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Unitarity Equations and Structure of the S-Matrix at the m-Particle Threshold in a Theory with Pure $m\to m$ Interaction

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Abstract. The structure of the S-matrix at the m-partical threshold $s=(m\mu)^2$ of a $m\to m$ process ($m\geqq 2$) in ν-dimensional space-time is determined in a theory with a simplified unitarity equation corresponding to a pure $m\to m$ interaction. If $(m-1)(\nu-1)$ is odd, a two-sheeted, square-root type structure is obtained as in the usual case of two-particle thresholds in dimension 4. The nature of the singularity is more complicated if $(m-1)(\nu-1)$ is even (e.g. $m=3$ in dimension 4). Results obtained in this case include an orthogonal decomposition of the scattering function T with nonholomic eigenvalues of the form $\left[\frac{1}{2i\pi}\ln\sigma+b_i(\sigma)\right]^{-1}$ [where $\sigma=(m\mu)^2-s$ and b_i is uniform around $\sigma=0$] and a related infinite expansion of T in powers of $\ln\sigma$ involving an on-shell irreducible kernel U which is the analogue for $(m-1)(\nu-1)$ even of Zimmerman's K-matrix.

1. Introduction

While substantial progress has been made in recent years in the analysis of the singularity structure of the S-matrix and of the Green's function of Quantum Field Theory, the knowledge of the exact nature of singularities has however remained limited so far; for instance, whereas the square-root nature of two-particle thresholds is an old result, there is no comparable information on the nature of the three-particle thresholds, even at a heuristic level. The present work gives a treatment of the m-particle threshold and an explicit description of the nature of its singularity in a simplified $m\to m$ scattering theory ($m\geqq 2$) with no subchannel interaction, in arbitrary space-time dimension ν. This treatment is based on the on-shell unitarity-type equation of this simplified theory. Complementary results obtained in the off-shell approach and based on a (simplified) Bethe-Salpeter type equation are described in [1], where the links between the two approaches are explained.

As discussed in [1a] the results accredit the idea that the nature of m-particle thresholds in the actual theory should be determined through an adequate analysis

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of integral equations in complex space, and that the degree of complexity of the singularities thus generated for the S-matrix should go in general far beyond the class of holonomic functions which incorporates previously known examples (square roots, poles, logarithms, ...) and was proposed a few years ago by Sato [2], as the natural framework of (standard[1]) scattering theory[2].

We consider for simplicity a theory with only one type of particle of spin zero and mass $\mu > 0$.

In the on-shell approach, namely that of a pure S-matrix theory, we consider an $m \to m$ scattering function $T(p_1, \ldots, p_m; p'_1, \ldots, p'_m)$ defined on the (complex) mass-shell variety $\mathcal{M}^c: p_i^2 = p_i'^2 = \mu^2$, $1 \leqq i \leqq m$, $\sum_{i=1}^{m} p_i = \sum_{i=1}^{m} p'_i$; the variables p_i and p'_i are ν-dimensional energy-momenta for the initial and final particles respectively. As usual, $p_i^2 = p_{i,0}^2 - \mathbf{p}_i^2$ where $p_{i,0}$ is the energy and $\mathbf{p}_i$ the $(\nu-1)$ dimensional momentum, and we denote by s the squared total center of mass energy variable $s = k^2$, $k = \sum_{i=1}^{m} p_i$. In the simplified theory, T is assumed to be locally analytic in a "cut neighbourhood" of the "threshold variety" $p_1 = \ldots = p_m = p'_1 = \ldots = p'_m$ [on which $s = (m\mu)^2$], and as usual we denote by "physical sheet" this initial (schlicht) domain (see Sect. 2 for a precise specification). The boundary values $T^{(0)}$ and $T^{(1)}$ of T at $s > (m\mu)^2$ from the respective sides $\operatorname{Im} s > 0$ and $\operatorname{Im} s < 0$ will be assumed to be continuous. The unitarity-type equation of the simplified theory is then:

$$T^{(0)} - T^{(1)} = T^{(0)} * T^{(1)}. \tag{1}$$

where $*$ denotes on-mass-shell integration over m internal energy-momenta. Namely (with a suitable normalization of T):

$$\begin{aligned} &T^{(0)} * T^{(1)}(p_1, \ldots, p_m; p'_1, \ldots, p'_m) \\ &\quad = \int T^{(0)}(p_1, \ldots, p_m; k_1, \ldots, k_m) T^{(1)}(k_1, \ldots, k_m; p'_1, \ldots, p'_m) \\ &\quad \cdot \delta^\nu\left(\sum_{i=1}^{m} k_i - \sum_{i=1}^{m} p_i\right) \prod_{i=1}^{m} \delta(k_i^2 - \mu^2)\theta(k_{i,0}) d^\nu k_i. \end{aligned} \tag{2}$$

It will also be convenient to assume that T is invariant through the inversion of all incoming and outgoing $(\nu-1)$ momenta in the center of mass system (see Sect. 2). This symmetry property is automatically fulfilled if T is invariant through the (full) Lorentz group in ν-dimensional space-time; however, Lorentz invariance is irrelevant in the present study and will not be assumed.

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1 Here we do not refer to the more recent important works by M. Sato and collaborators on "holonomic quantum fields," which apply to a certain class of two-dimensional models

2 At ordinary points of a Landau singularity $z_1 = 0$, holonomicity, if it holds, entails that the scattering function T is of the form: $T(z) = \sum_{\alpha, j} a_{\alpha,j}(z) z_1^\alpha (\ln z_1)^j$, where each j is a *positive* integer ($j \geqq 0$), the sum $\sum$ runs over a *finite* set and the coefficients $a_{\alpha,j}$ are uniform around $z_1 = 0$. It means correspondingly that the vector space generated by the successive determinations of T around $z_1 = 0$ is finite-dimensional (= "finite-determination property" of T)

The main results of the paper will be found in Sects. 5 and 6. Sections 2–4 present the basic preliminary results that will be needed there.

In Sect. 2, a new set of variables is introduced to describe the $m \to m$ particle mass shell, namely the total energy-momentum k of this channel and "angular" variables Ω (respectively Ω') which are associated with the incoming (respectively outgoing) relative momenta, and vary on the unit sphere $\mathbb{S}_{(m-1)\nu-m}$. In these variables, T appears as a kernel $T(k;\Omega,\Omega')$ in Ω-space which depends analytically on the (ν-dimensional) parameter k in a cut neighbourhood of $s \equiv k^2 = (m\mu)^2$; correspondingly Eq. (1) takes the form of an integral relation of the Fredholm type depending on the complex parameter k:

$$\begin{aligned} &T^{(0)}(k;\Omega,\Omega') - T^{(1)}(k;\Omega,\Omega') \\ &\quad = [(m\mu)^2 - k^2]^\beta \times \int T^{(0)}(k;\Omega,\Omega'')\,T^{(1)}(k;\Omega'',\Omega') \\ &\quad \cdot \hat{\alpha}(k,\Omega'')\,d\Omega'' , \end{aligned} \tag{3}$$

where $\hat{\alpha}$ is analytic at $k^2 = (m\mu)^2$ and $\beta = \dfrac{(m-1)\nu - m - 1}{2}$.

This allows one to show (Sect. 3) that, under the assumptions listed above, T satisfies "local maximal analyticity" around the threshold $s = (m\mu)^2$, i.e. is analytic or meromorphic in a certain covering of $\mathscr{V} - \{s = (m\mu)^2\}$, where $\mathscr{V}$ is a complex neighbourhood of $s = (m\mu)^2$. If $(m-1)(\nu-1)$ is odd, i.e. β half-integer, T is two-sheeted (i.e. the usual square-root singularity of the case $m=2$, $\nu=4$ is reobtained); however if $(m-1)(\nu-1)$ is even, i.e. β integer, an infinite number of determinations $T^{(r)}$ of T is obtained and the general relation

$$T^{(0)} - T^{(r)} = r T^{(0)} * T^{(r)}, \tag{4}$$

which entails under general conditions the non holonomicity of T at $s = (m\mu)^2$ (see Sect. 3 and [3]; see also Sect. 5 below), is derived from (1).

To further exploit the unitarity-type equations (1) and (4), we first give in Sect. 4 some useful results of Fredholm theory with complex parameters.

It is then shown in Sect. 5 that all solutions of the simplified theory satisfying the additional physical condition of hermitian analyticity admit, for $(m-1)(\nu-1)$ even and at s real below threshold, an orthogonal decomposition of the form:

$$T(k;\Omega,\Omega') = \sum_i \frac{1}{\dfrac{1}{2i\pi}\ln\sigma + b_i(k)} E_i(k;\Omega,\Omega'), \tag{5}$$

where $\sigma = (m\mu)^2 - s$, $\beta = \dfrac{(m-1)\nu - m - 1}{2}$, the kernels E_i satisfy $E_i * E_j \equiv E_i \delta_{i,j}$, and the functions E_i and b_i are uniform (analytic or meromorphic) around $\sigma = 0$; the functions $\hat{u}_i = 1/\sigma^\beta b_i$ are moreover analytic at $\sigma = 0$ under a boundedness condition on T at $\sigma = 0$, and for $\beta > 0$. The nonholonomocity[3] of the eigenvalue functions

3 An explicit proof of the **nonholonomicity** of these functions can be found e.g. in [1a]. See also in this connection footnote 2 of the present paper

$\left[\frac{1}{2i\pi}\ln\sigma + b_i(k)\right]^{-1}$ entails the nonholonomicity of T. (These eigenvalues reduce to partial waves at $m=2$, up to locally analytic factors.)

