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Derivations Vanishing on $S(\infty)$*

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Abstract. Let $S(\infty)$ be the group of finite permutations on countably many symbols. We exhibit an embedding of $S(\infty)$ into a UHF-algebra $\mathfrak{A}$ of Glimm type n^∞ such that, if δ is a *-derivation vanishing on $S(\infty)$ and satisfying $\tau \circ \delta = 0$, where τ is the unique trace on $\mathfrak{A}$, then δ admits an extension which is the generator of a C^*-dynamics.

1. Introduction

In [4] Goodman showed that if G is a locally compact group, and δ is a closed *-derivation on $C_0(G)$ commuting with the action of G as left translations on the algebra, then δ is a generator of a strongly continuous one-parameter group of *-automorphisms on $C_0(G)$. In a more recent paper, [5], Goodman and Jørgensen consider closed *-derivations on a C^*-algebra $\mathfrak{A}$ commuting with a strongly continuous representation α_G of a compact group G on $\mathfrak{A}$. They define a *-derivation δ to be *tangential* to α_G if it has the aforementioned property (i.e., $\delta \circ \alpha_g = \alpha_g \circ \delta$, for all $g \in G$) and if $\mathfrak{A}^\alpha$, the C^*-algebra of fixed elements of $\mathfrak{A}$, lies in the kernel of the derivation. Under certain restrictions on the system $(\alpha, G, \mathfrak{A})$ (e.g., $\mathfrak{A}$ is abelian, or the action of G on $\mathfrak{A}$ is ergodic) they prove that a derivation tangential to α_G is, in fact, the infinitesimal generator of a strongly continuous one-parameter group of automorphisms.

Suppose now that $\mathfrak{A}$ is a UHF (uniformly hyperfinite) C^*-algebra of Glimm type n^∞: i.e., $\mathfrak{A} = \overset{*}{\bigotimes}_{k \geqq 1} B_k$, where each B_k is a full $n \times n$ matrix algebra over the complex numbers $\mathbb{C}$. Define $S(\infty)$ to be the group of *finite* permutations on the symbols of $\mathbb{N}$, the positive integers. Then there exists a natural embedding of $S(\infty)$ into $\mathfrak{A}$ such that, if G is any compact group, and α_G a strongly continuous representation of *product* type, then $S(\infty)$ lies in the C^*-algebra $\mathfrak{A}^\alpha$ of fixed points of α_G (see [8]). Motivated by the results of [5], we show the following: if δ is a

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symmetric *-derivation vanishing on $S(\infty)$ and satisfying $\tau \circ \delta = 0$, where τ is the (normalized) trace on $\mathfrak{A}$, then δ extends to a generator $\bar{\delta}$ on $\mathfrak{A}$ whose associated one-parameter group is of product type.

2. Derivations Vanishing on $S(\infty)$

We shall make use of the following notation throughout. For n a fixed positive integer, let $B_1, B_2, \ldots$ be a sequence of $n \times n$ matrix algebras over $\mathbb{C}$, where B_k has identity I_k and matrix units $\{e^k_{ij} : 1 \leqq i, j \leqq n\}$ satisfying $e^k_{ij} e^k_{pq} = \delta_{jp} e^k_{iq}$. Let $\mathfrak{A}$ be the UHF-algebra formed as the infinite tensor product $\mathfrak{A} = \bigotimes_{k \geqq 1} B_k$. We write I for the identity of $\mathfrak{A}$. For finite subsets Λ of $\mathbb{N}$, there exists a canonical embedding $L_\Lambda : \bigotimes_{k \in \Lambda} B_k \to \mathfrak{A}$ which carries $\bigotimes_{k \in \Lambda} y_k$ into $\left(\bigotimes_{k \in \Lambda} y_k\right) \otimes \left(\bigotimes_{k \in \mathbb{N} \setminus \Lambda} I_k\right)$, and extends by linearity. Denote the image of L_Λ by $\mathfrak{A}_\Lambda$. (Whenever there is no danger of confusion we shall identify $\bigotimes_{k \in \Lambda} B_k$ with its image $\mathfrak{A}_\Lambda$ in $\mathfrak{A}$. In particular, we regard the algebras B_k as embedded in $\mathfrak{A}$.) For finite disjoint subsets Λ, Λ' of $\mathbb{N}$, $\mathfrak{A}_\Lambda$ and $\mathfrak{A}_{\Lambda'}$ are commuting subalgebras. For m a positive integer, let Λ_m denote the subset $\{1, 2, \ldots, m\}$ of $\mathbb{N}$, and denote $\mathfrak{A}_{\Lambda_m}$ by $\mathfrak{A}_m$. Then clearly $\mathfrak{A}_1 \subset \mathfrak{A}_2 \subset \ldots$, and the union $\mathfrak{A}_0 = \bigcup_{m=1}^{\infty} \mathfrak{A}_m$ is a uniformly dense subalgebra of $\mathfrak{A}$. We call $\mathfrak{A}_0$ the subalgebra of *local* elements of $\mathfrak{A}$. We refer the reader to [6] for the general theory of infinite tensor products of C^*-algebras.

