathscr{H}^{\mathrm{even}}$) before continuing the three body analysis above $4(m-\varepsilon)$. The problem is slightly different without the $\Phi\to-\Phi$ symmetry. (1.8′) would be replaced by

$$\mathrm{Im}(p_1^0+p_3^0)<3(m-\varepsilon)\,, \tag{(1.8'')}$$

which yields

$$\mathrm{Re}\,k<\tfrac{7}{2}m-\tfrac{3}{2}\varepsilon\,. \tag{1.25}$$

Knowing the analytic structure of R_3 up to $\frac{7}{2}(m-\varepsilon)$ would allow, in principle, replacement of (1.8″) by

$$\mathrm{Im}(p_1^0+p_3^0)<\tfrac{7}{2}(m-\varepsilon) \tag{1.26}$$

with a three body cut starting at $3m$, and to continue the analysis up to $\mathrm{Re}\,k<(4-\frac{1}{4})(m-\varepsilon)$. Iterating this procedure n times would permit us to reach

$$\mathrm{Re}\,k<(4-2^{-n})(m-\varepsilon)\,, \tag{1.27}$$

and to cover the whole three body region as $n\to\infty$. Such a program makes desirable a more global approach to asymptotic completeness. In a recent work [6–8], Cooper, Feldman, Rosen develop the Legendre transform (global) approach to particle irreducibility. It is not clear however that the new kernels so defined will be useful for studying asymptotic completeness.

We now come back to the Bethe Salpeter equation for an even theory with $\mathrm{Re}\,k<4(m-\varepsilon)$. In two dimensions $K_2^\alpha R_{02}^\alpha$ has an inverse square root divergence at the two body threshold, and K_3R_{03} has a logarithmic divergence at the three body threshold. These divergences can be eliminated from the equation by a resummation of two particle processes à la Faddeev. It is convenient for that purpose to split the Bethe Salpeter equation as follows:

$$R_3^{-1}=R_3'^{-1}-K_3 \tag{1.28}$$

and

$$R_3'^{-1}=R_{03}^{-1}-\tfrac{1}{3}\sum_\alpha K_2^\alpha\otimes R_1^{-1}, \tag{1.29}$$

or

$$R_{03}^{-1}R_3'-1=\sum_\alpha K_2^\alpha R_{02}^\alpha+\sum_\alpha K_2^\alpha R_{02}^\alpha(R_{03}^{-1}R_3'-1).$$

Let now M_α be the part of $R_{03}^{-1}R_3'-1$ which has a two body cut in the channel α. Then

$$R_{03}^{-1}R_3'-1=\sum_\alpha M_\alpha \tag{1.30}$$

and

$$M_\alpha=K_2^\alpha R_{02}^\alpha+K_2^\alpha R_{02}^\alpha\sum_\beta M_\beta, \tag{1.31}$$

or

$$(1-K_2^\alpha R_{02}^\alpha)M_\alpha=K_2^\alpha R_{02}^\alpha+K_2^\alpha R_{02}^\alpha\sum_{\beta\neq\alpha}M_\beta.$$

We can now resum two particle processes, i.e. use the two body Bethe Salpeter equation ($R_2^{-1}=R_{02}^{-1}-K_2$) to transform (1.31) into

$$M_\alpha=K_2^\alpha R_2^\alpha+K_2^\alpha R_2^\alpha\sum_{\beta\neq\alpha}M_\beta. \tag{1.32}$$

The improvement of (1.32) over (1.31) is that a suitable operator norm of $K_2^\alpha R_2^\alpha$ will be bounded by 1 in the absence of two body bound states. Let now

$$M=\begin{pmatrix}M^{23}\\M^{13}\\M^{12}\end{pmatrix},\quad A=\begin{pmatrix}0 & K_2^{23}R_2^{23} & K_2^{23}R_2^{23}\\ K_2^{31}R_2^{31} & 0 & K_2^{31}R_2^{31}\\ K_2^{12}R_2^{12} & K_2^{12}R_2^{12} & 0\end{pmatrix}. \tag{1.33}$$

A formal solution of (1.32) is then

$$M=(1-A)^{-1}\begin{pmatrix}K_2^{23}R_2^{23}\\K_2^{13}R_2^{13}\\K_2^{12}R_2^{12}\end{pmatrix}, \tag{1.32'}$$

which gives

$$R_3' = R_{03} + (R_{03}, R_{03}, R_{03}) \begin{pmatrix} M^{23} \\ M^{13} \\ M^{12} \end{pmatrix}$$

$$= R_{03} + (R_{03}, R_{03}, R_{03})(1-A)^{-1} \begin{pmatrix} K_2^{23} R_2^{23} \\ K_2^{13} R_2^{13} \\ K_2^{12} R_2^{12} \end{pmatrix}$$

$$= \sum_\alpha R_2^\alpha \otimes R_1 + (R_{03}, R_{03}, R_{03}) [(1-A)^{-1} - \tfrac{1}{3}] \begin{pmatrix} K_2^{23} R_2^{23} \\ K_2^{13} R_2^{13} \\ K_2^{12} R_2^{12} \end{pmatrix}$$

$$= \sum_\alpha R_2^\alpha \otimes R_1 + (R_2^{12} \otimes R_1 + R_2^{13} \otimes R_1, R_2^{12} \otimes R_1 + R_2^{23} \otimes R_1, R_2^{13} \otimes R_1 + R^{23} \otimes R_1)$$

$$\cdot (1-A)^{-1} \begin{pmatrix} K_2^{23} R_2^{23} \\ K_2^{13} R_2^{13} \\ K_2^{12} R_2^{12} \end{pmatrix} \tag{1.34}$$

where we have again used the two body Bethe Salpeter equation in the last two steps. Finally

$$R_3 = R_3'(1 - K_3 R_3')^{-1}. \tag{1.35}$$

This formal solution should of course be given a mathematical meaning. The main difficulty is to invert the operator $(1-A)$ near the three body threshold. Indeed a suitable norm of A, for $k < 3m$ in the first sheet, will satisfy $\|A\| \to 2$ as $k \to 3m$ and $\lambda \to 0$ and we prove that the spectral radius of the leading part of A is also strictly larger than one near $k = 3m$. Our positive result is

$$\|A^3\| < 1 \quad \text{if} \quad |k - 3m| > c\lambda^2,$$

and we deduce the announced results with $a(\lambda) = O(\lambda^2)$ and $b(\lambda) \approx \varepsilon(\lambda)$ of Spencer's cluster expansion.

Three body bound states are not expected [15] for weakly coupled even $\lambda P(\Phi)_2$ models with a strictly positive Φ^4 term in $P(\Phi)$. It seems, however, difficult to extract a dominant and repulsive part of A, as was done [21] in the simpler two body problem. The conjecture $\Gamma^6 < 0$ is an alternative approach [13] to this problem for $k < 3m$.

It would be more clearly possible, as in the work of Bros [2] (see also [4]) to use a Fredholm alternative to prove that $(1-A)^{-1}$ is meromorphic near $3m$. Moreover the boundedness of A should imply that the number of poles is finite. However such results are far below what one would like to have for a specific model like $(\Phi^4)_2$.

The plan of the article is as follows:

In Sect. 2 we study the operator $K_2^\alpha R_2^\alpha$ applied to functions of p_1, p_2, p_3 that have a two-body cut in a channel β different from α. Starting from the euclidean region, we establish its analytic continuation around the two-body threshold $(p_i+p_j)^2+4m^2=0$, $\alpha=(ij)$ and the three body threshold $(p_1+p_2+p_3)^2+9m^2=0$. We also prove the existence of only two Landau singular manifolds associated to three body processes. In Sect. 3 we consider the 2-particle-irreducible six point function in the center of mass frame, integrated over relative energy-momentum variables; we prove that it has analytic continuations with respect to the total energy variable across the interval $(3m+a(\lambda), 4m-b(\lambda))$. Section 4 is devoted to the proof of discontinuity formulae and of asymptotic completeness, again for total energy in the interval $(3m+a(\lambda), 4m-b(\lambda))$. The last section contains some remarks about the (unsolved!) question of three body bound states.

II. Analyticity of the Operator $K_2^\alpha R_2^\alpha$ in the Three Body Region

In this section we study the 4 point function $(K_2R_2)\,(p_1', p_2'; p_1, p_2)$ considered as the kernel of an integral operator

$$(K_2R_2f)(p_1', p_2', p_3) = \int dp_1 dp_2 (K_2R_2)(p_1', p_2'; p_1, p_2) f(p_1, p_2, p_3), \tag{2.1}$$

acting on functions f that belong to a suitable space of analytic functions. In order to motivate our choice, let us first describe the singularities which come up in the integrals: (we recall that $p=(p^0, \mathbf{p})\in\mathbb{C}^2$, and $p^2=(p^0)^2+\mathbf{p}^2$)

Poles: $p_j^2+m^2=0$ or $p_j^0=\pm i\omega(\mathbf{p}_j)$

[only $p_j^0=+i\omega(\mathbf{p}_j)$ will eventually enter our domain],

Two-Body Thresholds: $(p_i+p_j)^2=-4m^2$ or $p_i^0+p_j^0=\pm i\mu(\mathbf{p}_i+\mathbf{p}_j)$

[only $+i\mu(\mathbf{p}_i+\mathbf{p}_j)$ will enter our domain],

Landau Singularities: assume that poles (1) and (2) coincide together with the two body threshold $(2+3)$ (the case of the $1+3$ threshold is analogous)

$$\begin{cases} p_1^0=i\omega(\mathbf{p}_1) \\ p_2^0=i\omega(\mathbf{p}_2) \\ p_2^0+p_3^0=i\mu(\mathbf{p}_2+\mathbf{p}_3)\,. \end{cases}$$

If we perform the integral (2.1), where p_1+p_2 is kept fixed, the contour may (or may not) be pinched by these singularities. Assume

$$p_1+p_2+p_3=(ik, 0)\,, \tag{2.2}$$

and denote by $\pm\mathbf{p}(k)$ with $\mathrm{Im}\,\mathbf{p}(k)\leqq 0$ for k in the first sheet (and, in higher dimensions $\mathbf{p}(k)$ parallel to $\mathbf{p}_3$), the solutions of

$$k=\omega(\mathbf{p})+\mu(\mathbf{p})\,.$$

The location of the possible "first Landau singularities" is then given by

$$\begin{cases} k+ip_3^0=\omega(\mathbf{p}_1)+\omega(-\mathbf{p}_1-\mathbf{p}_3) \\ \qquad\quad k=\omega(\mathbf{p}_1)+\mu(\mathbf{p}_1) \end{cases}$$

or ($\mathbf{p}_1$ is the integration variable)

$$k+ip_3^0=\omega(\mathbf{p}(k))+\omega(\mathbf{p}_3\pm\mathbf{p}(k)).$$

Suppose now that one of these singularities is present in the integrand of (2.1) (in the p_1 variable), and that the contour is pinched between it and two poles present in K_2R_2. We then obtain "second Landau singularities" given by

$$\begin{cases} p_1^0=i\omega(\mathbf{p}_1) \\ p_2^0=i\omega(\mathbf{p}_2) \\ p_2^0+p_3^0=i\omega(\mathbf{p}(k))+i\omega(\mathbf{p}_1\pm\mathbf{p}(k)) \end{cases}$$

or

$$\begin{cases} k+ip_3^0=\omega(\mathbf{p}_1)+\omega(-\mathbf{p}_1-\mathbf{p}_3) \\ k=\omega(\mathbf{p}_1)+\omega(\mathbf{p}(k))+\omega(\mathbf{p}_1\pm\mathbf{p}(k)). \end{cases}$$

Eliminating $\mathbf{p}_1$ (which is the integration variable) between these two equations yields

$$k+ip_3^0=\omega\left(\frac{\mathbf{p}(k)}{2}\right)+\omega\left(\mathbf{p}_3\mp\frac{\mathbf{p}(k)}{2}\right).$$

What is remarkable is that the next step does not produce "third" Landau singularities, but only reproduces the first and second singularities. One can check the same property in higher dimensions.

We now describe a geometric limitation to the analyticity domain for $K_2^\alpha R_2^\alpha f$. We fix $ik=\sum_{j=1}^{3}p_j^0$, and work in the barycentric frame for $(\mathrm{Im}\,p_1^0, \mathrm{Im}\,p_2^0, \mathrm{Im}\,p_3^0)$ with origin at $\mathrm{Im}\,p_j^0=\frac{\mathrm{Re}\,k}{3}$, $j=1,2,3$. It follows from [20] that $K_2(p_1,p_2,p_1',p_2')$ is analytic in (1.8)–(1.13) and (1.15) for $i,j\in\{1,2\}$. Thus the expected analyticity domain for $K_2^{12}R_2^{12}f$ will be a subset of the hexagonal tube $\mathscr{D}(k)$ [see (1.17)] with a cut at $k+ip_3^0=\mu(\mathbf{p}_3)$. Assume now that $f=K_2^{23}R_2^{23}g$. Then if we want to apply $K_2^{12}R_2^{12}$ to it, the conserved variable p_3^0 should be chosen so as to allow us to deform the contour of integration in (2.1) (in the variable p_1^0):

1) above the cut $k+ip_1^0=\mu(\mathbf{p}_1)$,
2) above the pole $p_2^0=i\omega(\mathbf{p}_2)$,
3) below the pole $p_1^0=i\omega(\mathbf{p}_1)$.

A simple geometric construction (see Fig. 1) shows that p_3^0 should satisfy

$$\mathrm{Im}\,p_3^0<6(m-\varepsilon)-\mathrm{Re}\,k \tag{2.3}$$

and

$$\mathrm{Im}\,p_3^0>2\,\mathrm{Re}\,k-7(m-\varepsilon) \tag{2.4}$$

[taking into account the limitation (2.3) also for the integration variable p_1^0]. We thus define, for fixed k such that $\mathrm{Re}\,k<4(m-\varepsilon)$,

$$\mathscr{D}_i(k)=\mathscr{D}(k)\cap\{2\,\mathrm{Re}\,k-7(m-\varepsilon)<\mathrm{Im}\,p_i^0<6(m-\varepsilon)-\mathrm{Re}\,k\}. \tag{2.5}$$

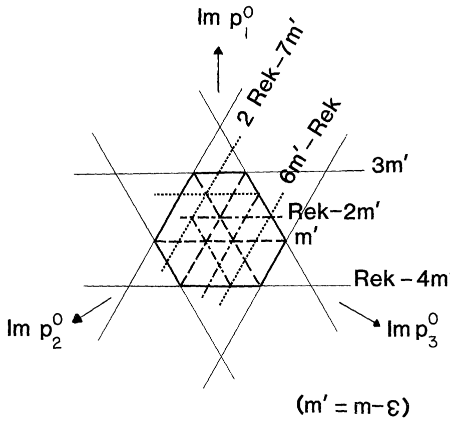

Fig. 1

We shall prove that $K_2^{12}R_2^{12}f$ is analytic in $\bigcup_{3(m-\varepsilon)<\mathrm{Re}k<4(m-\varepsilon)} \mathscr{D}_3(k)$ except for the two-body cut, three-body cut and Landau singularities, with suitable analytic continuations across the cuts, provided f has the same properties with the p_1 variable replaced by p_3, and δ is sufficiently small.