In Sect. 6, the decomposition (5) is extended to the physical-region $s>(m\mu)^2$ by introducing an on-shell irreducible kernel U which is the analogue for $(m-1)(v-1)$ even of Zimmerman's K matrix. If $\left[\frac{1}{2i\pi}\ln\sigma\right]^{-1}$ is not an eigenvalue function of T, U is defined, independently of the condition of hermitian analyticity, through the Fredholm equation:

$$T = U + \frac{i}{2\pi}\ln\sigma\, T * U. \tag{6}$$

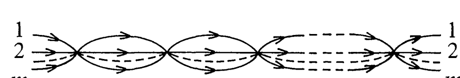
If $\left[\frac{1}{2i\pi}\ln\sigma\right]^{-1}$ is an eigenvalue function, T can be decomposed into a sum $T'+T''$, where T'' is the principal kernel associated with this eigenvalue and where T' and T'' are both solutions of the same unitarity-type equation as T. Then U is defined in terms of T' by an equation analogous to (6), with T replaced by T'. In both cases, the unitarity-type equation for T' is shown to be equivalent to the uniformity of U around $\sigma=0$. In the hermitian case, $U = \sum_{i;\, b_i \neq 0} u_i E_i$ for σ real, where $u_i = 1/b_i$, t_i and b_i being the functions occurring in (5).

If T is assumed to be uniformly bounded in the physical sheet near $\sigma=0$, $T=T'$, U is irreducible, i.e. locally analytic, at $\sigma=0$ and the functions $\hat{u}_i = u_i/\sigma^\beta$ are also locally analytic at $\sigma=0$. The irreducible character of U allows one on the other hand to express the nature of the threshold singularity of T through the following (convergent) expansion of T, which is the Neumann series of (6), valid for $\beta>0$ near $\sigma=0$ (and $\mathrm{Arg}\,\sigma$ bounded):

$$\begin{aligned} T(k;\Omega,\Omega') &= \sum_n U^{*(n+1)}(k;\Omega,\Omega')\left(\frac{i}{2\pi}\ln\sigma\right)^n \\ &= \sum_n U^{\hat{*}(n+1)}(k;\Omega,\Omega')\left(\frac{i}{2\pi}\sigma^\beta\ln\sigma\right)^n, \end{aligned} \tag{7}$$

where $\hat{*}$ is defined from $*$ by removing the factor σ^β generated by the latter, and $U^{*(n)} = U * U \ldots * U$ (n factors); all factors $U^{\hat{*}(n+1)}$ are, like U, locally analytic at $\sigma=0$ and are uniformly bounded by KC^n, for suitable constants K, C near $\sigma=0$.

In the off-shell approach to the simplified theory (see [1]), the individual terms in the expansion (7) appear naturally as "leading contributions" to generalized Feynman or G-convolution integrals associated with the graphs

$G_n^{(m)}$ = 

(8)

each vertex representing here an off-shell extrapolation of the irreducible kernel U.

Some complements to the present paper will also be found in [3], where in particular a general criterion implying the nonholonomicity of T (in this simplified theory) is proved (see Proposition 2 at the end of Sect. 3).

2. Choice of Variables

The (real) mass shell variety $\mathscr{M}$ can be described as the following bundle:

$$\mathscr{M}=\{(p_1,\ldots,p_m;p'_1,\ldots,p'_m)\in\mathbb{R}^{2\nu m}$$
$$k=\sum_{1\leqq i\leqq m}p_i=\sum_{1\leqq i\leqq m}p'_i=(k_0,\mathbf{k}),k_0>0,k^2\geqq(m\mu)^2;$$
$$(p_1,\ldots,p_m)\in e(k),(p'_1,\ldots,p'_m)\in e(k)\}\,, \tag{9}$$

where the "mass shell fiber" $e(k)=\left\{(k_1,\ldots,k_m);\ \sum_{1\leqq i\leqq m}k_i=k;\ k_i=(k_{i,0},\mathbf{k}_i),\ k_{i,0}>0,\right.$ $\left.k_i^2=\mu^2,\ 1\leqq i\leqq m\right\}$ will be parametrized through the following choice of variables: Let $k=\sqrt{s}\,v_0(k)$, where $v_0(k)$ varies in the connected hyperboloid shell $H^+_{\nu-1}=\{v_0=(v_{0,0},\mathbf{v}_0),\ \mathbf{v}_0\in\mathbb{R}^{\nu-1},\ v_{0,0}>0,\ v_{0,0}^2-\mathbf{v}_0^2=1\}$ and let $(v_0(k),\ \{v_\alpha(k);\ \alpha=1,\ldots,\nu-1)\})$ be a continously varying ν-dimensional Lorentz frame with analytic dependence in k. We put for each index i, $1\leqq i\leqq m$:

$$\left.\begin{aligned}&k_i=\underline{k}_i^{(0)}v_0(k)+\sum_{1\leqq\alpha\leqq\nu-1}\underline{k}_i^{(\alpha)}v_\alpha(k)\,,\\&\underline{\mathbf{k}}_i=(\underline{k}_i^{(1)},\ldots,\underline{k}_i^{(\nu-1)}),\ \underline{\mathbf{k}}_i^2=\sum_{1\leqq\alpha\leqq\nu-1}(\underline{k}_i^{(\alpha)})^2\end{aligned}\right\}. \tag{10}$$

Then $e(k)$ can be represented as a $[(m-1)\nu-m]$ dimensional manifold in the space $\mathbb{R}^{(m-1)(\nu-1)}$ of the $(\nu-1)$ momenta $\underline{\mathbf{k}}_1,\ldots,\underline{\mathbf{k}}_{m-1}$ through the equation:

$$\sqrt{s}-m\mu-\Phi(\underline{\mathbf{k}}_1,\ldots,\underline{\mathbf{k}}_{m-1})=0\,, \tag{11}$$

where

$$\Phi(\underline{\mathbf{k}}_1,\ldots,\underline{\mathbf{k}}_{m-1})=\sum_{1\leqq i\leqq m}(\omega(\underline{\mathbf{k}}_i)-\mu)\,, \tag{12}$$

$$\underline{k}_i^{(0)}=\omega(\underline{\mathbf{k}}_i)=[\underline{\mathbf{k}}_i^2+\mu^2]^{1/2},\qquad 1\leqq i\leqq m\,, \tag{13}$$

$$\underline{\mathbf{k}}_m=-(\underline{\mathbf{k}}_1+\ldots+\underline{\mathbf{k}}_{m-1})\,. \tag{14}$$

Here Φ is an even function of $\underline{\mathbf{k}}_1,\ldots,\underline{\mathbf{k}}_{m-1}$, which admits a critical point at the origin $\left(\text{i.e. } \frac{\partial\Phi}{\partial\underline{k}_i^{(\alpha)}}=0,\ \forall i,\ \forall\alpha\right)$ with critical value $\Phi(0)=0$, the associated quadratic form being positive definite. In view of Morse's reduction theorem applied to the analytic case (see Theorem 4-1 in [4]), it is possible to find a new system of local coordinates $x_1,\ldots,x_r$, $r=(m-1)(\nu-1)$ such that[4]:

a) the mapping φ:

$$(x_1,\ldots,x_r)\to\underline{\mathbf{k}}_i=\underline{\mathbf{k}}_i(x_1,\ldots,x_r)\,,\qquad 1\leqq i\leqq m-1 \tag{15}$$

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4 The method used in the proof of Theorem 4-1 of [4] allows one to deduce the symmetry property of the mapping φ from the even character of Φ assumed here

is biholomorphic at the origin and satisfies the relations:

$$\underline{\mathbf{k}}_i(-x_1, \ldots, -x_r) = -\underline{\mathbf{k}}_i(x_1, \ldots, x_r), \quad 1 \leqq i \leqq m-1, \tag{16}$$

b)
$$\sum_{1 \leqq i \leqq r} x_i^2 = \Phi(\underline{\mathbf{k}}_1, \ldots, \underline{\mathbf{k}}_{m-1}). \tag{17}$$

Finally, by putting $x_i = \sqrt{\tau}\,\Omega_i$, $\Omega = \{\Omega_i; 1 \leqq i \leqq m\} \in \mathbb{S}_{r-1}$, we obtain the following analytic parametrization of $e(k)$:

$$(k_1, \ldots, k_m) = \psi_0(k, \Omega), \tag{18}$$

where $\Omega = (\Omega_1, \ldots, \Omega_r) \in \mathbb{S}_{r-1}$, $k = \sqrt{s}\, v_0$, $v_0 \in H^+_{\nu-1}$, $s > (m\mu)^2$ and ψ_0 is defined by Eqs. (10), (13), and (14) together with:

$$\underline{\mathbf{k}}_i = \underline{\mathbf{k}}_i(\sqrt{\tau}\,\Omega_1, \ldots, \sqrt{\tau}\,\Omega_r), \tag{19}$$

$$\tau = \sqrt{s} - m\mu = (s - (m\mu)^2)(\sqrt{s} + m\mu)^{-1}. \tag{20}$$

In view of (9), this yields correspondingly by the following parametrization of the mass shell variety $\mathscr{M}$:

$$(p_1, \ldots, p_m; p'_1, \ldots, p'_m) = \Psi_0(k; \Omega, \Omega') : (p_1, \ldots, p_m) = \psi_0(k, \Omega)$$
$$(p'_1, \ldots, p'_m) = \psi_0(k, \Omega'). \tag{21}$$

Ψ_0 is an analytic mapping defined on the real set

$$\{(k; \Omega, \Omega'), k = \sqrt{s}\, v_0, v_0 \in H^+_{\nu-1}, s > (m\mu)^2, (\Omega \times \Omega') \in \mathbb{S}_{r-1} \times \mathbb{S}_{r-1}\}.$$

On the other hand if σ denotes the inversion of momenta in the center-of-mass system, namely the transformation $\underline{\mathbf{p}}_i \to -\underline{\mathbf{p}}_i$, $\underline{\mathbf{p}}'_i \to -\underline{\mathbf{p}}'_i$, $p_i^{(0)}$, $p_i'^{(0)}$ being unchanged $(1 \leqq i \leqq m)$, then Ψ_0 satisfies the following symmetry property [in view of (16), (19), and (20)]:

$$\sigma[(p_1, \ldots, p_m; p'_1, \ldots, p'_m)] = \sigma[\Psi_0(k; \Omega, \Omega')]$$
$$= \Psi_0(k; -\Omega, -\Omega'). \tag{22}$$

We now consider the variables $k \equiv (s, v_0)$, Ω, Ω' as *complex* and assume that s varies in a (complex) cut neighbourhood of the threshold $s = (m\mu)^2$, with the cut along $s \geqq (m\mu)^2$, while (v_0, Ω, Ω') varies in a complex neighbourhood of $H^+_{\nu-1} \times \mathbb{S}_{r-1} \times \mathbb{S}_{r-1}$ belonging to the corresponding complexified manifold. Let $\mathscr{W}$ be the set thus defined in the complex space of variables (k, Ω, Ω'); then formulae (18), (19), and (20) remain meaningful for (k, Ω, Ω') in $\mathscr{W}$, with the specification $0 < \arg \tau^{1/2} < \pi$; they respectively provide analytic continuations ψ and Ψ of the mappings ψ_0 and Ψ_0 and the image of $\mathscr{W}$ by Ψ is by definition a "cut neighbourhood" of the threshold variety[5] in the complex mass shell $\mathscr{M}^c$.