Let τ be the unique normalized trace on $\mathfrak{A}$, i.e., τ is the unique state on $\mathfrak{A}$ satisfying $\tau(xy) = \tau(yx)$, $x, y \in \mathfrak{A}$. If e^k_{ij} is a matrix unit of B_k, then $\tau(e^k_{ij}) = \delta_{ij}/n$; furthermore, for $x \in \mathfrak{A}_\Lambda$, $y \in \mathfrak{A}_{\Lambda'}$, and Λ, Λ' disjoint, $\tau(xy) = \tau(x)\tau(y)$. τ is a product state $\left(\tau = \bigotimes_{k \geqq 1} \tau_k\right.$, where τ_k is the normalized trace on $\left.B_k\right)$, hence [7, Theorem 2.5], a factor state, i.e., $\pi_\tau(\mathfrak{A})''$ is a factor in the associated GNS representation $(\pi_\tau, H_\tau, \Omega_\tau)$. For convenience we shall write $\pi_\tau = \pi$, $H_\tau = H$, $\Omega_\tau = \Omega$. That π is a faithful representation follows from the fact [3, Theorem 5.1] that $\mathfrak{A}$ is simple.

We now describe an embedding ϱ of the group $S(\infty)$ of finite permutations on the symbols of $\mathbb{N}$ into the group of unitary elements of $\mathfrak{A}$. We write e for the identity element of $S(\infty)$, and define $\varrho(e) = I$. Let $t = (kl) \in S(\infty)$ be a transposition $(k \neq l,\ k, l \in \mathbb{N})$, and define $\varrho(t)$ to be the operator $\varrho(t) = \sum_{i,j=1}^{n} e^k_{ij} \otimes e^l_{ji}$. Note that $\varrho(t)$ is self-adjoint and that $[\varrho(t)]^2 = I = \varrho(t^2)$, hence $\varrho(t)$ is unitary. Moreover, suppose $x \in \mathfrak{A}_0$, then x is a linear combination of elements of the form $e^{p_1}_{i_1 j_1} \otimes \ldots \otimes e^{p_r}_{i_r j_r}$. A straightforward calculation gives, for $t = (kl)$,

$$\varrho(t)[e^{p_1}_{i_1 j_1} \otimes \ldots \otimes e^{p_r}_{i_r j_r}]\varrho(t^{-1}) = e^{t(p_1)}_{i_1 j_1} \otimes \ldots \otimes e^{t(p_r)}_{i_r j_r}, \tag{1}$$

where $t(p)$ is the image of $p \in \mathbb{N}$ under the permutation t. In particular, Eq. (1) indicates that the mapping $x (\in B_p) \mapsto \varrho(t) x \varrho(t^{-1})$ is an isomorphism between B_p and $B_{t(p)}$.

Let $q \in S(\infty)$, then q may be written as a product of transpositions $q = t_1 t_2 \dots t_s$. We define $\varrho(q) = \varrho(t_1) \dots \varrho(t_s)$. To see that this is well-defined, suppose $q = e = t_1 \dots t_s$. Making repeated use of (1), we have, for $u = \varrho(t_s)\varrho(t_{s-1}) \dots \varrho(t_1)$,

$$\begin{aligned} u\{e^{p_1}_{i_1 j_1} \otimes \dots \otimes e_{i_r j_r}\}u^* &= \varrho(t_s) \dots \varrho(t_2)\{\varrho(t_1)[e^{p_1}_{i_1 j_1} \otimes \dots \otimes e^{p_r}_{i_r j_r}]\varrho(t_1^{-1})\}\varrho(t_2^{-1}) \dots \varrho(t_s^{-1}) \\ &= \varrho(t_s) \dots \varrho(t_2)\{e^{t_1(p_1)}_{i_1 j_1} \otimes \dots \otimes e^{t_1(p_r)}_{i_r j_r}\}\varrho(t_2^{-1}) \dots \varrho(t_s^{-1}) \\ &= \dots \\ &= e^{t_s \dots t_1(p_1)}_{i_1 j_1} \otimes \dots \otimes e^{t_s \dots t_1(p_r)}_{i_r j_r} \\ &= e^{e(p_1)}_{i_1 j_1} \otimes \dots \otimes e^{e(p_r)}_{i_r j_r} \\ &= e^{p_1}_{i_1 j_1} \otimes \dots \otimes e^{p_r}_{i_r j_r}. \end{aligned} \tag{2}$$

Hence for all $x \in \mathfrak{A}_0$, Eq. (2) yields $uxu^* = x$. By norm continuity, the same holds for all $x \in \mathfrak{A}$. Since $\mathfrak{A}$ has trivial center, however, and since u is unitary, $u = \lambda I$, for some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. But u is a product of operators of the form $\varrho(t) = \varrho((kl)) = \sum_{i,j=1}^{n} e^k_{ij} \otimes e^l_{ji}$, hence clearly $\lambda = \tau(u) > 0$. Thus $\lambda = 1$, $u = I = \varrho(e)$, and ϱ is well-defined.