We first define an analyticity domain D for the total energy k: D is a multi-sheeted manifold with a logarithmic branch point at $k=3m$, lying in the strip

$$3(m-\varepsilon)<\mathrm{Re}k<4(m-\varepsilon)$$

with $k \neq 3m$, and with all sheets but the first restricted by the curve

$$\{k=\omega(\varrho+i\delta')+\mu(\varrho+i\delta'), \varrho \in \mathbb{R}\} \qquad \delta'<\delta<\mathrm{Inf}(\sqrt{m\varepsilon}, \lambda), \tag{2.6}$$

where the labeling of the sheets is made respective to the cut $\{k \geqq 3m\}$.

We now describe a multisheeted manifold $D(k, \mathbf{p}_j)$ for the two-body energy $k+ip_j^0$ spreading around the two-body threshold $\mu(\mathbf{p}_j)$ and around the Landau singular points

$$k+ip_j^0=\omega(\mathbf{p}(k))+\omega(\mathbf{p}_j \pm \mathbf{p}(k)), \tag{2.7}$$

$$k+ip_j^0=\omega\left(\frac{\mathbf{p}(k)}{2}\right)+\omega\left(\mathbf{p}_j \pm \frac{\mathbf{p}(k)}{2}\right). \tag{2.8}$$

As a first step, take $\tilde{\tilde{D}}(k, \mathbf{p})$ to be a two sheeted manifold with branch point $z=\mu(\mathbf{p})$, lying in the strip

$$2\,\mathrm{Re}k-6(m-\varepsilon)<\mathrm{Re}z<-\mathrm{Re}k+7(m-\varepsilon) \tag{2.9}$$

with the second sheet restricted by the curves

$$z=\omega(\varrho \pm i\delta)+\omega(\varrho \pm i\delta+\mathbf{p}), \qquad \varrho \in \mathbb{R}, \tag{2.10}$$

where the labeling of the sheets is with respect to the cut

$$z=\omega\left(\frac{\mathbf{p}}{2}+\sigma\right)+\omega\left(\frac{\mathbf{p}}{2}-\sigma\right), \qquad \sigma \geqq 0, \tag{2.11}$$

and where in addition we exclude the possible "first Landau singular points" $z=\omega(\mathbf{p}(k))+\omega(\mathbf{p}\pm\mathbf{p}(k))$ from one sheet (not necessarily the same sheet for both) according to the following rule: If $|\mathrm{Im}\,\mathbf{p}|<2|\mathrm{Im}\,\mathbf{p}(k)|$, these Landau singular points are in the second sheet of $\tilde{\tilde{D}}(k,\mathbf{p})$ iff k is in an odd sheet in D. The general prescription follows by keeping track of the k and $\mathbf{p}$ dependence of the singularities.

Now we define $\tilde{D}(k,\mathbf{p})$ from $\tilde{\tilde{D}}(k,\mathbf{p})$ by a doubling of manifold around each Landau singular point present in $\tilde{D}(k,\mathbf{p})$, so as to obtain a monodromic structure where each path going twice around a singular point is identified with a point. The second sheets are restricted by the (parametric) curves

$$\begin{cases} z=\omega(\mathbf{p}')+\omega(\mathbf{p}+\mathbf{p}'), \text{ where} \\ k=\omega(\mathbf{p}')+\omega(\mathbf{p}'+\varrho\pm i\delta)+\omega(\varrho\pm i\delta),\ \varrho\in\mathbb{R}, \\ |\mathrm{Im}\,\mathbf{p}'|<\delta, \end{cases} \tag{2.12}$$

where the labeling of the sheets is made respective to the cut

$$\begin{cases} z=\omega(\mathbf{p}')+\omega(\mathbf{p}'+\mathbf{p}) \\ \omega(\mathbf{p}')+\omega\left(\frac{\mathbf{p}}{2}+\sigma\right)+\omega\left(\frac{\mathbf{p}}{2}-\sigma\right)=k,\ \sigma\geqq 0, \end{cases} \tag{2.13}$$

and where we exclude the possible "second Landau singular points"

$$z=\omega\left(\frac{\mathbf{p}(k)}{2}\right)+\omega\left(\mathbf{p}\pm\frac{\mathbf{p}(k)}{2}\right) \tag{2.14}$$

from one sheet according to the same rule as for $\tilde{\tilde{D}}$ $\left(\text{where } \mathbf{p}(k) \text{ is replaced by } \frac{\mathbf{p}(k)}{2}\right)$.

Now $D(k,\mathbf{p})$ is defined from $\tilde{D}(k,\mathbf{p})$ again by a doubling of manifold around those "second Landau singular points," where the second sheets are again restricted by the curves (2.12), and where the labeling of the sheets is made respective to the cut

$$\begin{cases} z=\omega(\mathbf{p}')+\omega(\mathbf{p}+\mathbf{p}'), \\ k=\omega(\mathbf{p}')+\omega(\mathbf{p}'')+\omega(\mathbf{p}'+\mathbf{p}''), \\ k=\omega(\mathbf{p}'')+\omega\left(\frac{\mathbf{p}'}{2}+\sigma\right)+\omega\left(\frac{\mathbf{p}'}{2}-\sigma\right),\ \sigma\geqq 0. \end{cases} \tag{2.15}$$

Remarks. (1) In the simple case where $|\mathrm{Re}(\omega(\mathbf{p})+\mu(\mathbf{p}))|<\mathrm{Re}\,k$, each Landau singular point in $D(k,\mathbf{p})$ goes around the two-body threshold in the same time as k goes around the three-body threshold.

(2) The relevant $\pm$ sign in (2.10) is the sign of $-\mathrm{Im}\,\mathbf{p}$.

We now define a domain $\mathscr{D}_l$ in (p_1,p_2,p_3) with a two-body threshold, and Landau singularities in the channel (ij) $(l\neq i,j)$.

Definition 2.1.

$$\mathscr{D}_l=\{(p_1,p_2,p_3)\in\mathbb{C}^6,\ p_1+p_2+p_3=(ik,0),$$

$$k\in D,\ |\mathrm{Im}\,\mathbf{p}_j|<\delta,\ 4(m-\varepsilon)-\mathrm{Re}\,k<\mathrm{Im}\,p_j^0<3(m-\varepsilon)$$

for any $j=1,2,3$ and $k+ip_l^0\in D(k,-\mathbf{p}_l)\}$.

We take as function spaces on $\mathscr{D}_l$:

$\mathscr{A}_l = \{f: \mathbb{C}^5 \to \mathbb{C}$ analytic and bounded on $\mathscr{D}_l$, continuous on the closure of $\mathscr{D}_l$, and symmetric with respect to the exchange of p_i and p_j and with respect to $\mathbf{p}_{i'} \to -\mathbf{p}_{i'}$, $i' = 1, 2, 3\}$ with the norm

$$|f|_l = |h|_\infty + |f|_\infty, \tag{2.16}$$

where h is defined by

$$\begin{aligned} &f(k; (i\omega(\mathbf{p}_l), \mathbf{p}_l); p_j - p_i) - f(k; (i\omega(\mathbf{p}_l), -\mathbf{p}_l); (p_j^0 - p_i^0, \mathbf{p}_j - \mathbf{p}_i + 2\mathbf{p}_l)) \\ &\quad \equiv \mathbf{p}_l h(k; \mathbf{p}_l; p_j - p_i). \end{aligned} \tag{2.17}$$

Here and in the following, we write either $f(p_1, p_2, p_3)$ or $f(k; p_l; p_i - p_j)$ for the same element f of $\mathscr{A}_l (l \neq i, j)$, and we denote equivalently $\mathscr{A}_l$ by $\mathscr{A}_{(ij)}$, $\mathscr{D}_l$ by $\mathscr{D}_{(ij)}$, p_l by $p_{(ij)}$.

We also define

Definition 2.2.

$$\begin{aligned} \mathscr{D}_0 = \{&(p_1, p_2, p_3) \in \mathbb{C}^6 : |\mathrm{Im}\,\mathbf{p}_j| < \delta, \mathrm{Im}\, p_j^0 > m - \varepsilon, \\ &j = 1, 2, 3, p_1 + p_2 + p_3 = (ik, 0), k \in D\}, \end{aligned}$$

$\mathscr{A}_0 = \{f: \mathbb{C}^5 \to \mathbb{C}$ analytic and bounded on $\mathscr{D}_0$, continuous on the closure of $\mathscr{D}_0$, and symmetric with respect to permutation of $p_1, p_2, p_3\}$ with the norm $|f|_\infty$.

The main tool of this section is the following:

Proposition 2.1. *Let* $f \in \mathscr{A}_1$, *and* $(p_1, p_2, p_3) \in \mathscr{D}_3$. *For*

$$\mathrm{Re}\, k - 2m + \frac{\delta^2}{3m} < \mathrm{Im}\, p_3^0 < 4m - \mathrm{Re}\, k - \frac{\delta^2}{6m},$$

let

$$F(p_1, p_2, p_3) = \int dp_1' dp_2' \delta(p_1 + p_2 - p_1' - p_2') \frac{f(p_1', p_2', p_3)}{(p_1'^2 + m^2)(p_2'^2 + m^2)}, \tag{2.18}$$

where the integration contour is $p_1' - p_2' \in \mathbb{R}^2$ *along which the integrand has no singularity. Then F can be analytically continued to the whole of* $\mathscr{D}_3$. *Moreover*

$$\begin{aligned} |F(p_1, p_2, p_3)| &< c|f|_\infty + c z_3'^{-1} \int d\mathbf{p}\, \delta(k + i p_3^0 - \omega(\mathbf{p}) - \omega(-\mathbf{p} - \mathbf{p}_3)) \\ &\quad \cdot |f((i\omega(\mathbf{p}), \mathbf{p}), (ik - p_3^0 - i\omega(\mathbf{p}), -\mathbf{p}_3 - \mathbf{p}), p_3)| \\ &\quad + c z_3'^{-1} \sup_{|\mathbf{p} \mp \mathbf{p}(k)| > c'|k - 3m|^{1/2}} (|f| + |h|)((i\omega(\mathbf{p}), \mathbf{p}), (ik - p_3^0 - i\omega(\mathbf{p}), -\mathbf{p}_3 - \mathbf{p}), p_3), \end{aligned} \tag{2.19}$$

where h is the "odd part" of f (see (2.17)),

$$z_3' \equiv \left| (k + i p_3^0 - \mu(\mathbf{p}_3)) \frac{\omega^3(\mathbf{p}_3/2)}{m^2} \right|^{1/2}, \tag{2.20}$$

and c and c' are constants only depending on δ, δ', *and* ε.

In particular

$$|F(p_1, p_2, p_3)| < c|f|_1 z_3'^{-1}. \tag{2.21}$$

Proof. Let $M \equiv \mathrm{Max}\{\mathrm{Re}k - 2(m-\varepsilon), \mathrm{Re}k - \mathrm{Im}p_3^0 - (m-\varepsilon)\}$. F can be written as

$$\begin{aligned} F(p_1, p_2, p_3) = & \int_{\mathrm{Re}\,\omega(\mathbf{p}_1') \geqq M} dp_1' dp_2' \delta(p_1 + p_2 - p_1' - p_2') \frac{f(p_1', p_2', p_3)}{(p_1'^2 + m^2)(p_2'^2 + m^2)} \\ & + \int_{\mathrm{Re}\,\omega(\mathbf{p}_1') < M} dp_1' dp_2' \delta(p_1' + p_2' - p_1 - p_2) \\ & \cdot \frac{f(p_1', p_2', p_3) - f((i\omega(\mathbf{p}_1'), \mathbf{p}_1'), p_2', p_3)}{(p_1'^2 + m^2)(p_2'^2 + m^2)} \\ & + \int_{\mathrm{Re}\,\omega(\mathbf{p}_1') < M} dp_1' dp_2' \delta(p_1' + p_2' - p_1 - p_2) \frac{f((i\omega(\mathbf{p}_1'), \mathbf{p}_1'), p_2', p_3)}{(p_1'^2 + m^2)(p_2'^2 + m^2)} . \end{aligned} \tag{2.22}$$

In the first two terms, the integration contour in $p_1'^0 - p_2'^0$ can be shifted to $\mathrm{Im}\,p_1'^0 \lesssim M$ as $\mathrm{Re}k$ increases up to and above $3m$. The corresponding contribution to F is bounded by $c(\varepsilon)|f|_\infty$ because the contour can be chosen at a distance $O(\varepsilon)$ from both poles.

In the third term, the integration over $p_1'^0$ can be done explicitly, yielding

$$\begin{aligned} 2i\pi & \int_{\mathrm{Re}\,\omega(\mathbf{p}_1') < M} d\mathbf{p}_1' dp_2' \delta(p_1' + p_2' - p_1 - p_2) f((i\omega(\mathbf{p}_1'), \mathbf{p}_1'), p_2', p_3) \\ & \cdot \{2i\omega(\mathbf{p}_1')(ik - p_3^0 - i\omega(\mathbf{p}_1') - i\omega(-\mathbf{p}_1' - \mathbf{p}_3))(ik - p_3^0 - i\omega(\mathbf{p}_1') + i\omega(-\mathbf{p}_1' - \mathbf{p}_3))\}^{-1} . \end{aligned} \tag{2.23}$$

When $\mathrm{Re}k$ increases up to and above $3m - \dfrac{\delta^2}{4m}$ in D, the integration contour $\mathrm{Im}\,\mathbf{p}_1' = -\frac{1}{2}\mathrm{Im}\,\mathbf{p}_3$ may be pinched by the zeros $\mathbf{p}_1' = -\dfrac{\mathbf{p}_3}{2} \pm z_3$ of $ik - p_3^0 - i\omega(\mathbf{p}_1') - i\omega(-\mathbf{p}_1' - \mathbf{p}_3) = 0$ and the singularities of f [taken at $p_1'^0 = i\omega(\mathbf{p}_1')$]: $\mathbf{p}_1' = \pm \mathbf{p}(k)$ and $\mathbf{p}_1' = \pm \dfrac{\mathbf{p}(k)}{2}$.

We then deform the contour so as to obtain the analytic continuation to $\mathscr{D}_3$ [it follows from Lemma 2.1 below that the contour cannot be pinched by the boundary of the analyticity domain of f for $p_1'^0 = i\omega(\mathbf{p}_1')$]. Note that given $p_1'^0 = i\omega(\mathbf{p}_1')$, when $k + ip_3^0$ varies so as to encircle once the value $\mu(\mathbf{p}_3)$, thereby changing sheet, the two poles $\mathbf{p}_1' = -\dfrac{\mathbf{p}_3}{2} \pm z_3$ perform half a circle around $-\dfrac{\mathbf{p}_3}{2}$. On the other hand, when $k + ip_3^0$ encircles once a Landau singular point (thereby also changing sheet) one of the two poles in $\mathbf{p}_1'$ encircles once the value $\mathbf{p}(k)$ $\left(\text{or } -\mathbf{p}(k) \text{ or } \pm \dfrac{\mathbf{p}(k)}{2}\right)$. Note also that f is bounded at the 2-body threshold and Landau singular points. It is therefore analytic there in terms of suitable square root variables. This implies some cancellations of residues taken in different sheets, at the corresponding thresholds. The desired analytic continuation of F to $\mathscr{D}_3$ follows easily.