5 Note that on the submanifold $s = (m\mu)^2$ of (k, Ω, Ω')-space $(\Psi)^{-1}$ defines a "blowing up" of the threshold subvariety, the latter being in fact a singular subset of $\mathscr{M}^c$

We notice that due to the occurrence of the factor $\tau^{1/2}$ [or $(s-(m\mu)^2)^{1/2}$] in (19), the limit Ψ_1 of Ψ on the cut $s>(m\mu)^2$ from the side $\mathrm{Im}\, s<0$ is different from Ψ_0 and such that:

$$\Psi_1(k;\Omega,\Omega')=\sigma[\Psi_0(k;\Omega,\Omega')]=\Psi_0(k;-\Omega,-\Omega'). \tag{23}$$

We can now express our *local analyticity assumption* on T as follows: T is analytic in a "cut neighbourhood of the threshold variety" of the form $\Psi(\mathcal{W})$. It will then be convenient in all the following to reexpress T in our new set of variables by putting:

$$T_\Psi(k;\Omega,\Omega')=T(\Psi(k;\Omega,\Omega'))\equiv T(p_1,\ldots,p_m;p'_1,\ldots,p'_m). \tag{24}$$

In view of the continuity assumption on $T^{(0)}$, $T^{(1)}$, the boundary values $T_\Psi^{(0)}$, $T_\Psi^{(1)}$ of T_Ψ at $s>(m\mu)^2$ from the respective sides $\mathrm{Im}\, s>0$, $\mathrm{Im}\, s<0$ exist as continuous functions and are linked to $T^{(0)}$, $T^{(1)}$ through the following relations:

$$T_\Psi^{(0)}(k;\Omega,\Omega')=T^{(0)}(\Psi_0(k;\Omega,\Omega'))=T^{(0)}(p_1,\ldots,p_m;p'_1,\ldots,p'_m), \tag{25}$$

$$T_\Psi^{(1)}(k;\Omega,\Omega')=T^{(1)}(\Psi_1(k;\Omega,\Omega'))=T^{(1)}(\sigma(p_1,\ldots,p_m;p'_1,\ldots,p'_m)), \tag{26}$$

the second equality in (26) being a consequence of relations (22) and (23).

At this point, we shall introduce our *symmetry assumption* on T, according to which T is invariant under the transformation σ. It entails [in view of (25), (26), and (22)] that:

$$T_\Psi^{(\varepsilon)}(k;\Omega,\Omega')=T^{(\varepsilon)}(p_1,\ldots,p_m;p'_1,\ldots,p'_m), \qquad \varepsilon=0,1, \tag{27}$$

$$T_\Psi(k;\Omega,\Omega')=T_\Psi(k;-\Omega,-\Omega'). \tag{28}$$

Throughout this section we shall then adopt without inconvenience (by an abuse of language) the notation $T(k,\Omega,\Omega')\equiv T_\Psi(k,\Omega,\Omega')$; in fact, the boundary values $T^{(0)}$, $T^{(1)}$ of $T(k,\Omega,\Omega')$ taken from the respective sides $\mathrm{Im}\, s>0$, $\mathrm{Im}\, s<0$ at $s=k^2>(m\mu)^2$ [and defined for each (v_0,Ω,Ω') fixed in the complex domain defined above] coincide in view of (27) and (28), for k,Ω,Ω' real, with the corresponding boundary values $T^{(0)}$, $T^{(1)}$, of $T(p_1,\ldots,p_m,p'_1,\ldots,p'_m)$ at the point $(p_1,\ldots,p'_m)=\Psi_0(k,\Omega,\Omega')$.

We are now in a position to prove

Proposition 1. *The unitarity equation* (1) *can be rewritten in terms of the new variables* (k,Ω,Ω') *as follows:*

$$T^{(0)}(k;\Omega,\Omega')-T^{(1)}(k;\Omega,\Omega') = \int_{\mathbb{S}_{(m-1)\nu-m}} T^{(0)}(k;\Omega,\Omega'')T^{(1)}(k;\Omega'',\Omega')\alpha(k;\Omega'')d\Omega'' \tag{29}$$

where

$$\alpha(k,\Omega)=((m\mu)^2-k^2)^{[(m-1)\nu-m-1]/2}\hat{\alpha}(k,\Omega), \tag{30}$$

$\hat{\alpha}$ *being an analytic function of* (k,Ω) *near* $s=k^2=(m\mu)^2$.

Proof. In formula (1), the substitution (18) is performed in the arguments of $T^{(0)}$, $T^{(1)}$, both for the external variables, by putting $(p_1,\ldots,p_m)=\psi_0(k,\Omega)$; $(p'_1,\ldots,p'_m)=\psi_0(k,\Omega')$, and for the integration variables by putting: $(k_1,\ldots,k_m)$

$=\psi_0(k,\Omega'')$. The left hand side of (1) is then, in view of (21) and (27), identical with the left hand side of (29). We shall now compute the integral (2) representing the right hand side of (1) by using the various equations corresponding to the mapping $(k_1,\ldots,k_m)=\psi_0(k;\Omega'')$: the integral (2) is first written in terms of the coordinates $(\underline{k}_i^{(0)},\underline{\mathbf{k}}_i)$ of $k_i\,(1\leqq i\leqq m)$ associated with the Lorentz frame $(v_0(k),\{v_\alpha(k)\})$ [see (10)], and the formal integration over the variables $\underline{k}_i^{(0)}$ $(1\leqq i\leqq m)$ and $\underline{\mathbf{k}}_m$ is done with the help of the δ-functions $\delta(k_i^2-\mu^2)$ $(1\leqq i\leqq m)$ and $\delta^{(\nu-1)}\left(\sum_{1\leqq i\leqq m}\underline{\mathbf{k}}_i\right)$. One is then led to the integral:

$$\int T^{(0)}T^{(1)}\left[\prod_{1\leqq i\leqq m}2\omega(\underline{\mathbf{k}}_i)\right]^{-1}\delta(\sqrt{s}-m\mu-\Phi(\underline{\mathbf{k}}_1,\ldots,\underline{\mathbf{k}}_{m-1})\\ d^{(\nu-1)}\underline{\mathbf{k}}_1,\ldots,d^{(\nu-1)}\underline{\mathbf{k}}_{m-1}\,, \tag{31}$$

in which we have used the notations introduced in (10)–(12).

By using the mapping φ [Eq. (15)] and taking formula (17) into account, one rewrites the integral (31) in the form:

$$\int T^{(0)}T^{(1)}J(x_1,\ldots,x_r)\delta\left(\sum_{1\leqq i\leqq r}x_i^2-(\sqrt{s}-m\mu)\right)dx_1,\ldots,dx_r\,, \tag{32}$$

where $J(x_1,\ldots,x_r)=\dfrac{D(\underline{\mathbf{k}}_1,\ldots,\underline{\mathbf{k}}_{m-1})}{D(x_1,\ldots,x_r)}\times\prod_{1\leqq i\leqq m}[2\omega(\underline{\mathbf{k}}_i)]^{-1}$ is a locally analytic function of $(x_1,\ldots,x_r)$; in view of (16), J is moreover even. Finally the substitution $x_i=\tau^{1/2}\Omega_i$, $(1\leqq i\leqq r)$ in (32) allows one to rewrite the latter as follows:

$$\int T^{(0)}T^{(1)}\hat{J}(\tau,\Omega'')\tau^{(r-2)/2}\delta(\tau-(\sqrt{s}-m\mu))d\tau\,d\Omega''\,, \tag{33}$$

where $d\Omega''$ is the canonical invariant measure on $\mathbb{S}_{r-1}$, and $\hat{J}(\tau,\Omega'')=\frac{1}{2}J(\tau^{1/2}\Omega''_1,\ldots,\tau^{1/2}\Omega''_r)$ is (since J is even) an analytic function of (τ,Ω'') at $\tau=0$. The formal integration of (33) over τ with the help of the δ-function $\delta(\tau-(\sqrt{s}-m\mu))$ then yields the right hand side of formula (29), with the following specification of $\hat{\alpha}$:

$$\hat{\alpha}(k,\Omega'')=\hat{J}(\sqrt{s}-m\mu,\Omega'')\times[-(\sqrt{s}+m\mu)]^{-(r-2)/2}\,, \tag{34}$$

$\hat{\alpha}$ being analytic at $s=(m\mu)^2$. QED.

In the following, we shall use again by convenience the notation $T^{(0)}*T^{(1)}$ for the right hand side of (29), so that Eq. (1) will represent also Eq. (29). More generally, being given two kernels $A(k;\Omega,\Omega')$, $B(k;\Omega,\Omega')$ depending analytically on the complex parameter k, we shall put:

$$(A*B)(k;\Omega,\Omega')=\int_{\mathbb{S}^{(m-1)\nu-m}}A(k;\Omega,\Omega'')B(k;\Omega'',\Omega')\alpha(k;\Omega'')d\Omega'' \tag{35}$$

and

$$(A\,\hat{*}\,B)(k;\Omega,\Omega')=\int_{\mathbb{S}^{(m-1)\nu-m}}A(k;\Omega,\Omega'')B(k;\Omega'',\Omega')\hat{\alpha}(k;\Omega'')d\Omega''\,, \tag{35'}$$

so that, in view of (30):

$$(A*B)(k;\Omega,\Omega')=\sigma^\beta(A\,\hat{*}\,B)(k;\Omega,\Omega')\,,$$

where $\sigma=(m\mu)^2-s$, $\beta=[(m-1)\nu-m-1]/2$.