To obtain the bound (2.19) we first prove the following result:

Lemma 2.1. *Let* $\varphi(\mathbf{p}) = k - \omega(\mathbf{p})$, *and let* $D(k)$ *be the intersection of the strip* $|\mathrm{Im}\,\mathbf{p}| < \delta$ *with the inverse image* $\varphi^{-1}(D(\mathbf{p}, k))$ *of* $D(\mathbf{p}, k)$. *Then* $D(k)$ *has a multi-*

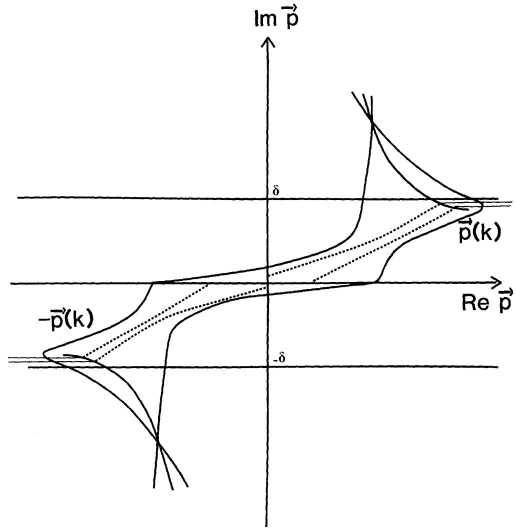

Fig. 2. Domain $D(k)$ of Lemma 2.1

sheeted analytic structure around the branch points $\mathbf{p}=\pm\mathbf{p}(k)$ *and* $\mathbf{p}=\pm\frac{\mathbf{p}(k)}{2}$ *induced by that of* $D(\mathbf{p},k)$; *the image of the second sheets of* $D(\mathbf{p},k)$ (*with respect to the two-body threshold and the Landau singular points*) *is restricted by the curves* (*see Fig.* 2)

$$\begin{aligned} k&=\omega(\mathbf{p})+\omega(\mathbf{p}+\varrho-i\delta)+\omega(\varrho-i\delta) \quad \text{if} \quad \operatorname{Im}\mathbf{p}\geqq 0, \\ k&=\omega(\mathbf{p})+\omega(\mathbf{p}+\varrho+i\delta)+\omega(\varrho+i\delta) \quad \text{if} \quad \operatorname{Im}\mathbf{p}\leqq 0, \end{aligned} \qquad \varrho\in\mathbb{R}. \tag{2.24}$$

In particular it contains a neighborhood of order $\delta-\delta'$ *of the curve*:

$$\begin{cases} \operatorname{Im}\mathbf{p}=\delta' & \text{if} \quad \frac{1}{2}\operatorname{Re}\mathbf{p}(k)<\operatorname{Re}\mathbf{p}\leqq\mathbf{p}_0, \\ \operatorname{Re}\mathbf{p}=\frac{1}{2}\operatorname{Re}\mathbf{p}(k) & \text{if} \quad 0<\operatorname{Im}\mathbf{p}<\delta', \\ \operatorname{Im}\mathbf{p}=0 & \text{if} \quad 0\leqq\operatorname{Re}\mathbf{p}<\frac{1}{2}\operatorname{Re}\mathbf{p}(k), \end{cases} \tag{2.25}$$

completed by symmetry $\mathbf{p}\to-\mathbf{p}$, *when k belongs to the second sheet in D and* $\operatorname{Im}k>0$, *where* $\mathbf{p}_0>0$ *is defined by*

$$k=\omega(\mathbf{p}_0+i\delta')+\omega\left(-\frac{\mathbf{p}_0+i\delta'}{2}+\sigma\right)+\omega\left(-\frac{\mathbf{p}_0+i\delta'}{2}-\sigma\right)$$

for some $\sigma\geqq 0$.

Proof of Lemma 2.1. The first statements follow easily from the analyticity of φ in the strip $|\operatorname{Im}\mathbf{p}|<\delta$. For the last statement, it is enough to consider the strip $0\leqq\operatorname{Im}\mathbf{p}<\delta$ (and therefore by remark (2) the curve

$$k=\omega(\mathbf{p})+\omega(\varrho-i\delta)+\omega(\mathbf{p}+\varrho-i\delta), \qquad \varrho\in\mathbb{R}, \tag{2.26}$$

which restricts the second sheets in this case) because the other case follows by symmetry $\mathbf{p}\to-\mathbf{p}$. It follows easily from expansions up to first orders that approximate solutions of (2.26) in parametric form are ($\mathbf{p}=x+iy$):

$$\begin{cases} x^2+x\varrho+\varrho^2=m(X-3m)+y^2-\delta y+\delta^2 \\ y=\dfrac{mY+\delta(x+2\varrho)}{2x+\varrho} \end{cases} \quad \text{if} \quad |k-3m|\ll m,$$

$$\begin{cases} X=\omega(x)+\omega(\varrho)+\omega(x+\varrho) \\ y=\left[Y+\delta\left(\dfrac{\varrho}{\omega(\varrho)}+\dfrac{x+\varrho}{\omega(x+\varrho)}\right)\right]\left[\dfrac{x}{\omega(x)}+\dfrac{x+\varrho}{\omega(x+\varrho)}\right]^{-1} \end{cases} \quad \text{if} \quad \delta\ll|m(k-3m)|^{1/2},$$

where $k=X+iY$.

An easy but tedious computation then implies

$$\begin{aligned} &\left|x-\frac{\mathrm{Re}\,\mathbf{p}(k)}{2}\right|>O(\delta-\delta') && \text{if} \quad y<\delta', \\ &y>\delta/5 && \text{if} \quad -\frac{\delta}{7}<x<\frac{\mathrm{Re}\,\mathbf{p}(k)}{2}, \\ &y<\mathrm{Im}\,\mathbf{p}(k) && \text{if} \quad \frac{\mathrm{Re}\,\mathbf{p}(k)}{2}<x<\mathbf{p}_0, \end{aligned}$$

which completes the proof of Lemma 2.1.

End of the Proof of Proposition 2.1. If k is not in the concavity of the curve (2.26), i.e. for $\mathrm{Im}\,k$ sufficiently large, the bound (2.19) is obvious because any contour in the strip $|\mathrm{Im}\,\mathbf{p}|<\delta$ remains in the first sheet with respect to the singularities of f, so that the problem reduces to a two-body problem.

Let now k be in the concavity of the curve (2.26); we can assume in addition that it is either in the lower half plane of the first sheet or in the upper half plane of the second sheet (the other case can be treated similarly). We now separate the two poles in (2.23), which yields

$$\frac{i\pi}{z_3}\int_{\mathrm{Re}\,\omega(\mathbf{p}_1')<M} d\mathbf{p}_1'\,\tilde{f}(\mathbf{p}_1',p_3)\left(\frac{1}{\mathbf{p}_1'+\dfrac{\mathbf{p}_3}{2}-z_3}-\frac{1}{\mathbf{p}_1'+\dfrac{\mathbf{p}_3}{2}+z_3}\right), \tag{2.27}$$

where

$$\tilde{f}(\mathbf{p}_1',p_3)=\frac{f((i\omega(\mathbf{p}_1'),\mathbf{p}_1'),(ik-p_3^0-i\omega(\mathbf{p}_1'),-\mathbf{p}_1'-\mathbf{p}_3),p_3)}{2i\omega(\mathbf{p}_1')(ik-p_3^0-i\omega(\mathbf{p}_1')+i\omega(-\mathbf{p}_1'-\mathbf{p}_3))}. \tag{2.28}$$

According to Lemma 2.1, we can define in the domain restricted by (2.24) two contours which remain at a distance of order $\delta-\delta'$ from each other except in a neighborhood of $\mathbf{p}=0$ where they meet at a given angle, e.g. $\frac{\pi}{2}$ (see Fig. 2). We can impose in addition that both contours be symmetric under $\mathbf{p}\to-\mathbf{p}$. For each term in (2.27), we then choose the contour which is the further from the corresponding pole. This may produce a residue which can be shown to satisfy the bound (2.19). The case of a pole at $\mathbf{p}=0$ is treated by a continuity argument. We are left with an integral of the form

$$\int_C \frac{\tilde{h}(\mathbf{p})+\mathbf{p}\tilde{g}(\mathbf{p})}{\mathbf{p}-a}\,d\mathbf{p}, \tag{2.29}$$

with $\tilde{h}$ and $\tilde{g}$ even in $\mathbf{p}$, analytic in $D(k) \cap \{\mathrm{Re}\,\omega(\mathbf{p}) < 2(m-\varepsilon)\}$ and bounded by a constant times $\frac{|f|_1}{\varepsilon}$. The factor ε corresponds to the minimum distance to $D(k) \cap \{\mathrm{Re}\,\omega(p) < 2(m-\varepsilon)\}$ of the zeros of the denominator in (2.28). The contour C being symmetric under $\mathbf{p} \to -\mathbf{p}$, (2.29) equals

$$a \int_C \frac{\tilde{h}(\mathbf{p})}{\mathbf{p}^2 - a^2} d\mathbf{p} + \int_C \tilde{g}(\mathbf{p})\, d\mathbf{p} + a^2 \int_C \frac{\tilde{g}(\mathbf{p})}{\mathbf{p}^2 - a^2} d\mathbf{p}.$$

The poles at $\mathbf{p} = \pm a$ either lie at a distance of order $\delta - \delta'$ (or more) from the contour (the bound is then obvious) or at a distance $\frac{a}{\sqrt{2}}$ (or more) from the contour. The bound for this last case follows by the change of variable $\mathbf{p} = |a|\mathbf{q}$. This completes the proof of Proposition 2.1.

The main result of this section is the following:

Theorem 2.1. *For any α and β, $K_2^\alpha R_2^\alpha$ is a bounded operator from $\mathscr{A}_\beta$ to $\mathscr{A}_\alpha$. More precisely for all f in $\mathscr{A}_\beta$:*

$$|(K_2^\alpha R_2^\alpha f)(p_1, p_2, p_3)| < c \left| \frac{\lambda d^2(\kappa_\alpha)}{1 + \lambda d^2(\kappa_\alpha)} \right| |f|_\beta,$$

where

$$\kappa_\alpha = (ik, 0) - (p_\alpha^0, \mathbf{p}_\alpha), \tag{2.30}$$

$$d^2(\kappa) = \int dp R_{02}(p; \kappa) = 2 \int dp R_1 \left(\frac{\kappa + p}{2}\right) R_1 \left(\frac{\kappa - p}{2}\right). \tag{2.31}$$

Proof. We first note that it follows from the proof of Proposition 2.1 that $K_2^\alpha R_{02}^\alpha f$ is analytic on $\mathscr{D}_\alpha$, because $K_2(p_i, p_j; p_i', p_j')$ is analytic on

$$\mathscr{D}(k) \times [\mathscr{D}(k) \cap (\{\mathrm{Im}\, p_i^0 > m - \varepsilon, \mathrm{Im}\, p_j^0 > m - \varepsilon\} \cup \{\mathrm{Im}\, p_i^0 < \mathrm{Re}\, k - 2(m-\varepsilon),$$
$$\mathrm{Im}\, p_j^0 < \mathrm{Re}\, k - 2(m - \varepsilon)\})].$$

Note that the integration variables p_1' and p_2' in the proof of Proposition 2.1 always lie in the second factor of the above product. We also note that the "anti-bound state," i.e. the pole of the operator $R_2^{12}(\kappa_3)$ in the second sheet of the variable κ_3 located at $2m - \frac{9}{4}\frac{\lambda^2}{m} + O(\lambda^3/m^2)$, does not belong to $\mathscr{D}_3$, due to the condition $\delta < \lambda$ [we recall that we have chosen the coefficient of Φ^4 in $P(\Phi)$ equal to $+1$].

Following Spencer and Zirilli [21], we now split the Bethe Salpeter kernel K_2 into a dominant repulsive part and a remainder

$$K_2 = -\lambda K_1' + K_2', \tag{2.32}$$

where K_2' is at least of order λ^2 and where K_1' is a positive constant equal to the coefficient of Φ^4 in $P(\Phi)$. Then if

$$R_2' = (R_{02}^{-1} + \lambda K_1')^{-1}, \tag{2.33}$$

the Bethe Salpeter equation is solved by

$$R_2 = R'_2(1-K'_2R'_2)^{-1}$$
$$= R'_2 + R'_2(1-K'_2R'_2)^{-1}K'_2R'_2 . \tag{2.34}$$

Therefore it is enough to prove that $\lambda K_1'^{\alpha}R_2'^{\alpha}$ and $K_2'^{\alpha}R_2'^{\alpha}$ are bounded operators from $\mathscr{A}_\beta$ to $\mathscr{A}_\alpha$ and that the norm of $K_2'^{\alpha}R_2'^{\alpha}$ as an operator in $\mathscr{A}_\alpha$ is $O(\lambda)$.

For $\beta=\alpha$, these statements are obvious consequences of the 2-body analysis, because no other singularity than the two poles occurs in the integral (2.18) for $f\in\mathscr{A}_\beta$. Thus we only consider the more difficult case $\beta\neq\alpha$; for example $\beta=(23)$, $\alpha=(12)$.

Let $p=p_1-p_2$, $p'=p'_1-p'_2$ and κ_3 be as in (2.30). Then the kernel of R'_2 is:

$$R'_2(p,p';\kappa_3)=R_{02}(p,\kappa_3)\delta(p-p')-\frac{\lambda}{1+\lambda d^2(\kappa_3)}R_{02}(p,\kappa_3)R_{02}(p',\kappa_3) . \tag{2.35}$$

Thus

$$\lambda K_1'^{12}R_2'^{12}(p,p')=\frac{\lambda}{1+\lambda d^2(\kappa_3)}R_{02}(p,\kappa_3)\delta(p-p') .$$

But it follows easily from 2-body analysis [21] that the integral (2.18) is the only divergent part of

$$\int R_{02}(p,\kappa_3)f(p_1,p_2,p_3)dp$$

near the two-body threshold. Therefore Proposition 2.1 and the fact that $\lambda z_3'^{-1}(1+\lambda d^2(\kappa_3))^{-1}$ is uniformly bounded in $\mathscr{D}_3$ imply that $\lambda K_1'^{12}R_2'^{12}$ is bounded from $\mathscr{A}_1$ to $\mathscr{A}_3$.