3. Local Maximal Analyticity and Unitarity Equations

Equation (29) expresses the fact that for each k real, with $k^2>(m\mu)^2$, $T^{(1)}$ is the Fredholm resolvent[6] of $T^{(0)}$ at the value $\lambda=1$ of the Fredholm parameter λ, with respect to the integration measure $\alpha(k;\Omega'')d\Omega''$ on $\mathbb{S}_{r-1}$ (see the review of Fredholm theory in Sect. 4). Equation (29) will then be extended for complex values of the parameter k in the neighborhood of $k^2=(m\mu)^2$, the extension $T(k;\Omega,\Omega')$ of $T^{(0)}$ being considered as a given Fredholm kernel on the sphere $\mathbb{S}^{(m-1)\nu-m}$ (i.e., in Ω-space), depending analytically on k. In view of the analyticity of T in the physical sheet domain $\mathscr{W}$, and of the continuity of the boundary values $T^{(0)}$, $T^{(1)}$, the Fredholm resolvent of T then defines an analytic (or meromorphic) function in $\mathscr{W}$ whose boundary value at k real, $s=k^2>(m\mu)^2$ from the side $\operatorname{Im} s>0$ is continuous and coincides with $T^{(1)}$; it thereby defines (by applying an elementary form of the edge-of-the-wedge theorem) a second-sheet determination for T in $\mathscr{W}$[7]. (This argument is the direct analogue of that given in Sect. 2 of [6] in the case $m=2$, $\nu=4$ and belongs to the general framework of Fredholm theory with complex parameters [7, 3].) The analysis of the unitarity equation (29) then provides the following preliminary indications.

If $(m-1)(\nu-1)$ is odd, the factor $\sigma^\beta=[(m\mu)^2-k^2]^{[(m-1)\nu-m-1]/2}$ involved in the right hand side of (29) [see (30)] changes its sign when $s=k^2$ terms around $(m\mu)^2$ in the complex plane and Eq. (29) becomes after one turn [by using our notation (35)]:

$$T^{(1)}-T^{(2)}=-T^{(1)}*T^{(2)}. \tag{36}$$

By comparing Eqs. (1) and (36), one sees that $T^{(0)}$ and $T^{(2)}$ satisfy the same Fredholm equation in terms of $T^{(1)}$ and therefore coincide, i.e. T is two-sheeted and has a square-root type singularity at $s=(m\mu)^2$, as in the usual case of two-particle thresholds in dimension 4.

If on the other hand $(m-1)(\nu-1)$ is even, the factor σ^β is uniform around $k^2=(m\mu)^2$ and Eq. (36) is replaced by:

$$T^{(1)}-T^{(2)}=T^{(1)}*T^{(2)}. \tag{37}$$

By an argument of analytic continuation similar to that given above for $T^{(1)}$, one obtains an infinite number of Riemann sheets for T around $s=(m\mu)^2$, with possible poles in the various unphysical sheets. The unitarity equation can then be continued in the full Riemann surface of T, the successive determinations $T^{(r)}$ or T at $s>(m\mu)^2$ satisfying

$$T^{(r-1)}-T^{(r)}=T^{(r-1)}*T^{(r)}. \tag{38}$$

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6 Fredholm theory entails in particular that the right hand side of Eq. (29) can then be written either $T^{(0)}*T^{(1)}$ or $T^{(1)}*T^{(0)}$ [e.g. from Eq. (49), with $\lambda=0$, $\mu=1$]

7 The continuity assumption on $T^{(0)}$ and $T^{(1)}$ plays here the same role as the (weaker) regularity condition introduced and used by Martin [5] at $m=2$, $\nu=4$, to show the local maximal analyticity of the partial waves

By interpreting Eq. (38) in terms of Fredholm resolvents and applying Eq. (49) of Sect. 4, or by a direct algebraic argument, one also has:

Lemma 1.

$$T^{(0)} - T^{(r)} = r T^{(0)} * T^{(r)}. \tag{39}$$

The following criterium, which entails the non holonomicity of the solutions of Eq. (1) as a general rule (see footnote 2), then holds:

Proposition 2. *T does not satisfy the finite-determination property at $s=(m\mu)^2$ unless $T^{(0)*q} \equiv 0$ for some >0 integer q ($T^{(0)*q} = T^{(0)} * T^{(0)} \ldots * T^{(0)}$, q factors).*

Proof. A general proof is given in [3]. We give here a proof valid for $\beta > 0$ under the assumption that $T^{(0)}$ is bounded near the threshold, namely that for any real K, $K^2 = (m\mu)^2$, $K_0 > 0$, $|T^{(0)}|$ is uniformly bounded by a constant C_K when k lies in a real neighborhood W_K of K. Under this condition, the Neumann series $\sum_{n=0}^{\infty} (-r)^n T^{(0)*(n+1)}$ of $T^{(r)}$ in (39) is absolutely convergent for

$$|\sigma|^\beta < d_r = (r C_K C'_K)^{-1}, \quad \text{where} \quad C'_K = \operatorname*{Max}_{k \in W_K} \int |\hat{\alpha}(k; \Omega)| d\Omega,$$

in view of the bounds $|T^{(0)*(n+1)}| \leq C_K (C'_K C_K |\sigma|^\beta)^n$ easily derived from the definition of $*$. $T^{(r)}$ is then equal to the sum of this series.

Any relation of the form $\sum_{r=0}^{r_0} a_r T^{(r)} = 0$, with given constants $a_0, \ldots, a_{r_0}$ ($a_0 \neq 0$, $a_{r_0} \neq 0$), entails if it holds the relation:

$$\sum_{n=0}^{\infty} c_n T^{(0)*(n+1)} = 0, \qquad c_n = \sum_{r=0}^{r_0} (-r)^n a_r \tag{40}$$

for $|\sigma|^\beta < d_{r_0}$. One c_n at least, $n = 1, \ldots, r_0$, is non zero since the determinant $|i^j|$, $i, j = 1, \ldots, r_0$ is non zero. Let $n_0 = \operatorname{Inf}\{n; c_n \neq 0\}$. Equation (40) gives:

$$-c_{n_0} T^{(0)*(n_0+1)} = T^{(0)*(n_0+1)} * \left[\sum_{n=0}^{\infty} c_{n+n_0+1} T^{(0)*(n+1)} \right]. \tag{41}$$

The bound $|c_n| < \left(\sum_{r=0}^{r_0} |a_r| \right) r_0^n$ and the previous bounds on the terms $|T^{(0)*(n+1)}|$ allow one to show that the sum $\sum$ in the right hand side is bounded in modulus e.g. by $2(\sum |a_r|) r_0^{n_0+1} C_K$ for $|\sigma|^\beta < d_{r_0}/2$. Hence the right hand side of (41) is itself bounded in modulus, up to a multiplicative constant, by $\operatorname{Max}_{\Omega, \Omega'} |T^{(0)*(n_0+1)}| \times |\sigma|^\beta$. Since $|\sigma|$ can be chosen arbitrarily small, Eq. (41) thus entails that $T^{(0)*q} \equiv 0$ with $q = n_0 + 1$. Conversely, if $T^{(0)*(q)} \equiv 0$ for some q, then all terms $T^{(0)*(n+1)}$, $n \geqq q$, vanish; the finite determination property and the holonomicity of T follow. Q.E.D.

4. Fredholm Theory with Complex Parameters: Review and Auxiliary Results

We give in this section a review of Fredholm theory, adapted to the present case of analytically dependent kernels, and a complementary lemma which will be useful throughout this section. The results apply to any compact measurable space of variables Ω with measure $d\Omega$ and to an integration operation $*$ of the form (35), with an arbitrary (continuous) weight $\alpha(k,\Omega)$ depending analytically on k.

Being given a Fredholm kernel $A(k;\Omega,\Omega')$, depending analytically on complex parameters k, its Fredholm resolvent, defined by

$$R_\lambda(k;\Omega,\Omega')=A(k;\Omega,\Omega')+\lambda(R_\lambda * A)(k;\Omega,\Omega') \tag{42}$$

is a meromorphic function of (λ,k) of the form:

$$R_\lambda(k;\Omega,\Omega')=\frac{N(\lambda,k;\Omega,\Omega')}{D(\lambda,k)}, \tag{43}$$

N and D being the standard Fredholm series, which are analytic with respect to k in the same domain as $A(k)$ and are entire functions of λ. In the following, we shall often identify (by convenience) kernels such as $A(k;\Omega,\Omega')$, $R_\lambda(k;\Omega,\Omega')$ with the corresponding k-dependent operators $A(k)$, $R_\lambda(k)$ and will also use the notation $A(k)*B(k)$ for the k-dependent operator whose kernel is $(A*B)(k;\Omega,\Omega')$. We denote by $a_i(k)$, i belonging to a finite or denumerable set I, the eigenvalues of the operator $A(k)$. The latter are the inverses of the characteristic values $\lambda_i(k)$, which are the poles of $R_\lambda(k)$ and are more precisely the solutions of the equation $D(\lambda,k)=0$. The functions $\lambda_i(k)$ are thus analytic in the same domain as $A(k)$, except possibly at a discrete set of branch points. Each associated projector $E_i(k)$ is a meromorphic function of k, with the same branch points as $\lambda_i(k)$, its kernel being defined by the Cauchy integral

$$E_i(k;\Omega,\Omega')=\frac{i}{2\pi}\oint R_\lambda(k;\Omega,\Omega')\,d\lambda \tag{44}$$

around the point $\lambda_i(k)$ in the complex plane of the Fredholm parameter λ.