Let us now consider $K_2'^{12}R_2'^{12}f$ for $f\in\mathscr{A}_1$. For points of $\mathscr{D}_3$ such that $z'_3>\delta$, it is easy to see that $K_2'^{12}R_{02}^{12}f$, and therefore $K_2'^{12}R_2'^{12}f$ is analytic and bounded. If $z'_3\leqq\delta$ (and thus $|z_3|\leqq\delta$), we use formula (2.35) for R'_2, and split K'_2 into a residue at one of the two poles of R_{02}, say $-\frac{\mathbf{p}_3}{2}+z_3$, plus a difference which vanishes at that pole:

$$\begin{aligned}
K_2'^{12}R_2'^{12}f=&\frac{1}{1+\lambda d^2(\kappa_3)}K'_2\left(\mathbf{p}_1,\mathbf{p}_2;-\frac{\mathbf{p}_3}{2}+z_3,-\frac{\mathbf{p}_3}{2}-z_3\right)\\
&\cdot\int d(p'_1-p'_2)R_{02}(p'_1-p'_2,\kappa_3)\\
&\cdot f(p'_1,p'_2,p_3)-\left\{c(p_1,p_2,p_3)-\frac{\lambda d^2(\kappa_3)}{1+\lambda d^2(\kappa_3)}\right.\\
&\left.K'_2\left(\mathbf{p}_1,\mathbf{p}_2;-\frac{\mathbf{p}_3}{2}+z_3,-\frac{\mathbf{p}_3}{2}-z_3\right)\right\}\\
&\cdot\int d(p'_1-p'_2)R_{02}(p'_1-p'_2;\kappa_3)f(p'_1,p'_2,p_3)\\
&+\int d(p'_1-p'_2)R_{02}(p'_1-p'_2;\kappa_3)\left\{K'_2(p_1,p_2;p'_1,p'_2)\right.\\
&\left.-K'_2\left(\mathbf{p}_1,\mathbf{p}_2;-\frac{\mathbf{p}_3}{2}+z_3,-\frac{\mathbf{p}_3}{2}-z_3\right)\right\}f(p'_1,p'_2,p_3) ,
\end{aligned}\tag{2.36}$$

where we have used the simplified notation:

$$K_2'(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_1', \mathbf{p}_2') \equiv K_2'(p_1, p_2; (i\omega(\mathbf{p}_1'), \mathbf{p}_1'), (p_1^0 + p_2^0 - i\omega(\mathbf{p}_1')), \mathbf{p}_2'), \quad (2.37)$$

and where $c(p_1, p_2, p_3)$ is given by

$$c(p_1, p_2, p_3) = \frac{\lambda}{1+\lambda d^2(\kappa_3)} \int d(p_1' - p_2') R_{02}(p_1' - p_2'; \kappa_3) K_2'(p_1, p_2; p_1', p_2'). \quad (2.38)$$

Now it is easy to see that the singular part of $\int d(p_1' - p_2') R_{02}(p_1' - p_2'; \kappa_3) f(p_1', p_2', p_3)$ is that given by Proposition 2.1, and that it is therefore bounded by $c|d^2(\kappa_3)|\,|f|_1$. For $|z_3| < \delta$, one has $\omega\left(-\frac{\mathbf{p}_3}{2} \pm z_3\right) \geqq m - \varepsilon$, so that the first term of (2.36) is analytic in $\mathscr{D}_3$ and is bounded by $c\frac{|K_2'|_\infty}{\lambda}|f|_1$.

On the other hand it follows from 2-body analysis that

$$(1+\lambda d^2(\kappa_3))^{-1} K_2'\left(\mathbf{p}_1, \mathbf{p}_2; -\frac{\mathbf{p}_3}{2} + z_3, -\frac{\mathbf{p}_3}{2} - z_3\right)$$

and

$$c(p_1, p_2, p_3) - \frac{\lambda d^2(\kappa_3)}{1+\lambda d^2(\kappa_3)} K_2'\left(\mathbf{p}_1, \mathbf{p}_2; -\frac{\mathbf{p}_3}{2} + z_3, -\frac{\mathbf{p}_3}{2} - z_3\right)$$

are analytic in $\mathscr{D}_3$ and that the latter is uniformly bounded by

$$c'\left|\frac{\lambda}{1+\lambda d^2(\kappa_3)}\right| |K_2'|_\infty .$$

Therefore the second term of (2.36) is analytic in $\mathscr{D}_3$ and bounded by $c'|K_2'|_\infty |f|_1$.

As in the beginning of the proof of Proposition 2.1, the last term can be decomposed into a possible residue at $p_1'^0 = i\omega(\mathbf{p}_1')$, plus an integral over $p_1'^0$ away from the poles, which does not give rise to a 2-body threshold, and is thereby bounded by $c''|K_2'|_\infty |f|_1$. The residue at $p_1'^0 = i\omega(\mathbf{p}_1')$ can be written

$$\int_{|\mathrm{Re}\,\omega(\mathbf{p}_1')| < M} d\mathbf{p}_1' \tilde{f}(\mathbf{p}_1', p_3) \cdot \frac{K_2'(\mathbf{p}_1, \mathbf{p}_2; +\mathbf{p}_1', -\mathbf{p}_1' - \mathbf{p}_3) - K_2'\left(\mathbf{p}_1, \mathbf{p}_2; -\frac{\mathbf{p}_3}{2} + z_3, -\frac{\mathbf{p}_3}{2} - z_3\right)}{\left(\mathbf{p}_1' + \frac{\mathbf{p}_3}{2}\right)^2 - z_3^2} \quad (2.39)$$

along a suitably chosen contour, where $\tilde{f}$ is given by (2.28). As in the proof of Proposition 2.1, this term is analytic in $\mathscr{D}_3$ and is bounded by $c'''|K_2'|_\infty |f|_1$.

So far we have proven that $K_2^{12} R_2'^{12} f$ has the desired analyticity properties in $\mathscr{D}_3$, and that it is bounded on $\mathscr{D}_3$. Now the bound for the odd part (2.17) of $K_2^{12} R_2'^{12} f$ will follow from the following lemma (for $\lambda K_1'^{12} R_2'^{12} f$, this odd part is zero):

Lemma 2.2. *Let* $(p_1, p_2, p_3) \in \mathscr{D}_3$ *with* $p_3^0 = i\omega(\mathbf{p}_3)$, *and let* f *belong to* $\mathscr{A}_1$. *Then*

$$\operatorname*{Sup}_{\mathscr{D}_3} |K_2'^{12} R_2'^{12} f((p_1^0, \mathbf{p}_1), (p_2^0, \mathbf{p}_2 + 2\mathbf{p}_3), (p_3^0, -\mathbf{p}_3)) - K_2'^{12} R_2'^{12} f(p_1, p_2, p_3)|$$

$$< c|\mathbf{p}_3|\,|f|_1 \frac{|K_2'|_\infty}{\lambda}. \tag{2.40}$$

Proof. We use decomposition (2.36) again. The integral in the first and second terms is invariant under $\mathbf{p}_3 \to -\mathbf{p}_3$. Thus the contribution of the first term to (2.40) is bounded by:

$$\left| K_2'\left(\mathbf{p}_1, \mathbf{p}_2 + 2\mathbf{p}_3 ; \frac{\mathbf{p}_3}{2} + z_3, \frac{\mathbf{p}_3}{2} - z_3\right) - K_2'\left(\mathbf{p}_1, \mathbf{p}_2 ; -\frac{\mathbf{p}_3}{2} + z_3, -\frac{\mathbf{p}_3}{2} - z_3\right)\right|$$

$$\cdot \left| \frac{d^2(\kappa_3)}{1 + \lambda d^2(\kappa_3)} \right| c|f|_1 < \tilde{c} \frac{|K_2'|_\infty}{\lambda} |\mathbf{p}_3|\,|f|_1 .$$

It follows from 2-body analysis that the first factor of the second term of (2.36) is $\lambda(1 + \lambda d^2(\kappa_3))^{-1}$ times a sum of integrals over suitable contours where the integrand is not singular, and it can therefore be differentiated with respect to $\mathbf{p}_3$ (at $\mathbf{p}_1$ fixed). Thus the contribution of this second term to (2.40) is bounded by $\tilde{c}' |K_2'|_\infty |\mathbf{p}_3|\,|f|_1$.

As previously, the third term is split into an integral over $p_1'^0$ having no 2-body threshold, thereby satisfying Lemma 2.2, and a residue at $p_1'^0 = i\omega(\mathbf{p}_1')$, whose contribution to (2.40) is:

$$\int \frac{d\mathbf{p}_1' d\mathbf{p}_2'}{\mathbf{p}_1' + \frac{\mathbf{p}_3}{2} + z_3} K_2''(\mathbf{p}_1, \mathbf{p}_2 + 2\mathbf{p}_3 ; \mathbf{p}_1', \mathbf{p}_2') \tilde{f}(\mathbf{p}_1', (i\omega(\mathbf{p}_3), -\mathbf{p}_3)) \delta(\mathbf{p}_1' + \mathbf{p}_2' - \mathbf{p}_3)$$

$$- \int \frac{d\mathbf{p}_1' d\mathbf{p}_2'}{\mathbf{p}_1' + \frac{\mathbf{p}_3}{2} + z_3} K_2''(\mathbf{p}_1, \mathbf{p}_2 ; \mathbf{p}_1', \mathbf{p}_2') \tilde{f}(\mathbf{p}_1', (i\omega(\mathbf{p}_3), \mathbf{p}_3)) \delta(\mathbf{p}_1' + \mathbf{p}_2' + \mathbf{p}_3)$$

$$= \int \frac{d\mathbf{p}_1' d\mathbf{p}_2'}{\mathbf{p}_1' + \frac{\mathbf{p}_3}{2} + z_3} \{ K_2''(\mathbf{p}_1, \mathbf{p}_2 + 2\mathbf{p}_3 ; \mathbf{p}_1', \mathbf{p}_2' + 2\mathbf{p}_3) \tilde{f}(\mathbf{p}_1', (i\omega(\mathbf{p}_3), -\mathbf{p}_3))$$

$$- K_2''(\mathbf{p}_1, \mathbf{p}_2 ; \mathbf{p}_1', \mathbf{p}_2') \tilde{f}(\mathbf{p}_1', (i\omega(\mathbf{p}_3), \mathbf{p}_3)) \} \, \delta(\mathbf{p}_1' + \mathbf{p}_2' + \mathbf{p}_3),$$

where K_2'' is defined by

$$\left(\mathbf{p}_1' + \frac{\mathbf{p}_3}{2} - z_3\right) K_2''(\mathbf{p}_1, \mathbf{p}_2 ; \mathbf{p}_1', \mathbf{p}_2') \equiv K_2'(\mathbf{p}_1, \mathbf{p}_2 ; \mathbf{p}_1', \mathbf{p}_2')$$

$$- K_2'\left(\mathbf{p}_1, \mathbf{p}_2 ; -\frac{\mathbf{p}_3}{2} + z_3, -\frac{\mathbf{p}_3}{2} - z_3\right).$$

Now this term can be treated similarly to (2.39) yielding a bound of the form $\tilde{c}''' |\mathbf{p}_3|\,|K_2'|_\infty |f|_1$, where the $\mathbf{p}_3$ factor comes from the analyticity of K_2'', and from the analyticity in $\mathbf{p}_2 - \mathbf{p}_3$ of the odd and even parts of f [decomposition (2.17)]. This completes the proof of Lemma 2.2.

III. Analyticity of the 2-Particle Irreducible 6 Point Function

In the Introduction [Eqs. (1.28)–(1.35)] we have already derived, at least formally, Faddeev type equations satisfied by the part R_3' of R_3 (the 2-particle irreducible 6-point function) constituted only from two body processes.

It is not difficult to make this derivation rigorous in the euclidean region, and to analytically continue the solution up to $\mathrm{Re}\,k = 3m - 2\varepsilon$. Indeed a tubular neighborhood of

$$\begin{aligned} &\mathrm{Im}\, p_j^0 = \tfrac{1}{3}\mathrm{Re}\,k\,, \\ &\mathrm{Im}\,\mathbf{p}_j = 0\,, \end{aligned} \qquad j = 1, 2, 3$$

remains free of any singularity in the whole range $|\mathrm{Re}\,k| < 3m - 2\varepsilon$. We shall now give precise definitions for the case $3(m-\varepsilon) < \mathrm{Re}\,k < 4(m-\varepsilon)$.

$$\tilde{\mathscr{A}} = \bigoplus_\alpha \mathscr{A}_\alpha \text{ is the set of vectors } \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \end{pmatrix}, \text{ where } \Phi_i \in \mathscr{A}_{(jl)} i \neq j, l.$$

We denote by $\|\cdot\|$ the norm of operators in $\mathscr{A}_\alpha$, $\mathscr{A}_0$ or $\tilde{\mathscr{A}}$ (the specification will be clear from the context). For suitable six-point functions $R(p_1, p_2, p_3, q_1, q_2, q_3)$, with a $\delta(p_1 + p_2 + p_3 - q_1 - q_2 - q_3)$ incorporated in it, we define the bilinear form $R(k)$ on $\mathscr{A}_0 \times \mathscr{A}_0$ by:

$$\begin{aligned} \langle f R(k) g \rangle = {} & \int \prod_1^3 (dp_j dq_j) \delta(p_1 + p_2 + p_3 - (ik, 0)) \\ & \cdot f(p_1, p_2, p_3) g(q_1, q_2, q_3) R(p_1, p_2, p_3, q_1, q_2, q_3)\,. \end{aligned} \tag{3.1}$$

Definition 3.1.

$$\begin{aligned} \mathscr{D} = \{ & (p_1, p_2, p_3) \in \mathbb{C}^6 : |\mathrm{Im}\,\mathbf{p}_j| < \delta, \mathrm{Re}\,k - 4(m-\varepsilon) < \mathrm{Im}\, p_j^0 \\ & < 3(m-\varepsilon), j = 1, 2, 3, p_1 + p_2 + p_3 = (ik, 0), k \in D\}\,. \end{aligned}$$

$\mathscr{A} = \{f : \mathbb{C}^5 \to \mathbb{C}$ analytic and bounded on $\mathscr{D}$, continuous on the closure of $\mathscr{D}$, symmetric with respect to permutations of $p_1, p_2, p_3\}$ with the norm $|f|_\infty$

$$\begin{aligned} \tilde{\mathscr{D}} = \Big\{ & (p_1, p_2, p_3) \in \mathbb{C}^6 : |\mathrm{Im}\,\mathbf{p}_j| < \delta, \mathrm{Re}\,k - 4(m-\varepsilon) < \mathrm{Im}\, p_j^0 \\ & < 3(m-\varepsilon), j = 1, 2, 3, \sum_1^3 \mathbf{p}_j = 0 \Big\}\,. \end{aligned}$$

$\tilde{\mathscr{A}} = \{f : \mathbb{C}^5 \to \mathbb{C}$ analytic and bounded on $\tilde{\mathscr{D}}$, continuous on the closure of $\tilde{\mathscr{D}}$, symmetric with respect to permutations of $p_1, p_2, p_3\}$ with the norm $|f|_\infty$.

It follows immediately from Theorem 2.1 that the operator A defined by (1.33) is a bounded operator in $\tilde{\mathscr{A}}$. Now we want (1) to study $(1-A)^{-1}$ as an operator in $\tilde{\mathscr{A}}$, and (2) to make sense of Eq. (1.34). This will be done by proving that each term in (1.34) is an analytic (in k) family of bounded bilinear forms in $\mathscr{A}_0 \times \mathscr{A}$. As regards the first aim, a result for k outside a neighborhood of order λ^2 of the three body threshold is given in Proposition 3.1 below, at least for an iterated form of (1.34). Given that, the second aim will follow from Theorem 3.1.

Proposition 3.1. *There exists some constant a such that if* $\lambda < a|k-3m|^{1/2}$, $\|A^3\| < 1$.

Proof. We first note that, from (2.32) and (2.34):

$$K_2R_2 = -\lambda K_1'R_2' + (1-\lambda K_1'R_2')(1-K_2'R_2')^{-1}K_2'R_2', \tag{3.2}$$

and therefore

$$\|K_2^\alpha R_2^\alpha + \lambda K_1'^\alpha R_2'^\alpha\|_{\mathscr{A}_\beta \to \mathscr{A}_\alpha} < O(\lambda). \tag{3.3}$$

Thus it is enough to prove that

$$\|(\lambda K_1'^\alpha R_2'^\alpha)(\lambda K_1'^\beta R_2'^\beta)(\lambda K_1'^\gamma R_2'^\gamma)\|_{\mathscr{A}_{(il)} \to \mathscr{A}_\alpha} < \left(1 + b\frac{|k-3m|^{1/2}}{\lambda}\right)^{-1} \tag{3.4}$$

for some $b>0$, and any $\alpha \neq \beta \neq \gamma \neq (il)$. But [e.g. $\alpha=(12)$, $\beta=(23)$, $\gamma=(12)$]

$$\lambda K_1'^{12} R_2'^{12} f = \lambda(1+\lambda d^2(\kappa_3))^{-1} R_{02}^{12} f, \tag{3.5}$$

and thus by Theorem 2.1

$$|\lambda K_1'^{12} R_2'^{12} f| < |\lambda d^2(\kappa_3)(1+\lambda d^2(\kappa_3))^{-1}| \, |f|_{(il)}. \tag{3.6}$$

When we apply $\lambda K_1'^{23} R_2'^{23}$ to this first result, we see that the second term in (2.19) is bounded by $c\left(1+d\frac{|k-3m|^{1/2}}{\lambda}\right)^{-1} |f|_{(il)}$. This bound is preserved by applying the third factor $(\lambda K_1'^{12} R_2'^{12})$.

The contribution of the first term in (2.19) to $(\lambda K_1'^{23} R_2'^{23})(\lambda K_1'^{12} R_2'^{12})f$ is bounded by

$$\left|\frac{\lambda d^2(\kappa_1)}{1+\lambda d^2(\kappa_1)}\right| \int d\mathbf{p}\,\delta(k+ip_1^0-\omega(\mathbf{p})-\omega(-\mathbf{p}-\mathbf{p}_1)) \left|\frac{\lambda d^2(k-\omega(\mathbf{p}),\mathbf{p})}{1+\lambda d^2(k-\omega(\mathbf{p}),\mathbf{p})}\right| |f|_{(il)}. \tag{3.7}$$

Applying the third factor $(\lambda K_1'^{12} R_2'^{12})$ gives an $O(\lambda)$ contribution plus a residue at $p_1^0 = i\omega(\mathbf{p}_1)$. The contribution of (3.7) to

$$\|\lambda K_1'^{12} R_2'^{12} (\lambda K_1'^{23} R_2'^{23})(\lambda K_1'^{12} R_2'^{12})\|_{\mathscr{A}_{(il)} \to \mathscr{A}_{(12)}}$$

is then bounded by

$$\sup_{\mathbf{p}_1} \left|\frac{1}{1+\frac{1}{\lambda}d^{-2}(k-\omega(\mathbf{p}_1),\mathbf{p}_1)}\right| \int d\mathbf{p}\,\delta(k-\omega(\mathbf{p}_1)-\omega(\mathbf{p})-\omega(-\mathbf{p}-\mathbf{p}_1))$$
$$\cdot \left|\frac{1}{1+\frac{1}{\lambda}d^{-2}(k-\omega(\mathbf{p}),\mathbf{p})}\right| |f|_{(il)}. \tag{3.8}$$

The function outside the integral equals one at $\mathbf{p}_1 = \pm\mathbf{p}(k)$ and is small away from these points. The integral equals one at $\mathbf{p}_1 = \pm\frac{\mathbf{p}(k)}{2}$ and is small away from these points. It is easy to show that the product is small away from

$$\mathbf{p}(k) = \frac{\mathbf{p}(k)}{2} = 0.$$

More precisely, (3.8) is bounded by $\left(1+\frac{b}{\lambda}\sqrt{|k-3m|}\right)^{-1}$, which completes the proof of Proposition 3.1.

Proposition 3.2. *For any α and β, $(R_2^\alpha \otimes R_1)(k)$ is a bounded family of bilinear forms on $\mathscr{A}_0 \times \mathscr{A}_\beta$, and on $\mathscr{A}_0 \times \mathscr{A}_0$, analytic for $k \in D$. More precisely there exist positive constants c and d such that for any $f \in \mathscr{A}_\beta$, $g \in \mathscr{A}_0$, $k \in D$*

$$|\langle g(R_2^\alpha \otimes R_1)(k) f\rangle| < \{|\mathrm{Log}(c|k-3m|^{1/2}+\lambda)|+d\}|f|_\beta |g|_\infty . \tag{3.9}$$

The same holds for $f, g \in \mathscr{A}_0$, with $|f|_\beta$ replaced by $|f|_\infty$.

Proof. We only give the proof for $f \in \mathscr{A}_1$, $\alpha=(12)$, $g \in \mathscr{A}_0$ and R_2 replaced by R_2' [the cases $f, g \in \mathscr{A}_0$, and $f \in \mathscr{A}_\alpha$, $g \in \mathscr{A}_0$ are easy consequences of 2-body analysis, and replacement of R_2' by R_2 will follow immediately by using (2.34) and Theorem 2.1]. Using (2.35) we have

$$\begin{aligned}\langle g(R_2'^{12} \otimes R_1)(k) f\rangle = & \int dp_1 dp_2 dp_3 \delta(p_1+p_2+p_3-(ik,0)) \frac{(fg)(p_1,p_2,p_3)}{(p_1^2+m^2)(p_2^2+m^2)(p_3^2+m^2)} \\ & + \int \frac{dp_3}{p_3^2+m^2} \frac{\lambda}{\lambda d^2(\kappa_3)+1} \\ & \cdot\left(\int dp_1 dp_2 \frac{g(p_1,p_2,p_3)}{(p_1^2+m^2)(p_2^2+m^2)} \delta(p_1+p_2+p_3-(ik,0))\right) \\ & \cdot\left(\int dp_1' dp_2' \frac{f(p_1',p_2',p_3)}{(p_1'^2+m^2)(p_2'^2+m^2)} \delta(p_1'+p_2'+p_3-(ik,0))\right),\end{aligned} \tag{3.10}$$

where we have omitted a factor $1+O(\lambda^2)$ for the field strength renormalization, and a remainder (coming from Lehmann formula) obviously bounded by $c|f|_\infty |g|_\infty$. We now prove (3.9). The first term of (3.10) can be rewritten as

$$\begin{aligned}& \int \frac{dp_3}{p_3^2+m^2} g(i\omega(\mathbf{p}_1), \mathbf{p}_1; i\omega(\mathbf{p}_2), \mathbf{p}_2; p_3) \int dp_1 dp_2 \delta(p_1+p_2+p_3-(ik,0)) \\ & \cdot \frac{f(p_1,p_2,p_3)}{(p_1^2+m^2)(p_2^2+m^2)} \\ & + \int \frac{dp_3}{p_3^2+m^2} \int \frac{dp_1 dp_2 f(p_1,p_2,p_3)}{(p_1^2+m^2)(p_2^2+m^2)} \delta(p_1+p_2+p_3-(ik,0)) \\ & \cdot \{g(p_1,p_2,p_3) - g(i\omega(\mathbf{p}_1), \mathbf{p}_1; i\omega(\mathbf{p}_2), \mathbf{p}_2; p_3)\} .\end{aligned} \tag{3.11}$$

But it follows from 2-body analysis and Proposition 2.1 that the second term of (3.10) equals

$$\begin{aligned}& \int \frac{dp_3}{p_3^2+m^2} \frac{\lambda d^2(\kappa_3)}{1+\lambda d^2(\kappa_3)} g(i\omega(\mathbf{p}_1), \mathbf{p}_1; i\omega(\mathbf{p}_2), \mathbf{p}_2; p_3) \\ & \cdot \int dp_1 dp_2 \delta(p_1+p_2+p_3-(ik,0)) f(p_1,p_2,p_3)(p_1^2+m^2)^{-1}(p_2^2+m^2)^{-1}\end{aligned} \tag{3.12}$$

plus a remainder that is uniformly bounded by $c|f|_1 |g|_\infty$. Now the second term in (3.11) can be rewritten as

$$\int \frac{dp_1 d^2(\kappa_1)}{p_1^2+m^2} [f(p_1,p_2,p_3)\{g(p_1,p_2,p_3)-g(i\omega(\mathbf{p}_1),p_1;i\omega(\mathbf{p}_2),p_2;p_3)\}]\Big|_{\substack{p_2^0=i\omega(\mathbf{p}_2)\\ p_3^0=i\omega(\mathbf{p}_3)}} \tag{3.13}$$

plus a remainder uniformly bounded by $c|f|_\infty |g|_\infty$. Now from analyticity properties of g, we have

$$\begin{aligned} &g(p_1;i\omega(\mathbf{p}_2),\mathbf{p}_2;i\omega(\mathbf{p}_3),\mathbf{p}_3)-g(i\omega(\mathbf{p}_1),\mathbf{p}_1;i\omega(\mathbf{p}_2),\mathbf{p}_2;i\omega(\mathbf{p}_3),\mathbf{p}_3)\\ &\quad=(p_1^0-i\omega(\mathbf{p}_1))g'(p_1;i\omega(\mathbf{p}_2),\mathbf{p}_2;i\omega(\mathbf{p}_3),(\mathbf{p}_3),\end{aligned}$$

where g' is analytic in a section of $\mathscr{D}_0$, and $|g'|_\infty < c|g|_\infty$. Thus the integration contour in (3.13) can be shifted to $p_1^0 = k-2(m-\varepsilon)$, which yields a bound $c|f|_\infty |g|_\infty$. Now the first term of (3.11) combined with (3.12) equals

$$\begin{aligned} &\int \frac{dp_3}{p_3^2+m^2}\frac{1}{1+\lambda d^2(\kappa_3)} g(i\omega(\mathbf{p}_1),\mathbf{p}_1;i\omega(\mathbf{p}_2),\mathbf{p}_2;p_3)\\ &\quad\cdot\int dp_1dp_2\delta(p_1+p_2+p_3-(ik,0))f(p_1,p_2,p_3)(p_1^2+m^2)^{-1}(p_2^2+m^2)^{-1},\end{aligned}$$

which is norm bounded by

$$\int_{\mathrm{Re}\,\omega(\mathbf{p}_3)<\mathrm{Re}\,k-2(m-\varepsilon)} d\mathbf{p}_3|(2\omega(\mathbf{p}_3))^{-1}\tilde{g}(i\omega(\mathbf{p}_3),\mathbf{p}_3)|, \tag{3.14}$$

with

$$|\tilde{g}(i\omega(\mathbf{p}_3),\mathbf{p}_3)| < (\lambda+|k-\omega(\mathbf{p}_3)-\mu(\mathbf{p}_3)|^{1/2})^{-1}|g|_\infty |f|_1 .$$

The bound (3.9) follows easily after choosing a suitable contour in (3.14).

Let now D' be the connected part of $D\backslash\{k||k-3m|^{1/2}\leqq\lambda/a\}$ which meets the first sheet. We have proven:

Theorem 3.1. *For λ sufficiently small, $R_3'(k)$ given by Eq.* (1.34) *is a bounded family of bilinear forms on $\mathscr{A}\times\mathscr{A}_0$ analytic in D'. Furthermore if $f\in\mathscr{A}_0$, $g\in\mathscr{A}$, there exist positive constants c and d such that*

$$|\langle gR_3'(k)f\rangle| < \{|\mathrm{Log}(c|k-3m|^{1/2}+\lambda)|+d\}|f|_\infty |g|_\infty . \tag{3.15}$$

Remark 3.1. For this result we have used Theorem 2.1 and the fact that $\mathscr{A}\subset\mathscr{A}_\alpha$; this implies that, while the first term in (1.34) is a good bilinear form on $\mathscr{A}_0\times\mathscr{A}_0$, the second is only a bilinear form on $\mathscr{A}_0\times\mathscr{A}$. Actually one could have gone through the whole program of this paper with $\mathscr{D}_\alpha$ replaced by a neighborhood of the triangle of cuts, i.e. $\mathrm{Im}\,p_i^0<\mathrm{Re}\,k-2(m-\varepsilon)$. The analog of Theorem 3.3 would have been less satisfactory because $\mathscr{D}_\alpha$ does not have the natural boundaries (1.8) and (1.9). However it is easy to check in this frame that $K_2^\alpha R_2^\alpha$ is a bounded operator from $\mathscr{A}_0$ to the modified $\mathscr{A}_\alpha$, so that R_3' and R_3 are bounded families of bilinear forms on $\mathscr{A}_0\times\mathscr{A}_0$ for $|k-3m|>\lambda^2/a^2$, which is a stronger version of Theorems 3.1 and 3.2.

Now the full 2-particle irreducible six-point function R_3 is obtained by a further easy step, using Eq. (1.35) and Proposition 3.3 below:

Proposition 3.3. *Let $\mathscr{A}'_0$ and $\mathscr{A}'$ be defined in analogy with $\mathscr{A}_0$ and $\mathscr{A}$ in Definitions 2.2 and 3.1 respectively, but with D replaced by D'. Then $K'_3R'_3$ is a bounded operator in $\mathscr{A}'$ or in $\mathscr{A}'_0$ satisfying*

$$\|K_3R'_3\|_{\substack{\mathscr{A}'_0\to\mathscr{A}'_0\\ \mathscr{A}'\to\mathscr{A}'}}<c|\mathrm{Log}\lambda|\,|K_3|_\infty \tag{3.16}$$

$$\left(\text{where } |K_3|_\infty = \sup_{\substack{p_i\in\mathscr{D}(k)\\ p'_i\in\mathscr{D}_0(k)\\ 3(m-\varepsilon)<\mathrm{Re}\,k<4(m-\varepsilon)}} |K_3(p_1,p_2,p_3;p'_1,p'_2,p'_3)|\right).$$

Proof. This follows immediately from the fact that K_3 is analytic in $\mathscr{D}(k)\times\mathscr{D}_0(k)$ [see (1.16) and (1.17)] and from (3.15).

Theorem 3.2. *For λ sufficiently small, $R_3(k)$ is a bounded family of bilinear forms on $\mathscr{A}\times\mathscr{A}_0$ analytic in D'. More precisely for any $f\in\mathscr{A}_0$, $g\in\mathscr{A}$, there exist positive constants c and d such that*

$$|\langle gR_3(k)f\rangle|<(|\mathrm{Log}(c|k-3m|^{1/2}+\lambda)|+d)|f|_\infty\,|g|_\infty\,. \tag{3.17}$$

Proof. $|K_3|_\infty$ is at least of order λ, and thus the norm of $K_3R'_3$ is small when λ is sufficiently small. This allows us to invert the operator $(1-K_3R'_3)$ in $\mathscr{A}'$ or in $\mathscr{A}'_0$, and to obtain

$$R_3=R'_3(1-K_3R'_3)^{-1}\,. \tag{3.18}$$

Theorem 3.2 then follows from Theorem 3.1.

We can now study the analytic structure of $R_{03}^{-1}R_3$ (the full 2-particle irreducible six point function amputated on its left).

Theorem 3.3. *Let $\mathscr{A}'_\alpha$ be defined in analogy with $\mathscr{A}_\alpha$ in Definition 2.1 with D replaced by D'. Then (see* (1.30), (1.35)*)*

$$R_{03}^{-1}R_3\equiv\left(1+\sum_\alpha M_\alpha\right)(1-K_3R'_3)^{-1}$$

is a bounded operator from $\mathscr{A}$ to $\sum_\alpha\mathscr{A}'_\alpha$ (i.e. for any $f\in\mathscr{A}$, $R_{03}^{-1}R_3f$ is a sum of terms, each term being analytic in some $\mathscr{A}'_\alpha$).

The proof is an immediate consequence of Theorem 2.2, of Propositions 3.1 and 3.3 and of Eq. (1.32').