With each eigenvalue function $a_i(k)=1/\lambda_i(k)$ is associated a unique decomposition of $A(k)$:

$$A(k)=A_i(k)+A'_i(k), \tag{45}$$

where A_i and A'_i depend analytically on k, with possible branch points; the "principal kernel" $A_i(k)$ [relative to the eigenvalue function $a_i(k)$] is of finite rank and admits $a_i(k)$ as its single eigenvalue; the associated "regular-kernel" $A'_i(k)$ has all the remaining eigenvalues of $A(k)$ and

$$A_i(k)*A'_i(k)=A'_i(k)*A_i(k)=0. \tag{46}$$

The resolvent $R_\lambda(k)$ admits the corresponding decomposition:

$$R_\lambda(k)=R_{\lambda,i}(k)+R'_{\lambda,i}(k), \tag{47}$$

where $R_{\lambda,i}$, respectively $R'_{\lambda,i}$, is the Fredholm resolvent of A_i, respectively A'_i, and

$$\forall \lambda, \mu: R_{\lambda,i}(k) * R'_{\mu,i}(k) = 0. \tag{48}$$

Finally, the following relation is satisfied by the resolvent $R_\lambda(k)$ of $A(k)$ at different values of the Fredholm parameter:

$$\begin{aligned}\forall \lambda, \mu: R_\lambda(k) - R_\mu(k) &= (\lambda - \mu) R_\lambda(k) * R_\mu(k) \\ &= (\lambda - \mu) R_\mu(k) * R_\lambda(k).\end{aligned} \tag{49}$$

Details will be found in [8] and in [7, 3].

The following result then holds:

Lemma 2. *Let $A(k)$ and $B(k)$ be two Fredholm operators linked by the relation*

$$A(k) = B(k) + \varphi(k) A(k) * B(k), \tag{50}$$

A, B and the function φ depending analytically on k. Then

(i) *There exists a bijection between the sets of eigenvalue functions a_i and b_i of $A(k)$ and $B(k)$ given by the relation:*

$$a_i(k) = b_i(k) + \varphi(k) a_i(k) b_i(k). \tag{51}$$

(ii) *The operators $A(k)$ and $B(k)$ admit the same system of projectors $E_i(k)$ associated with the respective eigenvalues $a_i(k)$ or $b_i(k)$ satisfying* (51).

(iii) *The principal kernels $A_i(k)$, respectively $B_i(k)$, of $A(k)$, respectively $B(k)$, associated with the given eigenvalues $a_i(k)$, respectively $b_i(k)$, satisfy the relation*

$$A_i(k) = B_i(k) + \varphi(k) A_i(k) * B_i(k). \tag{52}$$

The corresponding relation between the associated regular kernels $A'_i(k)$ and $B'_i(k)$ in the decomposition (45) *relative to $A(k)$ and $B(k)$ also holds:*

$$A'_i(k) = B'_i(k) + \varphi(k) A'_i(k) * B'_i(k). \tag{53}$$

Proof. We denote below by $R_{\lambda,A}(k)$ and $R_{\lambda,B}(k)$ the respective resolvents of $A(k)$ and $B(k)$. By noticing that $A(k)$ is the Fredholm resolvent of $B(k)$ for the value $\varphi(k)$ of the Fredholm parameter, one easily obtains, in view of (49):

$$R_{\lambda,A}(k) = R_{\lambda+\varphi(k),B}(k). \tag{54}$$

The characteristic values $\lambda_{i,A}(k)$ and $\lambda_{i,B}(k)$ of $A(k)$ and $B(k)$ (which are the respective poles of $R_{\lambda,A}$ and $R_{\lambda,B}$) are therefore linked by the relations

$$\forall i, \ \lambda_{i,A}(k) = \lambda_{i,B}(k) - \varphi(k), \tag{55}$$

from which (51) follows.

Property (ii) of the lemma then follows from (54) by taking into account the expression (44) of the projectors ($E_{i,A} = E_{i,B}$).

To show property (iii), we write for $B(k)$ a decomposition of the form (45), namely $B(k) = B_i(k) + B'_i(k)$, where $B_i(k)$ is the principal kernel associated with the eigenvalue $b_i(k)$ of $B(k)$. The corresponding decomposition (47), relative to $R_{\lambda,B}(k)$ for $\lambda = \varphi(k)$ can be written: $A(k) = A_i(k) + A'_i(k)$. The relations (52) and (53) then express the fact that $A_i(k)$, respectively $A'_i(k)$, is the resolvent of $B_i(k)$, respectively $B'_i(k)$, for $\lambda = \varphi(k)$. Finally, applying the previous result (i) to each relation (52) or

(53) shows that $A_i(k)$ admits the single eigenvalue $a_i(k)$ while $A_i'(k)$ has all the remaining eigenvalues $a_j(k)$. This shows that the finite rank operator $A_i(k)$ is indeed the principal kernel associated with $a_i(k)$. Q.E.D.

The following Fredholm norm of the operator $A(k)$ (for each given k) will be used:

$$\|A(k)\| = [\int |A(k;\Omega,\Omega')|^2 \times |\alpha(k,\Omega)\alpha(k,\Omega')| \, d\Omega \, d\Omega']^{1/2}. \tag{56}$$

A standard result due to Schur (see [8b]), asserts that the eigenvalues $a_i(k)$ of $A(k)$ satisfy the inequality:

$$\sum_i |a_i(k)|^2 \leqq \|A(k)\|^2. \tag{57}$$

The following bound on each individual eigenvalue is obtained as a weak by product of the latter:

$$\forall i, \ |a_i(k)| \leqq \|A(k)\|. \tag{58}$$

[Better bounds on each a_i follow from the general theory, but (58) will be sufficient for our purposes.]

5. The Hermitian Case: Orthogonal Decomposition and Nonholonomicity of T

In this section, we come back to the simplified theory defined in Sects. 1 and 2, and we study the solutions T of the unitarity equation (1) which satisfy the additional condition of (anti)-hermitian analyticity in the physical sheet, near $s=(m\mu)^2$:

$$T(k;\Omega,\Omega') + \overline{T(\bar{k};\Omega,\Omega')} = 0. \tag{59}$$

A condition of this type holds in field theory for the Green's functions and follows for the $m \to m$ scattering functions by restriction to the complex mass shell (assuming that the corresponding analyticity has been established).

We then have:

Theorem 1. "*If the scattering function T satisfies the local analyticity, unitarity and symmetry properties of our simplified theory, together with condition* (59) *of (anti)-hermitian analyticity, and if $(m-1)(v-1)$ is even, then:*

i) *T admits in the neighbourhood of $k^2=(m\mu)^2$ a complete denumerable set of (nonholonomic) eigenvalues $t_i(k)$ of the form:*

$$t_i(k) = \left[\frac{1}{2i\pi}\ln((m\mu)^2 - k^2) + b_i(k)\right]^{-1} \tag{60}$$

and a corresponding system of orthogonal projectors $E_i(k)$ $(E_i(k) * E_j(k) = E_i(k)\delta_{ij})$, where the functions $b_i(k)$ and the kernels $E_i(k;\Omega,\Omega')$ depend analytically on k and are uniform around $k^2=(m\mu)^2$; the functions b_i satisfy $b_i(k)+\overline{b_i(\bar{k})}=0$.

ii) *The following expansion holds near the threshold for any k real, $k^2 \neq (m\mu)^2$, in each Riemann sheet*[9], *in the sense of L^2-convergence on $\mathbb{S}^{(m-1)v-m} \times \mathbb{S}^{(m-1)v-m}$:*

$$T(k;\Omega,\Omega') = \sum_i t_i(k) E_i(k;\Omega,\Omega'). \tag{61}$$

9 With the exception of the possible real poles for the unphysical sheet determinations of T

iii) *T has a nonholonomic singularity at $k^2=(m\mu)^2$.*

iv) *If T is uniformly bounded near $k^2=(m\mu)^2$ in the physical sheet, and if $\beta=\frac{(m-1)\nu-m-1}{2}>0$ the functions $\hat{u}_i(k)=u_i(k)/\sigma^\beta$, where $u_i(k)=1/b_i(k)$ and $\sigma=(m\mu)^2-s$, are locally analytic at $k^2=(m\mu)^2$".*

Proof. We prove below the assertions i), iii), and iv) of the theorem and the decomposition (61) of ii) in the region $k^2<(m\mu)^2$. An alternative proof of i) and the proof of (61) in the region $k^2>(m\mu)^2$ will be given at the end of Sect. 6.

Let $\{t_i(k), E_i(k); i\in I\}$ be the system of eigenvalue functions and associated projectors of $T(k)$, which are analytic in k near $s=(m\mu)^2$ $(s\neq(m\mu)^2)$ and could a priori have a discrete set of branch points (see Sect. 4). For any given $i\in I$, let us consider any path of analytic continuation in the physical sheet around $s=(m\mu)^2$ that starts and ends at a given real point s_0, $s_0>(m\mu)^2$, and stays away from the possible singularities of t_i and E_i. We denote by $t_i^{(0)}$, $E_i^{(0)}$, respectively $t_i^{(1)}$, $E_i^{(1)}$ the respective determinations of t_i, and E_i at $s>(m\mu)^2$ from the sides $\mathrm{Im}\,s>0$ and $\mathrm{Im}\,s<0$. The unitarity equation (1) and (29) and Lemma 2 [applied to the case $\varphi(k)=1$] entail the existence of an element $j\in I$ (independent of s in the neighbourhood of s_0) such that:

$$t_j^{(0)}(k)-t_i^{(1)}(k)=t_j^{(0)}(k)\,t_i^{(1)}(k)\,, \tag{62}$$

$$E_i^{(1)}(k;\Omega,\Omega')=E_j^{(0)}(k;\Omega,\Omega')\,. \tag{63}$$

A particular possible solution of (62), with $j\equiv i$, is $t_i(k)=\left[\frac{1}{2i\pi}\ln((m\mu)^2-k^2)\right]^{-1}$ in which case E_i is uniform around $s=(m\mu)^2$ [in view of (63)]; such an eigenvalue, if it exists, satisfies Eq. (60) with $b_i\equiv 0$. For a different eigenvalue function $t_i(k)$, it is convenient to introduce the function $u_i(k)$ defined through the relation:

$$\frac{1}{u_i(k)}=\frac{1}{t_i(k)}-\frac{1}{2i\pi}\ln((m\mu)^2-k^2)\,. \tag{64}$$

It then follows from (62) that (near s_0):

$$u_i^{(1)}(k)=u_j^{(0)}(k)\,. \tag{65}$$

The condition of (anti) hermitian analyticity (59) ensures that t_i, and hence u_i, is purely imaginary in the physical sheet at $k^2<(m\mu)^2$ and that E_i is symmetric in that region $(E_i(k;\Omega,\Omega')=\overline{E_i(k,\Omega',\Omega)})$. We show below that these properties (also applied to u_j, E_j) necessarily entail that $j\equiv i$, i.e. in view of (63), (65) that u_i and E_i are uniform around $s=(m\mu)^2$.