IV. Asymptotic Completeness in the Three Body Region

In this section we prove asymptotic completeness for the models under consideration, in the center of mass frame and on the odd subspace of energy between $3m+a(\lambda)$ and $4m-b(\lambda)$, denoted $\mathscr{H}^{\mathrm{odd}}_{]3m+a(\lambda),4m-b(\lambda)[}$ [we recall that $a(\lambda)$ and $b(\lambda)$ are positive and tend to zero with λ]. Asymptotic completeness for all states with *mass* in the same interval will follow by Lorentz invariance. We first express asymptotic completeness on $\mathscr{H}^{\mathrm{odd}}_{]3m+a(\lambda),4m-b(\lambda)[}$ under the form of "asymptotic completeness relations" satisfied by a set of irreducible functions, and then deduce these relations from the analyticity properties of R_3 established in Sect. 3. The first step is a rather standard job in quantum field theory [1, 2, 22], but we include it in the beginning of this section for the sake of completeness.

Theorem 4.1. *Asymptotic completeness on* $\mathscr{H}^{\text{odd}}_{]3m+a(\lambda),4m-b(\lambda)[}$ *for* $\lambda P(\Phi)_2$ *models is implied by the set of relations* (1.4)–(1.6) *for any* $k\in]3m+a(\lambda), 4m-b(\lambda)[$ *and any functions* f *and* g *in* $\tilde{\mathscr{A}}$ *(see Definition 3.1).*

Remark 4.1. Not the whole analytic structure of R_3 is needed at this stage. However some regularity properties of $\langle fR_3(k)g\rangle_\pm$ as boundary values of analytic functions for k on the real axis will be used (which imply in particular the absence of CDD zeros). This is the reason why, even for this step, we restrict our attention to energies in $]3m+a(\lambda), 4m-b(\lambda)[$, although Theorem 4.1 is true on $\mathscr{H}^{\text{odd}}_{]0,5(m-\varepsilon)[}$ (under rather weak assumptions, but we do not want to go into more details here).

Proof of Theorem 4.1. It is easy to see that only particles of mass m will be present in this range of energy (three body bound states near $3m$, if any, will of course not contribute). Therefore we introduce a complete orthonormal basis $\{h_l(\mathbf{p})\}$ in $L^2(\mathbb{R}, d\mathbf{p}/2\omega(\mathbf{p}))$

$$\int(2\omega(\mathbf{p}))^{-1}\overline{h_l(\mathbf{p})}h_{l'}(\mathbf{p})d\mathbf{p}=\delta_{ll'}, \tag{4.1}$$

$$\sum_l \bar{h}_l(\mathbf{p})h_l(\mathbf{q})(4\omega(\mathbf{p})\omega(\mathbf{q}))^{-1/2}=\delta(\mathbf{p}-\mathbf{q}), \tag{4.2}$$

and an associated orthonormal basis on $\mathscr{H}^{\text{ex}}$

$$n_N^{\text{ex}}=\prod_l (n_l^N!)^{-1/2}(A_{h_l}^{\text{ex}*})^{n_l^N}\Omega, \tag{4.3}$$

where $n_N=(n_l^N)_l$, $N=\sum_l n_l^N<\infty$, "ex" denotes either "incoming" or "outgoing", Ω is the physical vacuum and $A_{h_l}^{\text{ex}*}$ is the creation operator for a free asymptotic particle of mass m with wave function h_l. Then asymptotic completeness on $\mathscr{H}_I^{\text{odd}}$ (where I denotes the interval $]3m+a(\lambda), 4m-b(\lambda)[$) reads

$$E_I=E_I\sum_{n_3} n_3^{\text{ex}}\rangle\langle n_3^{\text{ex}}E_I, \tag{4.4}$$

where E_I is the spectral projector of the energy operator on the interval I (at zero total momentum).

Now, following a result due to Glimm et al. [16], (note that appealing to this result is not necessary but allows for a simplified formulation) the closure of the span of

$$\mathscr{F}=\{\Phi_0(f_0)\Omega, e^{itP^0}\Phi_0(f_3)\Phi_0(f_2)\Phi_0(f_1)\Omega, t\in\mathbb{R}, f_j\in\mathscr{C}_0^\infty(\mathbb{R}, d\mathbf{x})\} \tag{4.5}$$

contains $\mathscr{H}^{\text{odd}}_{]0,5(m-\varepsilon(\lambda))[}$ for λ sufficiently small and $\Phi_0(f)$ the physical time zero field, smeared by the function f. Thus asymptotic completeness on $\mathscr{H}_I^{\text{odd}}$ is equivalent to:

$$\langle\theta, E_{I'}\theta'\rangle=\sum_{n_3}\langle\theta, E_{I'}n_3^{\text{ex}}\rangle\langle n_3^{\text{ex}}, E_{I'}\theta'\rangle \tag{4.6}$$

for any θ and θ' in $\mathscr{F}$, and any interval $I' \subset I$, where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathscr{H}$. We now use the "reduction formulae" ([12], see also [1, 3, 22]) to rewrite the right hand side of (4.6). (Note that the "reduction formulae" usually written in terms of the so-called "time ordered expectation values" of the axiomatic QFT can equivalently be expressed in terms of analytic continuations of momentum space Schwinger functions [11].) Thus taking

$$\theta = e^{itP^0}\Phi_0(\bar{f}_3)\Phi_0(\bar{f}_2)\Phi_0(\bar{f}_1)\Omega, \theta' = \text{idem}(t \to t', \bar{f}_j \to g_j),$$

(4.6) reads

$$\int_{k^0 \in I'} e^{ik^0(t'-t)} d\langle \Omega, \Phi_0(f_1)\Phi_0(f_2)\Phi_0(f_3)E(k^0,0)\Phi_0(g_3)\Phi_0(g_2)\Phi_0(g_1)\Omega\rangle$$

$$= 3! \sum_{l_1,l_2,l_3} i^6(2\pi Z)^3 \int_{I'} dk^0 e^{+ik^0t'} \prod_1^3 (dp_j h_{l_j}(\mathbf{p}_j)\delta_+(p_j))\delta\left(\tilde{k} - \sum_1^3 p_j\right)$$

$$\cdot \int (R_{03}^{-1}\,{}^TS_{3,3})_\pm(p_1,p_2,p_3,q_1,q_2,q_3) \prod_1^3 (dq_j g_j(\mathbf{q}_j)) \times \overline{\text{Idem}}(t' \to t, g_j \to \bar{f}_j), \quad (4.7)$$

where

$$\delta_+(p) = \delta(p^2+m^2)\theta(\operatorname{Im} p^0), \quad (4.8)$$

$$\theta(x) = \frac{x+|x|}{2|x|}, \quad (4.9)$$

$$\tilde{k} = (ik^0, 0), \quad (4.10)$$

upper bar denotes complex conjugation, $dE(k^0, \mathbf{k})$ is the spectral resolution of the energy momentum operator,

$$^TS_{3,3}(p_1,p_2,p_3,q_1,q_2,q_3) = S_6(p_1,p_2,p_3,q_1,q_2,q_3)$$
$$- S_2(p_1,p_2)S_4(p_3,q_1,q_2,q_3) - \text{perm}(p_i), \quad (4.11)$$

$$(R_{03}^{-1}\,{}^TS_{3,3})_\pm(p_1,p_2,p_3,q_1,q_2,q_3) = \lim_{\eta \downarrow 0} (R_{03}^{-1}\,{}^TS_{3,3})(p_1 \pm (\eta,0), p_2 \pm (\eta,0), p_3 \pm (\eta,0),$$
$$q_1 \pm (\eta,0), q_2 \pm (\eta,0), q_3 \pm (\eta,0)), \quad (4.12)$$

where all arguments are in the first sheet with respect to all variables.

But using (4.2) (namely completeness of the basis $\{h_l\}_l$), the right hand side of (4.7) reads:

$$\int_{I'} dk^0 e^{ik^0(t'-t)} 3!(2i\pi Z)^3 \int \prod_1^3 (dp_j \delta_+(p_j)) \int \prod_1^3 (dq_j dq'_j f_j(\mathbf{q}_j) g_j(\mathbf{q}'_j))$$

$$\cdot (R_{03}^{-1}\,{}^TS_{3,3})_\pm(p_1,p_2,p_3;q_1,q_2,q_3)$$

$$\cdot (R_{03}^{-1}\,{}^TS_{3,3})_\mp(p_1,p_2,p_3;q'_1,q'_2,q'_3)\delta\left(\tilde{k} - \sum_1^3 p_j\right). \quad (4.13)$$

Note that $\pm$ subscripts in (4.7) correspond to "ex" $=\binom{\text{out}}{\text{in}}$. Now (4.13) holds for any t and t', any f_j, g_j, and any $I' \subset I$. Therefore for almost all k^0 in I, we have

$$\frac{d}{dk^0}\langle \Omega, \Phi_0(f_1)\Phi_0(f_2)\Phi_0(f_3)E(k^0,0)\Phi_0(g_3)\Phi_0(g_2)\Phi_0(g_1)\Omega\rangle$$
$$=3!(2i\pi Z)^3 \int \prod_1^3 (dp_j\delta_+(p_j))\left((R_{03}^{-1}{}^TS_{3,3})_\pm\left(\bigotimes_1^3 f_j\right)\right)(p_1,p_2,p_3)$$
$$\cdot\delta\left(\tilde{k}-\sum_1^3 p_j\right)\left((R_{03}^{-1}{}^TS_{3,3})_\mp\left(\bigotimes_1^3 g_j\right)\right)(p_1,p_2,p_3), \qquad (4.14)$$

where the right hand side is continuous in k^0 for $k^0 \in I$ [continuity follows from the analyticity properties of R_3 established in Sect. 3, and from Eqs. (1.1), (1.3), (1.18), and (1.19) connecting Green's functions to irreducible functions]. For simplicity we write the right hand side of (4.14) as

$$\left\langle\left(\bigotimes_1^3 f_j\right)(S_{3,3}^T)^{\mp} *^{(3)}({}^TS_{3,3})^{\pm}\left(\bigotimes_1^3 g_j\right)\right\rangle(k^0).$$

Remark 4.2. We shall use the fact that the left hand side of (4.14) equals almost everywhere a continuous function of k^0 when we want to relate it to the discontinuity along the three body cut of $\left\langle\left(\bigotimes_1^3 f_j\right)S_6(k^0)\left(\bigotimes_1^3 g_j\right)\right\rangle$. This property looks similar to that used by Bros [1] as an extra postulate in the proof of the equivalence between two body asymptotic completeness and suitable analyticity properties of the Bethe-Salpeter kernel. Actually it is only needed as an extra assumption when going from asymptotic completeness to analyticity properties, whereas (as in the present paper) it appears as a subproduct of the analysis in the converse direction.

We now proceed with the proof of Theorem 4.1. Assume $\tilde{k}=(ik^0,0)\in\mathbb{R}^2$. Then

$$\int \prod_{j=1}^3 (dp_j dq_j \hat{f}_j(\mathbf{p}_j)\hat{g}_j(\mathbf{q}_j))S_6(p_1,p_2,p_3,q_1,q_2,q_3)\delta\left(\tilde{k}-\sum_1^3 p_j\right)$$
$$=\frac{1}{2\pi}\int d\tau e^{i\tau\cdot\tilde{k}}\langle\Omega,\Phi_0(f_1)\Phi_0(f_2)\Phi_0(f_3)e^{-|\tau^0|P^0+i\tau\cdot\mathbf{P}}\Phi_0(g_3)\Phi_0(g_2)\Phi_0(g_1)\Omega\rangle$$
$$=\frac{1}{2\pi}\int\delta(\mathbf{k}')\left\{\frac{1}{k'^0-k^0}+\frac{1}{k'^0+k^0}\right\}d\langle\Omega,\Phi_0(f_1)\Phi_0(f_2)\Phi_0(f_3)$$
$$\cdot E(k'^0,\mathbf{k}')\Phi_0(g_3)\Phi_0(g_2)\Phi_0(g_1)\Omega\rangle \qquad (4.15)$$

has an analytic continuation to each of the half planes $\mathrm{Im}\,k^0 \lessgtr 0$, and the discontinuity with respect to k^0 along the interval I

$$\Delta\left\langle\left(\bigotimes_1^3 f_j\right)S_6(k^0)\left(\bigotimes_1^3 g_j\right)\right\rangle \equiv \lim_{\eta\downarrow 0}\left\{\int\prod_{j=1}^3(dp_j dq_j\hat{f}_j(\mathbf{p}_j)\hat{g}_j(\mathbf{q}_j))\right.$$
$$\left.\cdot S_6(p_1,p_2,p_3,q_1,q_2,q_3)\delta\left((i(k^0+i\eta),0)-\sum_1^3 p_j\right)-(\eta\to-\eta)\right\} \qquad (4.16)$$

is nothing but

$$\lim_{\eta\downarrow 0}\frac{i\eta}{\pi}\int\frac{dk'^0}{(k'^0-k^0)^2+\eta^2}\left\langle\left(\bigotimes_1^3 f_j\right)(S^T_{3,3})^{\mp}*^{(3)}({}^TS_{3,3})^{\pm}\left(\bigotimes_1^3 g_j\right)\right\rangle(k'^0)$$

$$=i\left\langle\left(\bigotimes_1^3 f_j\right)(S^T_{3,3})^{\mp}*^{(3)}({}^TS_{3,3})^{\pm}\left(\bigotimes_1^3 g_j\right)\right\rangle(k^0) \quad (4.17)$$

because of (4.14), and of continuity of the right hand side of (4.14) for $k^0\in I$. This implies for any k^0 in I:

$$\Delta\left\langle\left(\bigotimes_1^3 f_j\right)S_6(k^0)\left(\bigotimes_1^3 g_j\right)\right\rangle=i\left\langle\left(\bigotimes_1^3 f_j\right)(S^T_{3,3})^{\mp}*^{(3)}({}^TS_{3,3})^{\pm}\left(\bigotimes_1^3 g_j\right)\right\rangle(k^0). \quad (4.18)$$

Varying θ and θ' in $\mathscr{F}$ allows one to prove two analogous results with S_2 and S_4 in the left hand side instead of S_6. The system of three equations thus obtained expresses asymptotic completeness on $\mathscr{H}_I^{\text{odd}}$. But this system easily follows from the analogous system of Eqs. (1.4)–(1.6) satisfied by the 1-particle irreducible functions. (That both formulations are equivalent was established in [3], at least for total energy momenta in the complementary set of the zeros of the two point function (CDD zeros).) Indeed $R_1(ik,0)$ and $R_1^{-1}(ik,0)$ have bounded boundary values in I [this follows immediately from (1.19) and the bound (3.17) for λ small]. Thus

$$\begin{aligned}\Delta R_1(ik,0)&=-R_1^-(ik,0)\,\Delta R_1^{-1}(ik,0)\,R_1^+(ik,0)\\&=i(R_1K^{(1)}_{1,3})^{\pm}(ik,0)*^{(3)}(K^{(1)}_{3,1}R_1)^{\mp}(ik,0)\\&=iS^{T}_{1,3}{}^{\mp}*^{(3)}S^{T}_{3,1}{}^{\pm}\end{aligned}$$

and similarly for the others. This completes the proof of Theorem 4.1.

We now prove the "asymptotic completeness relations" (1.4)–(1.6).

Theorem 4.2. *Let f and g belong to $\tilde{\mathscr{A}}$, and $3m+\lambda^2/a^2<k<4(m-\varepsilon)$. Let $\langle fR_3(k)g\rangle_\pm$ be the limits as $\eta\downarrow 0$ of $\langle fR_3(k\pm i\eta)g\rangle$ with $k\pm i\eta$ in the first sheet of D. Define*

$$\langle f\Delta R_3(k)g\rangle=\langle fR_3(k)g\rangle_+-\langle fR_3(k)g\rangle_- . \quad (4.19)$$

Then

$$\langle f\Delta R_3(k)g\rangle=3!(2\pi Z)^3\int\prod_{j=1}^{3}(dp_j\delta(p_j^2+m^2)\theta(\operatorname{Im}p_j^0))$$
$$\cdot\delta(p_1+p_2+p_3-(ik,0))(R_{03}^{-1}R_3f)_{\mp}(p_1,p_2,p_3)(R_{03}^{-1}R_3g)_{\pm}(p_1,p_2,p_3), \quad (4.20)$$

where the range of integration is bounded and where

$$(R_{03}^{-1}R_3f)_\pm(p_1,p_2,p_3)=\left\{\left(1+\sum_\alpha M_\alpha\right)(1+K_3R_3)f\right\}_\pm(p_1,p_2,p_3)$$

is a sum of functions analytic in $\mathscr{D}_\alpha$, *limits as* $\eta\downarrow 0$ *of the same expression evaluated at* $p_i \mp(\eta,0)$ *in the first sheet.*

The proof of this theorem splits into a series of intermediate results:

Lemma 4.1. *Let* $\langle f\Delta R_{03}(k)g\rangle$ *be defined similarly to* (4.19). *Then*

$$\frac{1}{3!}\langle f\Delta R_{03}(k)g\rangle=(2\pi Z)^3\int\prod_{j=1}^{3}dp_j\delta(p_j^2+m^2)\theta(\operatorname{Im}p_j^0)$$
$$\cdot\delta(p_1+p_2+p_3-(ik,0))f(p_1,p_2,p_3)g(p_1,p_2,p_3)\,. \qquad (4.21)$$

Proof. The only nonzero contribution to $\langle f\Delta R_{03}(k)g\rangle$ comes from the term $Z^3\prod_{j=1}^{3}(p_j^2+m^2)^{-1}\delta^{\otimes 3}(p,p')$ in R_{03}, where

$$Z\equiv Z(\lambda)=(p^2+m^2)S_2(p)|_{p^0=i\omega(\mathbf{p})}\,.$$

Now

$$\lim_{\eta\downarrow 0}\left\{\int dp_1dp_2dp_3\frac{(fg)(p_1,p_2,p_3)\delta(p_1+p_2+p_3-i(k+i\eta,0))}{(p_1^2+m^2)(p_2^2+m^2)(p_3^2+m^2)}-(\eta\to-\eta)\right\}$$
$$=\lim_{\eta\downarrow 0}(2i\pi)^2\int d\mathbf{p}_1\int\frac{d\mathbf{p}_2}{2i\omega(\mathbf{p}_2)}\frac{d\mathbf{p}_3}{2i\omega(\mathbf{p}_3)}\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)\frac{2i\eta}{\left(\sum_1^3\omega(\mathbf{p}_j)-k\right)^2+\eta^2}$$
$$\cdot\frac{(fg)(i(k-\omega(\mathbf{p}_2)-\omega(\mathbf{p}_3)),\mathbf{p}_1;i\omega(\mathbf{p}_2),\mathbf{p}_2;i\omega(\mathbf{p}_3),\mathbf{p}_3)}{\omega(\mathbf{p}_1)+k-\omega(\mathbf{p}_2)-\omega(\mathbf{p}_3)}$$
$$=i(2\pi)^3\int\frac{d\mathbf{p}_1}{2\omega(\mathbf{p}_1)}\frac{d\mathbf{p}_2}{2\omega(\mathbf{p}_2)}\frac{d\mathbf{p}_3}{2\omega(\mathbf{p}_3)}\delta(k-\omega(\mathbf{p}_1)-\omega(\mathbf{p}_2)-\omega(\mathbf{p}_3))$$
$$\cdot\delta(\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3)(fg)(i\omega(\mathbf{p}_1),\mathbf{p}_1;i\omega(\mathbf{p}_2),\mathbf{p}_2;i\omega(\mathbf{p}_3),\mathbf{p}_3)\,.$$

Given any function g in, say, $\mathscr{A}_{23}$, we define its "2-body and 3-body discontinuities" as follows:
Let p_j^0 be pure imaginary (i.e. Minkowskian) and let

$$g_-^\pm\equiv\lim_{\eta\downarrow 0}g(p_1+(5\eta,0),p_2\mp(\eta,0),p_3\mp(\eta,0))\,, \qquad (4.22)$$

$$g_+^\pm\equiv\lim_{\eta\downarrow 0}g(p_1-(5\eta,0),p_2\mp(\eta,0),p_3\mp(\eta,0))\,, \qquad (4.23)$$

where all the arguments of g are taken in the first sheet in $\mathscr{D}_{23}$.

$$(\Delta_{23}g)_\pm\equiv g_\pm^+-g_\pm^-\,, \qquad (4.24)$$

$$\Delta g=\lim_{\eta\downarrow 0}\int d(p_1+p_2+p_3)g(p_1,p_2,p_3)\{\delta(p_1+p_2+p_3-(i(k+i\eta),0))-(\eta\to-\eta)\}\,, \qquad (4.25)$$

where the contour lies in the first sheet of the total energy. (Note that the superscripts $\pm$ refer to 2-body energies, and the subscripts $\pm$ refer to the 3-body energy.)

We then prove:

Lemma 4.2. *Let $f\in\tilde{\mathscr{A}}$, $g\in\mathscr{A}_{23}$, and $\langle f\Delta R_{03}(k)g\rangle$ be defined similarly to (4.19). Then the following identity holds for $3m<k<4(m-\varepsilon)$:*

$$\begin{aligned}\langle f\Delta R_{03}(k)g\rangle &= 3!(2\pi Z)^3\int\prod_{j=1}^{3}(dp_j\delta_+(p_j))\delta(p_1+p_2+p_3-(ik,0))\\ &\quad\cdot f(p_1,p_2,p_3)g_+^+(p_1,p_2,p_3)+\lim_{\eta\downarrow 0}\langle fR_{03}(k-i\eta)\Delta g\rangle\\ &\quad+3(2\pi Z)\int dp_1\delta_+(p_1)\lim_{\eta\downarrow 0}\int dp_2dp_3R_{02}(p_2,p_3)\\ &\quad\cdot\delta(p_1+p_2+p_3-i(k-i\eta,0))(f(\Delta_{23}g)_+)(p_1,p_2,p_3)\,. \end{aligned}\tag{4.26}$$

The same result holds with $+\to-$, $g_+^+\to g_-^-$, $\Delta_{23}g_+\to\Delta_{23}g_-$.

Analogous results hold for $\mathscr{A}_{12}$ or $\mathscr{A}_{31}$ instead of $\mathscr{A}_{23}$.

Proof. $\langle fR_{03}(k\pm i\eta)g\rangle$ can be replaced by

$$Z\int\frac{dp_1}{p_1^2+m^2}\int dp_2dp_3R_{02}(p_2,p_3)\delta(p_1+p_2+p_3-i(k\pm i\eta,0))(fg)(p_1,p_2,p_3)\,. \tag{4.27}$$

We first assume that g has no three body cut [but has a two body cut in the channel (23)]. Then, after getting rid of the contribution to (4.27) of $\mathbf{p}_1$ such that $\omega(\mathbf{p}_1)>\mathrm{Re}k-2(m-\varepsilon)$, the integral over p_1^0 can be split into a residue at $p_1^0=i\omega(\mathbf{p}_1)$ plus a contour integral which has no cut in k. Thus

$$\begin{aligned}\langle f\Delta R_{03}(k)g\rangle &= \lim_{\eta\downarrow 0}3(2\pi Z)\int dp_2dp_3(fg)((i\omega(-\mathbf{p}_2-\mathbf{p}_3),-\mathbf{p}_2-\mathbf{p}_3),p_2,p_3)\\ &\quad\cdot R_{02}(p_2,p_3)[\delta(p_2+p_3-i(k+i\eta-\omega(\mathbf{p}_2+\mathbf{p}_3),0))-(\eta\to-\eta)]\\ &=3!2\pi Z\int dp_1\delta_+(p_1)\Big\{(2\pi Z)^2\int dp_2dp_3\delta_+(p_2)\\ &\quad\cdot\delta_+(p_3)g^+(p_1,p_2,p_3)f(p_1,p_2,p_3)\\ &\quad+\lim_{\eta\downarrow 0}3\int dp_2dp_3R_{02}(p_2,p_3)\delta(p_1+p_2+p_3-i(k-i\eta,0))\\ &\quad\cdot(f\Delta_{23}g)(p_1,p_2,p_3)\Big\}. \end{aligned}\tag{4.28}$$

If g has in addition a three body cut, denote by $\tilde{g}$ its analytic continuation through the three body cut from the first sheet from above. Then

$$\langle f\Delta R_{03}(k)g\rangle=\lim_{\eta\downarrow 0}\langle fR_{03}(k-i\eta)(g-\tilde{g})\rangle+\langle f\Delta R_{03}(k)\tilde{g}\rangle\,, \tag{4.29}$$

$\tilde{g}$ is analytic in k in the neighborhood of the interval $3m<k<4(m-\varepsilon)$ and $\langle f\Delta R_{03}(k)\tilde{g}\rangle$ can therefore be treated as above. This completes the proof of the lemma.

Lemma 4.3. *Let* $f\in\tilde{\mathscr{A}}$. *Then*

(i) $(\Delta_\beta M_\beta f)_+ = +2(2\pi Z)^2(1+K_2^\beta R_2^\beta)_-^- K_2^\beta \delta_+^{\otimes 2}\Big((1+\sum_\alpha M_\alpha)f\Big)_+^+$.

(ii) *Let* ΔMf *(respectively* I*) be the column vector in* $\bigoplus_\alpha \mathscr{A}_\alpha$ *whose components are* $\Delta M_\alpha f$ *(respectively*

$$2(2\pi Z)^3(1+K_2^\alpha R_2^\alpha)\sum_{\beta\neq\alpha} K_2^\alpha\delta_+(p_l)R_1(p_j)(1+K_2^\beta R_2^\beta)_-^- K_2^\beta\delta_+^{\otimes 2}\Big(\Big(1+\sum_\gamma M_\gamma\Big)f\Big)_+^+\Big),$$

where $\alpha=(jl)$, j *being the element common to pairs* α *and* β *and where* $\delta_+^{\otimes 2}$ *in the channel* $\beta=(ij)$ *is nothing but* $\delta_+(p_i)\delta_+(p_j)$. *Then*

$$\Delta Mf=(1-A_-)^{-1}I.$$

Proof. It follows from Eq. (1.32) that the discontinuity of $M_{23}\,f$ in the variable $p_2^0+p_3^0$ only comes from R_2^{23}. But from two-body analysis [21] it follows that:

$$\begin{aligned}1/2(2\pi Z)^{-2}\{(R_2^{23})^+-(R_2^{23})^-\}&=[1+K_2^{23}(R_2^{23})^+]^*\delta_+^{\otimes 2}[1+K_2^{23}(R_2^{23})^+]\\ &=[1+(R_2^{23})^-K_2^{23}]\delta_+^{\otimes 2}[1+K_2^{23}(R_2^{23})^+]\,.\end{aligned}\tag{4.30}$$

Thus

$$\begin{aligned}(\Delta_{23}M_{23}f)_\pm &= +2(2\pi Z)^2(1+K_2^{23}(R_2^{23})^-)K_2^{23}\delta_+^{\otimes 2}(1+K_2^{23}(R_2^{23})^+)\\ &\quad\cdot\Big(\sum_{\beta\neq(23)}(M_\beta f)_\pm+f_\pm\Big)\\ &=2(2\pi Z)^2(1+K_2^{23}R_2^{23})^-K_2^{23}\delta_+^{\otimes 2}\Big(f_\pm+(M_{23}f)_\pm^\pm+\sum_{\beta\neq(23)}(M_\beta f)_\pm^\pm\Big)\end{aligned}\tag{4.31}$$

because the factor $\delta_+^{\otimes 2}$ before $M_\beta f$ implies that the two-body and three-body energies have the same imaginary parts. This completes the proof of (i).

In order to prove (ii) we start from (1.32) again. As f has no two-body cut by assumption, the inhomogeneous term in (1.32) does not produce a 3-body cut. Thus we need only consider the three body discontinuity of a term of the form

$$Z\int\frac{dp_1\,dp_2\,R_1(p_2)}{(p_1^2+m^2)}\delta(p_1+p_2+p_3-(ik,0))g_{23}(p_1,p_2,p_3)\tag{4.32}$$

with $g_{23}\in\mathscr{A}_{23}$. Assume first that g_{23} only has a two body cut in the channel 23, but no three body cut. Then it follows easily from the proof of Proposition 2.1 that the three body discontinuity of (4.32) is

$$2\pi Z(\delta_+(p_1)R_1(p_2))\Delta_{23}g_{23}\,.$$

In the presence of a three body cut in g_{23}, we write

$$(\Delta K_2^{12}R_2^{12}g_{23})=(K_2^{12}R_2^{12})_-(g_{23}-\tilde{g}_{23})+(\Delta K_2^{12}R_2^{12}\tilde{g}_{23}),\tag{4.33}$$

where $\tilde{g}_{23}$ denotes the analytic continuation of g_{23} through the three body cut from the first sheet from above. Then

$$(\Delta K_2^{12} R_2^{12} g_{23}) = (K_2^{12} R_2^{12})_- \Delta g_{23} + 2\pi Z((1 - K_2^{12} R_2^{12}) K_2^{12} \delta_+(p_1) R_1(p_2)) (\Delta_{23} g_{23})_+ . \tag{4.34}$$

Applying to $g_{23} = M_{23} f$, and more generally to $M_\beta f$ with $\beta \neq \alpha$ for α instead of (12), and using (i), we get (ii).

Proposition 4.1. *Let $f, g \in \tilde{\mathscr{A}}$ and $\langle f \Delta R_3'(k) g\rangle$ be defined similarly to* (4.19). *Then, for $3m + \lambda^2/a^2 < k < 4(m - \varepsilon)$*

$$\langle f \Delta R_3'(k) g\rangle = 3!(2\pi Z)^3 \int \prod_{j=1}^{3} (dp_j \delta_+(p_j)) \delta(p_1 + p_2 + p_3 - (ik, 0)) \cdot (R_{03}^{-1} R_3' g)_-^- (p_1, p_2, p_3) (R_{03}^{-1} R_3' f)_+^+ (p_1, p_2, p_3) . \tag{4.35}$$

Proof. For any k in D'

$$\langle f R_3'(k) g\rangle = \langle f R_{03}(k) g\rangle + \sum_\alpha \langle f R_{03}(k) (M_\alpha g)\rangle .$$

Now using Lemmas 4.1–4.3, we have

$$\begin{aligned}\langle f \Delta R_3'(k) g\rangle = (2\pi Z)^3 \Big\{ & 3! \Big\langle f \delta_+ \otimes \delta_+ \otimes \delta_+ \Big(\Big(1 + \sum_\alpha M_\alpha\Big) g\Big)_+^+ \Big\rangle \\ & + 3! \sum_\beta \Big\langle f (R_2^\beta K_2^\beta)_-^- \delta_+ \otimes \delta_+ \otimes \delta_+ \Big(\Big(1 + \sum_\alpha M_\alpha\Big) g\Big)_+^+ \Big\rangle \Big\} \\ & + \Big\langle f (R_{03})_- \sum_\alpha \Delta M_\alpha g \Big\rangle .\end{aligned} \tag{4.36}$$

But the third term is

$$3! \Big\langle f (R_{03})_-^- \sum_\beta (M_\beta - K_2^\beta R_2^\beta)_-^- R_{03}^{-1} \delta_+ \otimes \delta_+ \otimes \delta_+ \Big(\Big(1 + \sum_\alpha M_\alpha\Big) g\Big)_+^+ \Big\rangle (2\pi Z)^3 .$$

Now due to the relation

$$R_{02}^\beta K_2^\beta R_2^\beta (R_{02}^\beta)^{-1} = R_2^\beta K_2^\beta ,$$

we have a cancellation with the second term of (4.36). We are left with

$$\begin{aligned} 3!(2\pi Z)^3 \int & \prod_{j=1}^{3} (dp_j \delta_+(p_j)) \delta(p_1 + p_2 + p_3 - (ik, 0)) \\ & \cdot \Big\{ f(p_1, p_2, p_3) \Big(\Big(1 + \sum_\alpha M_\alpha\Big) g\Big)_+^+ (p_1, p_2, p_3) \\ & + \sum_\beta (M_\beta f)_-^- (p_1, p_2, p_3) \Big(\Big(1 + \sum_\alpha M_\alpha\Big) g\Big)_+^+ (p_1, p_2, p_3) \Big\} \end{aligned}$$

because $(R_{03} M_\beta R_{03}^{-1})(p_1, p_2, p_3; p_1', p_2', p_3') = (M_\beta)\ (p_1', p_2', p_3';\ p_1, p_2, p_3)$. This completes the proof of Proposition 4.1.