In fact, since u_i is purely imaginary at $k^2<(m\mu)^2$, one has, in the region $k^2>(m\mu)^2$:

$$u_i^{(0)}(k)+\overline{u_i^{(1)}(k)}=0\,, \tag{66}$$

and thus, in view of (65)

$$u_i^{(0)}(k)+\overline{u_j^{(0)}(k)}=0\,. \tag{67}$$

The same arguments as above applied to the element j of I show the existence of an element $i' = i(j) \in I$ such that:

$$u_j^{(1)}(k) = u_{i'}^{(0)}(k), \tag{68}$$

$$E_j^{(1)}(k) = E_{i'}^{(0)}(k), \tag{69}$$

$$u_j^{(0)}(k) + \overline{u_{i'}^{(0)}(k)} = 0. \tag{70}$$

The comparison of (67) and (70) entails that $u_i^{(0)}(k) \equiv u_{i'}^{(0)}(k)$ near s_0, and hence that $i = i'$. If $j \neq i$, let us then introduce the operator:

$$U_{ij}(k; \Omega, \Omega') = u_i(k) E_i(k; \Omega, \Omega') + u_j(k) E_j(k; \Omega, \Omega'). \tag{71}$$

This operator is uniform around $s = (m\mu)^2$ in view of (65), (68), (63), and (69) and of the identity $i = i'$. Since u_i, u_j are purely imaginary in the physical sheet in the region $k^2 < (m\mu)^2$ and since E_i, E_j are symmetric in this region, U_{ij} is antisymmetric there. In view of its uniformity, it is therefore also antisymmetric at $k^2 > (m\mu)^2$, and hence its eigenvalues u_i, u_j are again purely imaginary there. Since u_i, u_j are purely imaginary both at $k^2 < (m\mu)^2$ and at $k^2 > (m\mu)^2$, they are necessarily uniform, which contradicts the assumption $j \neq i$. Since Eq. (64) can be rewritten under the form (60) with $b_i(k) = 1/u_i(k)$, this achieves the proof of property (i).

Let now $\underline{T}^{(r)}$ be the r^{th} determination of T in the region $k^2 < (m\mu)^2$, $\underline{T}^{(0)}$ being the physical sheet determination. Similarly let $\underline{t}_i^{(r)}(k)$, be the corresponding r^{th} determination of $t_i(k)$. Formula (61) readily follows for $\underline{T}^{(0)}(k)$ from the Hilbert-Schmidt decomposition of the anti-hermitian operator $\underline{T}^{(0)}(k)$ (the eigenvalues and projectors occurring in that decomposition being obviously identical with those obtained above from Fredholm theory). We now establish it for all other determinations $\underline{T}^{(r)}(k)$.

At each real point k, $k^2 < (m\mu)^2$, which is not a pole of $\underline{T}^{(r)}$, Shur's result [see (57)] applied to $\underline{T}^{(r)}(k)$ shows that $\Sigma |\underline{t}_i^{(r)}(k)|^2 \leqq \| T^{(r)}(k) \|^2 < \infty$ and thus entails, in view of the hermitian property of the orthogonal projectors $E_i(k)$ that the series $\Sigma_i \underline{t}_i^{(r)}(k) \underline{E}_i(k, \Omega, \Omega')$ is convergent [in the sense of L^2-convergence in (Ω, Ω')-space]. Let $\underline{\Sigma}^{(r)}(k; \Omega, \Omega')$ be the sum of this series. We check below that $\underline{\Sigma}^{(r)} \equiv \underline{T}^{(r)}$; this follows from the identity $\underline{\Sigma}^{(0)} = \underline{T}^{(0)}$ already proved above, and from the fact that $\underline{\Sigma}^{(r)}$ and $\underline{T}^{(r)}$ are solutions of the same Fredholm resolvent equation with given kernel $\underline{\Sigma}^{(0)} \equiv \underline{T}^{(0)}$.

We have indeed:

$$\underline{T}^{(0)} - \underline{T}^{(r)} = r \underline{T}^{(0)} * \underline{T}^{(r)}, \tag{72}$$

$$\underline{\Sigma}^{(0)} - \underline{\Sigma}^{(r)} = r \underline{\Sigma}^{(0)} * \underline{\Sigma}^{(r)}. \tag{73}$$

Equation (72) is obtained by analytic continuation from Eq. (39); Eq. (73) can be derived by standard arguments of convergence from the hermitian property and orthogonality of the $\underline{E}_i(k)$ and from the relations:

$$\underline{t}_i^{(0)}(k) - \underline{t}_i^{(r)}(k) = r \underline{t}_i^{(0)}(k) \underline{t}_i^{(r)}(k) \tag{74}$$

that directly follow from (64) and (60).

This achieves the proof of property (ii) at $k^2 < (m\mu)^2$, in all Riemann sheets.

The nonholonomicity of T follows from that of the coefficients $t_i(k)$: in view of the decomposition (61) established for each $\underline{T}^{(r)}$, any condition of the form $\sum_{r=0}^{r_0} a_r \underline{T}^{(r)} = 0$ would yield $\sum_{r=0}^{r_0} a_r \underline{t}_i^{(r)}(k) = 0$ for each i, which contradicts (60) (see e.g. [1a]). (Note that condition (59) in fact forbids, in view of (61), the pathology $T^{(0)*q} \equiv 0$ considered in Sect. 3, which would yield $[t_i(k)]^q \equiv 0$, $\forall i$, hence $T \equiv 0$.)

Finally, if T is uniformly bounded near $\sigma = (m\mu)^2 - k^2 = 0$ in the physical sheet, its eigenvalues $t_i(k)$ are uniformly bounded in modulus, near $\sigma = 0$, [see (58)] by

$$\mathrm{Sup}_k \| T(k) \| \leqq |\sigma^\beta| \times \mathrm{Sup}_{k,\Omega,\Omega'} |T(k;\Omega,\Omega')| \times \mathrm{Sup}_k \int |\hat{\alpha}(k;\Omega)| d\Omega < \infty .$$

Hence, in view of (60), $\hat{u}_i(k) = [\sigma^\beta b_i(k)]^{-1}$ is bounded near $\sigma = 0$; $\hat{u}_i$ is thus analytic at $\sigma = 0$ since it is uniform around $\sigma = 0$. Q.E.D.

Remark. An alternative method, based on the explicit form of holonomic functions indicated in footnote 2, could be used to derive the nonholonomicity of T, for $(m-1)(\nu-1)$ even, from the same assumptions as in Theorem 1. In particular, elementary arguments show that (anti) hermitian holonomic functions with regular singularities (i.e. with coefficients $a_{\alpha,j}$ analytic at $z = 0$ cannot satisfy the unitarity equation (1).

As an introduction to Sect. 6, we conclude with the following heuristic comment. Let us assume for simplicity that all eigenvalues $t_i(k)$ are such that $b_i(k)$ is not identically zero: this is the case for instance if T is bounded near $\sigma = 0$ in the physical sheet. The proof of Theorem 1 then suggests the introduction of the kernel $U = \Sigma_i u_i E_i$. If this sum is convergent, it follows from our previous analysis that U is uniform when k^2 turns around $(m\mu)^2$. In view of the orthogonality relations $E_i * E_j = \delta_{i,j} E_i$ and of the relation (64), written in the form:

$$t_i(k) = u_i(k) + \frac{i}{2\pi} \mathrm{In}((m\mu)^2 - k^2) t_i(k) u_i(k) . \tag{75}$$

U satisfies on the other hand the integral relation:

$$T = U + \frac{i}{2\pi} \mathrm{In}((m\mu)^2 - k^2) T * U . \tag{76}$$

Equation (76) will be used in Sect. 6 to define rigorously the kernel U in terms of T independently of the condition of (anti) hermitian analyticity of T, and its uniformity will be proved to be equivalent to the unitarity equation. Closely related results will also be obtained if

$$t_0(k) = \left[\frac{i}{2\pi} \mathrm{In}((m\mu)^2 - k^2) \right]^{-1}$$

is one of the eigenvalue functions of T.

6. On-Shell Irreducible Kernel U

Being given the scattering function T, Eq. (76) defines U in a unique way for each given k through Fredholm theory, provided that $\lambda_0(k) = -\frac{i}{2\pi} \mathrm{In}((m\mu)^2 - k^2)$ is not a characteristic value of $T(k)$. Hence, by Fredholm theory with complex parameters (see Sect. 4), $U(k;\Omega,\Omega')$ is a well-defined meromorphic function of k in the same Rieman domain as $T(k,\Omega,\Omega')$ provided that $t_0(k)=1/\lambda_0(k)$ is not one of the eigenvalue functions $t_i(k)$ of T.

If $t_0(k)$ is an eigenvalue function of T, there exists (see Sect. 4) a decomposition of T of the form:

$$T(k;\Omega,\Omega')=T'(k;\Omega,\Omega')+T''(k;\Omega,\Omega'), \tag{77}$$

where T'' is the principal kernel of T corresponding to the eigenvalue $t_0(k)$ and the associated regular kernel T' does not have the eigenvalue function $t_0(k)$; T' and T'' are analytic in the same domain as T and $T' * T''=0$. U can then be defined (as a meromorphic function in the same domain as T) through the Fredholm equation:

$$T'=U+\frac{i}{2\pi}\ln((m\mu)^2-k^2)\,T' * U. \tag{78}$$

In the following, we shall consider Eq. (78) as the general definition of U, since it reduces to (76) ($T''=0$, $T=T'$) if $t_0(k)$ is not an eigenvalue function of T.

The following results can then be derived from Lemma 2 of Sect. 4:

Lemma 3. *The unitarity equation* (1) *and* (29) *implies the analogous equation for* T':

$$T'^{(0)}-T'^{(1)}=T'^{(0)} * T'^{(1)}, \tag{79}$$

where $T'^{(0)}$ *and* $T'^{(1)}$ *are the physical sheet boundary values of* T' *at* $k^2>(m\mu)^2$ *from the respective sides* $\operatorname{Im}s>0$ *and* $\operatorname{Im}s<0$.

Proof. Part (iii) of Lemma 2, applied to $A=T^{(0)}$, $B=T^{(1)}$, $\varphi=1$, shows that the regular kernels of $T^{(0)}$ and $T^{(1)}$ associated respectively with the eigenvalue functions $t_0(k)$ and $t_{0,1}(k)=\left[\frac{i}{2\pi}\ln((m\mu)^2-k^2)-1\right]^{-1}$ satisfy the same integral equation as $T^{(0)}$ and $T^{(1)}$. In view of the analytic dependence on k of T, and hence of T', these kernels are respectively $T'^{(0)}$ and $T'^{(1)}$. QED

Let $\{t_i(k), E_i(k); i\in I\}$ denote, as in Sect. 5, the system of eigenvalue functions and associated projectors of T. In view of Eq. (78), we have:

Lemma 4. *U admits the same system of projectors $\{E_i(k); i\in I, i \neq 0\}$ as T' and its eigenvalues $u_i(k)$ are analytic functions of k around $s=(m\mu)^2$ that satisfy the relations:*

$$\forall i\in I, i\neq 0;\; t_i(k)=u_i(k)+\frac{i}{2\pi}\ln((m\mu)^2-k^2)\,t_i(k)u_i(k). \tag{80}$$

Lemma 4 follows directly from Part (i) of Lemma 2.

We next prove the following algebraic relation[10], which will allow one to show the equivalence between the unitarity equation (79) and the uniformity of U with respect to k around $s=(m\mu)^2$.

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10 An analogous algebraic relation, which allows one to prove the equivalence between off-shell unitarity equations and the uniformity of the Bethe-Salpeter kernel at threshold, holds for $m=2,3$ in the exact theory (see [9]) and for every m in the simplified theory [1]

Lemma 5. *If* $(m-1)(\nu-1)$ *is even, then:*

$$(T'^{(0)}-T'^{(1)}-T'^{(0)}*T'^{(1)})*\left(\mathbb{1}-\left[\frac{i}{2\pi}\ln((m\mu)^2-k^2)-1\right]U^{(1)}\right)$$
$$=\left(\mathbb{1}+\frac{i}{2\pi}\ln((m\mu)^2-k^2)\,T'^{(0)}\right)*(U^{(0)}-U^{(1)}), \tag{81}$$

where $U^{(0)}$ *and* $U^{(1)}$ *are the physical sheet boundary values of* U *at* $k^2>(m\mu)^2$ *from the respective sides* $\operatorname{Im} s>0$ *and* $\operatorname{Im} s<0$.

Proof. The relations

$$T'^{(0)}=U^{(0)}+\frac{i}{2\pi}\ln((m\mu)^2-k^2)\,T'^{(0)}*U^{(0)}, \tag{82}$$

$$T'^{(1)}=U^{(1)}+\left[\frac{i}{2\pi}\ln((m\mu)^2-k^2)-1\right]T'^{(1)}*U^{(1)} \tag{83}$$

yield [since $T'^{(0)}*U^{(0)}-T'^{(1)}*U^{(1)}=(T'^{(0)}-T'^{(1)})*U^{(1)}+T'^{(0)}*(U^{(0)}-U^{(1)})$]:

$$(T'^{(0)}-T'^{(1)})*\left(\mathbb{1}-\left[\frac{i}{2\pi}\ln((m\mu)^2-k^2)-1\right]U^{(1)}\right)$$
$$=\left(\mathbb{1}+\left[\frac{i}{2\pi}\ln((m\mu)^2-k^2)-1\right]T'^{(0)}\right)*(U^{(0)}-U^{(1)})+T'^{(0)}*U^{(0)}. \tag{84}$$

Equation (81) follows by subtracting from both sides of (84)

$$T'^{(0)}*T'^{(1)}*\left(\mathbb{1}-\left[\frac{i}{2\pi}\ln((m\mu)^2-k^2)-1\right]U^{(1)}\right),$$

which is also equal in view of (83) to $T'^{(0)}*U^{(1)}$. QED.

Remark. If $(m-1)(\nu-1)$ is odd, the analogue of Eq. (76) is a direct extension of Zimmermann's equation (of the case $m=2$, $\nu=4$), namely:

$$T=K+\tfrac{1}{2}T*K. \tag{85}$$

We assume here for simplicity that $-\frac{1}{2}$ is not a fixed characteristic value of $T(k;\Omega,\Omega')$; this condition is guaranteed in particular if the (anti) hermitian analyticity property (59) is assumed. (Otherwise, T should be replaced by the kernel T' obtained by subtracting from T the principal kernel associated with the characteristic value $-1/2$.)

The analogue of Eq. (81) is then:

$$(T^{(0)}-T^{(1)}-T^{(0)}*T^{(1)})*(\mathbb{1}+\tfrac{1}{2}K^{(1)})=(\mathbb{1}+\tfrac{1}{2}T^{(0)})*(K^{(0)}-K^{(1)}). \tag{86}$$

The proof of Eq. (86) from Eq. (85) is similar to the proof given above for Eq. (81).

We now state:

Theorem 2. "*The following properties are equivalent if* $(m-1)(\nu-1)$ *is even:*

a) T' *satisfies the unitarity equation* (79).

b) *The associated kernel* $U(k;\Omega,\Omega')$ *is uniform around the threshold* $s=(m\mu)^2$.

If $\beta=\dfrac{(m-1)\nu-m-1}{2}>0$, *conditions* a) *and* b) *can be replaced by the stronger conditions:*

a') *Equation* (79) *still holds in the limit* $k^2=(m\mu)^2$ *and* $T'^{(0)}=T'^{(1)}$ *in this limit.*

b') U *is analytic at* $k^2=(m\mu)^2$".

Proof. In view of the invertibility[11] of the kernels $\mathbb{1}-\frac{i}{2\pi}\ln((m\mu)^2-k^2)U^{(0)}$ and $\mathbb{1}+\left[\frac{i}{2\pi}\ln((m\mu)^2-k^2)-1\right]T'^{(1)}$, Lemma 5 entails the equivalence between Eq. (79) and the relation $U^{(0)}-U^{(1)}=0$, i.e. the uniformity of U. This proves the equivalence between a) and b).

The equivalence between a') and b') is obtained by the same methods as those used below in the proof Theorem 3'.

We first state the following direct corollary of the first part of Theorem 2 and of Lemma 3.

Theorem 3. *If* $(m-1)(\nu-1)$ *is even and if* T *satisfies the local analyticity, unitarity and symmetry properties of our simplified theory, then the associated kernel* U *(defined through* Eq. (77), (78)*) is uniform around* $k^2=(m\mu)^2$.

The following more refined version of Theorem 3 also holds:

Theorem 3'. "*Let* $(m-1)(\nu-1)$ *be even and let* $\beta=\dfrac{(m-1)\nu-m-1}{2}$ *be* >0. *If* T *satisfies the local analyticity, unitarity and symmetry properties of our simplified theory, and if* T *is moreover uniformly bounded in the physical sheet near* $k^2=(m\mu)^2$, *then*

(i) U *is analytic with respect to* k *at* $k^2=(m\mu)^2$.

(ii) $T=T'$, *and* T *admits following expansion*[12] *in the neighborhood of* $k^2=(m\mu)^2$:

$$\mathrm{T}(\mathrm{k};\Omega,\Omega')=\sum_n U^{*(n+1)}(k;\Omega,\Omega')\left(\frac{i}{2\pi}\ln\sigma\right)^n$$

$$=\sum_n U^{\circledast(n+1)}(k;\Omega,\Omega')\left(\frac{i}{2\pi}\sigma^\beta\ln\sigma\right)^n, \tag{87}$$

($\sigma=(m\mu)^2-k^2$), *which is uniformly convergent in the Riemann surface of* T *for* $\operatorname{Arg}\sigma$ *bounded,* $|\sigma|$ *sufficiently small and* $(\Omega,\Omega')\in\mathbb{S}_{2\beta+1}\times\mathbb{S}_{2\beta+1}$.

(iii) $$\lim_{k\to\underline{k},\,\underline{k}^2=(m\mu)^2} T(k;\Omega,\Omega')=U(\underline{k};\Omega,\Omega') \tag{88}$$

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11 For instance the inverse of $\mathbb{1}+\frac{i}{2\pi}\ln((m\mu)^2-k^2)T'^{(0)}$ is in view of Eq. (82): $\mathbb{1}-\frac{i}{2\pi}\ln((m\mu)^2-k^2)U^{(0)}$

12 The following related result is given in [10]: if T is a convergent sum $\sum a_n(p)\left(\frac{i}{2n}\sigma^\beta\ln\sigma\right)^n$ of dominant contributions to the Freyman integrals $I(G_n^{(m)})$ of the graphs (8) multiplied by locally analytic coefficients, then T satisfies Eq. (1) if (and only if) the a_n are of the form $a^{\circledast(n+1)}$ for some locally analytic kernel a

for any $\underline{k}$ in the threshold manifold $k^2=(m\mu)^2$ and from any direction of the Riemann surface of T".

Proof. The eigenvalues $t_i(k)$ of $T(k)$ are uniformly bounded in view of (58) and of the boundedness condition of the theorem by

$$\| T(k)\| < |\sigma(k)|^{\beta} \operatorname{Sup}_{k,\Omega,\Omega'} |T(k;\Omega,\Omega')| \times \operatorname{Sup}_k \int |\alpha(k;\Omega)| d\Omega \tag{89}$$

when k varies in the physical sheet near $k^2=(m\mu)^2$. Therefore

$$t_0(k) = \left[\frac{i}{2\pi} \ln \sigma(k)\right]^{-1}$$

cannot be an eigenvalue function of T. [$\sigma^{\beta} \ln \sigma \to 0$ when $k^2 \to (m\mu)^2$ since $\beta > 0$.] Hence $T = T'$ and U is defined through Eq. (76).