In order to complete the proof of Theorem 4.2 we use the equation $R_3 = R_3'(1 - K_3 R_3')^{-1}$. Given $f, g \in \tilde{\mathscr{A}}$, we have

$$\langle f \Delta R_3(k) g\rangle = \langle f \Delta R_3'(k) ((1 - K_3 R_3')^{-1} g)_+\rangle + \lim_{\eta \downarrow 0} \langle f R_3'(k - i\eta) \Delta (1 - K_3 R_3')^{-1} g\rangle .$$

But as elements of $\mathscr{A}'$

$$\Delta(1-K_3R_3')^{-1}g=(1-K_3R_3')_-^{-1}\{(1-K_3R_3')_- -(1-K_3R_3')_+\}(1-K_3R_3')_+^{-1}g$$
$$=((1-K_3R_3')^{-1}K_3)_- \Delta R_3'(1-K_3R_3')_+^{-1}g$$

because $\lim_{\eta\downarrow 0}(K_3(k+i\eta)-K_3(k-i\eta))R_3'(k)_+ \to 0$ strongly in $\tilde{\mathscr{A}}$. Thus

$$\langle f\,\Delta R_3(k)g\rangle=\langle(1+K_3R_3)_+\,f\,\Delta R_3'(k)(1+K_3R_3)_+\,g\rangle$$

because $(1-K_3R_3')^{-1}=1+K_3R_3$ as operators in $\mathscr{A}'$. Now using Proposition 4.1 completes the proof of Theorem 4.2.

The last result of this section is the following:

Corollary 4.1. *Let $f\in\tilde{\mathscr{A}}$, and $3m+\lambda^2/a^2<k<4(m-\varepsilon)$. Let $(K_{1,3}^{(1)}f)_\pm(k)$ (respectively $R_1^{-1}(k)_\pm$) be the limits as $\eta\downarrow 0$ of $(K_{1,3}^{(1)}f)$ $(k\pm i\eta)$ (respectively $R_1^{-1}(k\pm i\eta)$) with $k\pm i\eta$ in the first sheet of D. Define*

$$\Delta(K_{1,3}^{(1)}f)(k)=(K_{1,3}^{(1)}f)_+(k)-(K_{1,3}^{(1)}f)_-(k),$$
$$\Delta R_1^{-1}(k)=R_1^{-1}(k)_+-R_1^{-1}(k)_-.$$

Then

$$\Delta(K_{1,3}^{(1)}f)(k)=3!(2\pi Z)^3\int dp\int\prod_{j=1}^{3}(dp_j\delta(p_j^2+m^2)\theta(\operatorname{Im}p_j^0))$$
$$\cdot\delta(p_1+p_2+p_3-(ik,0))(R_{03}^{-1}K_{3,1}^{(1)})_\mp(p_1,p_2,p_3;p)(R_{03}^{-1}R_3f)_\pm(p_1,p_2,p_3),$$
$$-\Delta R_1^{-1}(k)=3!(2\pi Z)^3\int dp\,dp'\int\prod_{1}^{3}(dp_j\delta(p_j^2+m^2)\theta(\operatorname{Im}p_j^0))$$
$$\cdot\delta(p_1+p_2+p_3-(ik,0))(R_{03}^{-1}K_{3,1}^{(1)})_\mp(p_1,p_2,p_3;p)\,(R_{03}^{-1}K_{3,1}^{(1)})_\pm(p_1,p_2,p_3;p').$$

We do not give the proof which is an obvious consequence of Eqs. (1.18) and (1.19) and of Theorem 4.2.

V. How to Exclude Three Body Bound States?

The problem is essentially to invert the operator $(1-A)$ near the three body threshold or, alternatively, to give a meaning to $\sum A^n$. We note that $K_2^\alpha R_2^\alpha$ is of order one near the two body threshold but of order λ away from it. One could hope that integrating over the two body energy κ_α^0 in a convolution $K_2^\beta R_2^\beta K_2^\alpha R_2^\alpha$ would make a small average and that, by iteration, the norm of A^n would be small. We prove that such is not the case near the three body threshold, at least for the leading part of A when $\lambda\to 0$.

For comparison we recall that in the presence of a two-body bound state, the dominant (and of course unbounded) part of K_2R_2 near the two body threshold is the pole at m_B. A study of three body bound states [19] is then essentially a two-body problem with unequal masses m_B and m. Even in that case however, difficulties similar to those described below prevent us from showing boundedness of R_3 inside a small neighborhood of $k=3m$; this point was overlooked in [19].

We restrict our attention to Minkowski space and $k<3m$. Then (see Sect. 1)

$$\begin{aligned}(K_2^{12}R_2^{12}f)(p_1,p_2,p_3)&=\frac{-\lambda}{1+\lambda d^2(\kappa_3)}\int dp_1'\,dp_2'\\&\cdot f(p_1',p_2',p_3)\,(p_1'^2+m^2)^{-1}(p_2'^2+m^2)^{-1}\delta(p_1+p_2-p_1'-p_2')+O(\lambda|f|_\infty)\\&=\frac{-\pi\lambda}{1+\lambda d^2(\kappa_3)}\int_{\omega(\mathbf{p}_1')<2(m-\varepsilon)}d\mathbf{p}_1'\,dp_2'\,\delta(p_1+p_2-p_1'-p_2')\,f((i\omega(\mathbf{p}_1'),\mathbf{p}_1'),p_2',p_3')\\&\cdot\omega^{-1}(\mathbf{p}_1')\left[-(k+ip_3^0-\omega(\mathbf{p}_1'))^2+\omega^2(-\mathbf{p}_1'-\mathbf{p}_3)\right]^{-1}+O(\lambda|f|_\infty).\end{aligned}\tag{5.1}$$

We are interested in $(1-A)^{-1}=\sum_0^\infty(A)^n$. This involves iterating the above formula where, by induction, the leading part of f will depend only upon p_2+p_3, and where the result will be used at the next step for $p_3^0=i\omega(\mathbf{p}_3)$ only. Also the α-channel term in A^n is $(K_2^\alpha R_2^\alpha)$ times a chain of $n-1$ factors with different consecutive channels; but each such chain gives an equal contribution to A^n, yielding a factor 2^{n-1}. Therefore we define an operator acting on functions of one (space) variable:

$$\begin{aligned}(\tilde{B}f)(p)&=\frac{-2\pi\lambda}{1+\lambda d^2(k-\omega(p),p)}\int_{\omega(p')<2(m-\varepsilon)}dp'\,\omega^{-1}(p')\,f(p')\\&\cdot\left[-(k-\omega(p)-\omega(p'))^2+\omega^2(p+p')\right]^{-1}\end{aligned}\tag{5.2}$$

$$\begin{aligned}&=\frac{-2\pi\lambda(k-m)^{-1}}{1+\lambda\pi^2(k-m)^{-1}\left[m\left(3m-k+\dfrac{3p^2}{4m}\right)\right]^{-1/2}}\\&\cdot\int dp'\frac{f(p')}{\left(p'+\dfrac{p}{2}\right)^2+m\left(3m-k+\dfrac{3}{4m}p^2\right)}+O(\lambda\log\lambda|f|_\infty)\\&=\frac{-2}{1+x\sqrt{1+\frac34q^2}}\,\frac{1}{\pi}\int dq'\frac{f(q'\sqrt{1+\frac34q^2}\,\sqrt{m(3m-k)})}{1+\left(q'+\dfrac{q}{2\sqrt{1+\frac34q^2}}\right)^2}+O(\lambda\log\lambda|f|_\infty),\end{aligned}\tag{5.3}$$

where $q=p[m(3m-k)]^{-1/2}$ and $x=\pi^{-2}(k-m)\dfrac{\sqrt{m(3m-k)}}{\lambda}$.

(5.3) follows by an expansion (up to second or fourth order) in p and p' near zero, and by noting that

$$p\frac{\lambda d^2(k-\omega(p),p)}{1+\lambda d^2(k-\omega(p),p)}=O(\lambda).$$

We now define an operator B_x acting on even functions of one variable

$$(B_xg)(q)=\frac{-2}{1+x\sqrt{1+\frac34q^2}}\,\frac{1}{\pi}\int dq'\frac{g(q'\sqrt{1+\frac34q^2})}{1+\left(q'+\dfrac{q}{2\sqrt{1+\frac34q^2}}\right)^2},\tag{5.4}$$

which is such that if S is the dilation operator

$$(Sf)(p)=f\left(\frac{p}{\sqrt{m(3m-k)}}\right), \tag{5.5}$$

$$2(K_2^{12}R_2^{12}f)(p_1,p_2,(i\omega(\mathbf{p}_3),\mathbf{p}_3))=(S^{-1}B_xSf_1)(\mathbf{p}_3)+O(\lambda\log\lambda|f|_\infty), \tag{5.6}$$

where $f\in\mathscr{A}_1$ is such that there exists a function $f_1:\mathbb{R}\to\mathbb{R}$ with

$$f((i\omega(\mathbf{p}_1),\mathbf{p}_1);p_2;p_3)=f_1(\mathbf{p}_1)+O(\lambda\log\lambda|f|_\infty).$$

We now prove that $\sum_0^\infty B_x^n$, which could approximate $\sum_0^\infty A^n$, is not an absolutely convergent series.

Theorem 5.1. (i) $\|B_x\|_{L^\infty\to L^\infty}=\dfrac{2}{1+x}$,

$$\|B_x^2\|_{L^\infty\to L^\infty}=\frac{2}{1+O(x\log x)}.$$

(ii) *For x sufficiently small, there exists $c>1$ such that for all positive integers n*

$$\|B_x^n\|_{L^\infty\to L^\infty}>c^n.$$

Proof. (i) is obvious, using an explicit calculation for $(B_x^2 1)(q)$.

(ii) is proven in five steps:

a) $-B_x$ is positivity preserving,

$$\|B_x^n\|_{L^\infty\to L^\infty}=[(-B_x)^n 1](0),$$

$(-B_x)^n 1$ is a positive even function, decreasing for $q>0$.

b) Let

$$(-B'_xf)(q)=\left(1+x+x\frac{\sqrt{3}}{2}|q|\right)^{-1}\frac{2}{\pi}\int dq'\frac{f\left(|q'|\left(1+\frac{\sqrt{3}}{2}|q|\right)\right)}{1+\left(q'+\frac{1}{\sqrt{3}}\right)^2}, \tag{5.7}$$

$(-B'_x)^n 1$ is a positive even function, decreasing and convex for $q>0$.

c) $((-B'_x)^n 1)(0)>c(x)^n$, where

$$c(x)=\sup_{1<a<\frac{2}{\sqrt{3}}}\frac{4\dfrac{\operatorname{Arctg}a}{\pi}}{1+\dfrac{2x}{1-a\dfrac{\sqrt{3}}{2}}}. \tag{5.8}$$

d) Let f be a positive even function, decreasing for $q>0$. Then $-B_xf\geqq -B'_xf$.

e) $(-1)^n(B_x^n 1 - B_x'^n 1) = (-1)^n \sum_{p=0}^{n-1} B_x^p (B_x - B_x') B_x'^{n-1-p} 1 \geqq 0$.

We only give the proof of c). Let f be a positive even function, decreasing and convex for $q>0$. Then

$$-B_x' f > \frac{2}{1+x+\frac{x\sqrt{3}}{2}|q|} \frac{1}{\pi} \int_{-\frac{1}{\sqrt{3}}-a}^{-\frac{1}{\sqrt{3}}+a} dq' \frac{f\left(q'\left(1+\frac{\sqrt{3}}{2}|q|\right)\right)}{1+\left(q'+\frac{1}{\sqrt{3}}\right)^2}$$

$$> \frac{2}{1+x+\frac{x\sqrt{3}}{2}|q|} \frac{\operatorname{Arctg} a}{\pi}$$

$$\cdot\left\{f\left(\left(1+\frac{\sqrt{3}}{2}|q|\right)\left(a+\frac{1}{\sqrt{3}}\right)\right)+f\left(\left(1+\frac{\sqrt{3}}{2}|q|\right)\frac{1}{\sqrt{3}}\right)\right\}$$

$$> \frac{2}{1+x+\frac{x\sqrt{3}}{2}|q|} \frac{2\operatorname{Arctg} a}{\pi} f\left(\left(1+\frac{\sqrt{3}}{2}|q|\right)\left(\frac{a}{2}+\frac{1}{\sqrt{3}}\right)\right).$$

Let $a' = \frac{a}{2} + \frac{1}{\sqrt{3}}$

$$((-B_x')^n 1)(q) > \left(\frac{4\operatorname{Arctg} a}{\pi}\right)^n \frac{1}{1+x\left(1+\frac{\sqrt{3}}{2}|q|\right)} \frac{1}{1+x+\frac{x\sqrt{3}}{2}a'\left(1+\frac{\sqrt{3}}{2}|q|\right)}$$

$$\cdots \frac{1}{1+x+\frac{x\sqrt{3}}{2}a'+\ldots+x\left(\frac{\sqrt{3}}{2}a'\right)^{n-2}+x\left(\frac{\sqrt{3}}{2}a'\right)^{n-1}\left(1+\frac{\sqrt{3}}{2}|q|\right)}.$$

Then there exists a function $b(q)$ with $b(0)=1$ such that for all n:

$$((-B_x')^n 1)(q) > b(q)\, c(x)^n,$$

where $c(x)$ is given by (5.8).

We finally remark that $c(x) \to \frac{4}{\pi} \operatorname{Arctg} \frac{2}{\sqrt{3}} > 1$ when $x \to 0$. This completes the proof of Theorem 5.1.

As a conclusion, we note that it is necessary to use the repulsivity of Φ^4 [i.e. the alternate character of the series $(1-A)^{-1}$] to prove that Φ^4-like models do not have three body bound states. One hope would be to decompose B_x into a rank one operator associated to an eigenvalue between one and two (for x small) plus a remainder of norm less than one. The eigenvalue should be $b = \lim_n \frac{B_x^{n+1} 1}{B_x^n 1}$ and the eigenfunction $f_b(q) = \lim_n b^{-n} B_x^n 1$. One could then invert explicitly $1-B_x$.

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