It will be convenient for the present purpose to rewrite Eq. (76) in the form:

$$T = U + \frac{i}{2\pi} \sigma^{\beta} \ln \sigma \, T \hat{*} U \tag{90}$$

when $\hat{*}$ is defined in Eq. (35'). The Neumann series

$$\sum_n \left[-\frac{i}{2\pi} \sigma^{\beta} \ln \sigma\right]^n T^{\hat{*}(n+1)}(k;\Omega,\Omega')$$

of U in terms of T is uniformly convergent for σ in the physical sheet, $|\sigma|$ sufficiently small and $(\Omega,\Omega') \in \mathbb{S}_{2\beta+1} \times \mathbb{S}_{2\beta+1}$ since

$$|T^{\hat{*}(n+1)}(k;\Omega,\Omega')| < (\operatorname{Sup}_{k;\Omega,\Omega'} |(T(k;\Omega,\Omega')|)^{n+1} \times (\operatorname{Sup}_k \int |\alpha(k;\Omega)| d\Omega)^n .$$

Therefore U, which coincides with the sum of this series, is itself uniformly bounded in modulus in the physical sheet near $\sigma=0$. Since U is uniform around $\sigma=0$, it is therefore analytic in k at $\sigma=0$. This proves property (i).

The expansion (87) of property (ii) is then obtained in the same way as above, by considering the Neumann series of T in terms of U relative to Eq. (90). Since U is analytic at $\sigma=0$, the argument is now valid also in unphysical sheets provided that $|\sigma^{\beta} \ln \sigma|$ be sufficiently small.

Finally, property (iii) follows from Eq. (90) in the limit $\sigma \to 0$ since T and U are both uniformly bounded near $\sigma=0$ for $|\sigma^{\beta} \ln \sigma|$ sufficiently small. QED

The Hermitian Case. The previous analysis and results of this subsection are independent of the condition of (anti) hermitian analyticity of T. We now impose again this condition as in Sect. 5. We then give below an alternative proof of part (i) of Theorem 1 and the proof of the decomposition (61) at $k^2 > (m\mu)^2$.

If $t_0(k)$ is one of the eigenvalue functions of T, one first deduces from (59) the same (anti) hermitian analyticity property for $T' = T - T''$; in fact, in the present case $T''(k;\Omega,\Omega') = t_0(k) E_0(k;\Omega,\Omega')$, $E_0(k)$ being the projector associated with the eigenvalue $t_0(k) = \left[\frac{i}{2\pi} \ln((m\mu)^2 - k^2)\right]^{-1}$, as follows from the Hilbert-Schmidt decomposition of T at $k^2 < (m\mu)^2$ in the physical sheet. Equation (78) that defines U in terms of T' as a meromorphic function of k can be rewritten:

$$\mathscr{C}(k) * \mathscr{U}(k) = \mathscr{U}(k) * \mathscr{C}(k) = \mathbb{1}, \tag{91}$$

where

$$\mathscr{C}(k) = \mathbb{1} + t_0(k)\, T'(k)\,, \tag{92}$$

$$\mathscr{U}(k) = \mathbb{1} - t_0(k)\, U(k)\,. \tag{93}$$

The antihermitian analyticity property of T' entails that $\mathscr{C}(k)$, and hence in view of (91) $\mathscr{U}(k)$, is a self-adjoint operator for any real k in the region $k^2 < (m\mu)^2$ (near the threshold and except at the possible polar singularities of U). Formula (93) then ensures in turn that U satisfies for $k^2 < (m\mu)^2$ the antihermitian condition:

$$U(k;\Omega,\Omega') + \overline{U(k;\Omega,\Omega')} = 0\,. \tag{94}$$

Since U is uniform around $k^2 = (m\mu)^2$ (Theorem 2), condition (94) also holds at $k^2 > (m\mu)^2$. The eigenvalues $u_i(k)$ of $U(k)$ are thus purely imaginary both at $k^2 < (m\mu)^2$ and at $k^2 > (m\mu)^2$, and are therefore uniform. The uniformity of the associated projectors $E_i(k;\Omega,\Omega')$ is proved similarly. In view of Lemma 4, part (i) of Theorem 1 is therefore reobtained (with $b_i = 1/u_i$).

We now prove part (ii) of Theorem 1 at $k^2 > (m\mu)^2$ for the physical-sheet determination $T^{(0)}$ of T. [The decomposition (61) for the other determinations $T^{(r)}$ of T will then follow from Lemma 1 [Eq. (39)] in the same way as that presented in Sect. 5 for $\underline{T}^{(r)}$, starting from the decomposition (61) of $\underline{T}^{(0)}$.] Being given any real point k with $k^2 > (m\mu)^2$ near threshold, let $\Sigma(k;\Omega,\Omega')$ be the sum of the series $\sum_{i\in I} t_i^{(0)}(k) E_i^{(0)}(k;\Omega,\Omega')$, which is known to be convergent (in the L^2-sense in Ω-space) in view of Schur's result (57) and of the hermitian property of the orthogonal projectors $E_i^{(0)}$. In order to prove the decomposition (61) of $T^{(0)}$, we have to prove that $T^{(0)} \equiv \Sigma$. The proof given below applies to any fixed real point $k = K$, $K^2 > (m\mu)^2$, near the threshold.

We first assume for simplicity that $t_0(K) = \left[\frac{i}{2\pi}\ln((m\mu)^2 - K^2)\right]^{-1}$ is not an eigenvalue of $T^{(0)}(K)$. This condition is satisfied if $t_0(k)$ is not an eigenvalue function of T and if K does not belong to a polar singularity of $U(k)$. Since U is antihermitian at $k^2 > (m\mu)^2$ (see above), it admits at K the Hilbert-Schmidt decomposition.

$$U(K;\Omega,\Omega') = \sum_i u_i(K) E_i(K;\Omega,\Omega')\,. \tag{95}$$

By the same convergence arguments as those used in Sect. 5, one then checks, in view of Lemma 4 [Eq. (80)] that

$$\Sigma(K) = U(K) + \frac{i}{2\pi}\ln((m\mu)^2 - K^2)\,\Sigma(K) * U(K)\,, \tag{96}$$

i.e. $\Sigma(K)$ satisfies the same Fredholm equation in terms of $U(K)$ as $T^{(0)}(K)$. Hence $T^{(0)}(K) \equiv \Sigma(K)$.

If $t_0(K)$ is one of the eigenvalues of $T(K)$, let I_0 be the (finite) set of indices i such that $t_i(K) = t_0(K)$. It is then convenient to introduce the decomposition:

$$T(k) = \hat{T}'(k) + \hat{T}''(k)\,, \tag{97}$$

where $\hat{T}''(k)$ is the sum of the principal kernels of $T(k)$ associated with the eigenvalues $t_i(k)$, $i \in I_0$, and $\hat{T}'(k)$ does not have the eigenvalues $t_i(k)$; $\hat{T}'$ and $\hat{T}''$ coincide respectively with T' and T'' unless one or more eigenvalue functions $t_i(k)$ different from $t_0(k)$ are equal to $t_0(K)$ at the point K considered. We first consider $\hat{T}''$. In the region $k^2 < (m\mu)^2$ of the physical sheet, $\hat{T}''$ is equal, in view of the Hilbert-Schmidt decomposition of T to $\sum_{i \in I_0} t_i(k) E_i(k; \Omega, \Omega')$. Since the sum is finite, the same result still holds by analytic continuation in the region $k^2 > (m\mu)^2$, in particular at the given point K. On the other hand, a well-defined kernel $\hat{U}(k; \Omega, \Omega')$ is associated to $\hat{T}'(K)$ through the equation analogous to (78):

$$\hat{T}'(K) = \hat{U}(K) + \frac{i}{2\pi} \ln((m\mu)^2 - K^2)\, \hat{T}'(K) * \hat{U}(K), \tag{98}$$

since $t_0(K)$ is not an eigenvalue of $\hat{T}'(K)$. By the same arguments as those used in the previous case for the sum $\Sigma(K)$, one shows here that the sum $\sum_{i, i \in I \setminus I_0} t_i(K) E_i(K; \Omega, \Omega')$ coincides with $\hat{T}'(K; \Omega, \Omega')$ as the unique solution of (98) in terms of $\hat{U}$.

By combining the results thus obtained for $\hat{T}'(K)$ and $\hat{T}''(K)$, the decomposition (61) of $T^{(0)}$ at $k = K$ is therefore proved. QED

Remark. In the hermitian case and if T is moreover assumed to be uniformly bounded in the physical sheet near $\sigma = 0$, then the orthogonal decomposition (61) can be recovered from (and is in fact equivalent to) the expansion (87) of T in powers of $\sigma^\beta \ln \sigma$. Starting from (87), one may introduce for k real near the threshold the Hilbert-Schmidt decomposition $U = \sum_{i \in I} u_i E_i = \sum_{i \in I} \hat{u}_i \hat{E}_i$ of U, where $u_i = \sigma^\beta \hat{u}_i$, $E_i = \sigma^{-\beta} \hat{E}_i$. In view of the orthogonality of the E_i and the corresponding relations $\hat{E}_i \hat{*} \hat{E}_j = \hat{E}_i \delta_{i,j}$, one gets:

$$U^{\hat{*}(n+1)} = \sum_{i \in I} \hat{u}_i^{n+1} \hat{E}_i \tag{99}$$

from which (61) follows, by resummation for each i of the series

$$\sum_n \left[\frac{i}{2\pi} \sigma^\beta \ln \sigma \right]^n \hat{u}_i^{n+1},$$

in the form:

$$T(k; \Omega, \Omega') = \sum_i \frac{1}{\dfrac{1}{2i\pi} \sigma^\beta \ln \sigma + 1/\hat{u}_i(k)} \hat{E}_i(k; \Omega, \Omega'). \tag{100}$$

The converse is proved by expansion for each i of the functions

$$\left[\frac{1}{2i\pi} \sigma^\beta \ln \sigma + 1/\hat{u}_i(k) \right]^{-1}$$

in powers of $\sigma^\beta \ln \sigma$ (and resummation over i, for each n).

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