The faithfulness of ϱ is apparent from Eq. (1), and thus we have

Lemma 1. *The mapping ϱ of $S(\infty)$ into the unitaries of $\mathfrak{A}$ is a faithful group representation.*

In what follows, we shall identify $S(\infty)$ with its embedding $\varrho(S(\infty))$ in $\mathfrak{A}$ given above. Under this identification, the map $\mathrm{Ad}: S(\infty) \to \mathrm{Aut}(\mathfrak{A})$ defined by $\mathrm{Ad}(p)(x) = pxp^{-1}$, $p \in S(\infty)$, $x \in \mathfrak{A}$, forms a group of inner automorphisms of $\mathfrak{A}$. Moreover, if x is local, i.e., $x \in \mathfrak{A}_l$ for some $l \in \mathbb{N}$, and $p(k) = i_k$, $1 \leqq k \leqq l$, an application of Eq. (1) yields $pxp^{-1} \in \mathfrak{A}_\Lambda$, where $\Lambda = \{i_1, i_2, \dots, i_l\}$. By [9, Lemma 2.1], $\mathfrak{A}$ is asymptotically abelian with respect to this group action.

If G is a compact group, and $g \mapsto \alpha'_g \in \mathrm{Aut}(\mathsf{M})$ is a strongly continuous representation of G as *-automorphisms on an $n \times n$ matrix algebra M, then define corresponding representations $g \mapsto \alpha^k_g \in \mathrm{Aut}(B_k)$ as follows: if $\{e_{ij} : 1 \leqq i,j \leqq n\}$ are matrix units for M, and if $\alpha'_g(e_{ij}) = \sum_{s,t=1}^{n} \beta_{ijst} e_{st}$, define $\alpha^k_g(e^k_{ij}) = \sum_{s,t=1}^{n} \beta_{ijst} e^k_{st}$. We may then construct a strongly continuous group of product automorphisms $\{\alpha_g : g \in G\}$ of $\mathfrak{A}$ by forming the tensor product $\alpha_g = \bigotimes_{k \geqq 1} \alpha^k_g$. Let $t \in S(\infty)$ and let $g \in G$; then it is clear, using (1), that $\alpha_g(txt^{-1}) = t\alpha_g(x)t^{-1}$, all $x \in \mathfrak{A}$. Thus $(t^{-1})(\alpha_g(t))$ is a central unitary element of $\mathfrak{A}$, and since $\mathfrak{A}$ has trivial center, we must have $\alpha_g(t) = \lambda t$, some $\lambda \in \mathbb{C}$, $|\lambda| = 1$. But $\tau = \tau \circ \alpha_g$, by the uniqueness of the trace on $\mathfrak{A}$, and a slight modification of the argument preceding Lemma 1 shows that $\tau(t) > 0$, so that $\tau(t) = \tau(\alpha_g(t)) = \lambda\tau(t)$, or $\lambda = 1$. Thus $\alpha_g(t) = t$, all $t \in S(\infty)$, and therefore $S(\infty) \subset \mathfrak{A}^\alpha$, the subalgebra of $\mathfrak{A}$ of fixed elements of α_G. Hence if δ is any derivation vanishing on $\mathfrak{A}^\alpha$, then certainly $\delta p = 0$, all $p \in S(\infty)$, and thus we are led by [5] to consider symmetric *-derivations δ on $\mathfrak{A}$ [i.e., $D(\delta)$ is a dense *-subalgebra of $\mathfrak{A}$, and $\delta(x^*) = (\delta x)^*$, all $x \in D(\delta)$] which vanish on $S(\infty)$. If we impose the restriction $\tau \circ \delta = 0$, then it follows (Theorem 6) that δ has an extension $\hat{\delta}$ which is a generator.

As a preliminary to proving this we make a definition and establish some results on strong convergence in $\pi(\mathfrak{A})''$.

Definition 1. Let $r>m$ be non-negative integers, then define $S_{r,m}\subset S(\infty)$ to be the subgroup [of order $(r-m)!$] of permutations which fix the symbols of $\mathbb{N}\setminus\{m+1,\dots,r\}$.

Lemma 2. *Let x be a fixed element of $\mathfrak{A}$. Define, for $r>0$,*

$$x_r=(1/r!)\cdot\sum_{p\in S_{r,0}} pxp^{-1}.$$

Let (π,H,Ω) be the GNS *construction for τ. Then the sequence $\{\pi(x_r)\}$ has a strong limit in $\pi(\mathfrak{A})''$, and* $\operatorname{st-lim}_{r\to\infty}\pi(x_r)=\tau(x)\pi(I)$.

Proof. Without loss of generality we may assume x to be self-adjoint. Furthermore, we may assume x to be local, i.e., $x\in\mathfrak{A}_0$. For suppose $x\in\mathfrak{A}$, and $\operatorname{st-lim}_{r\to\infty}\pi(x'_r)$ exists for all $x'\in\mathfrak{A}_0$. If $x'\in\mathfrak{A}_0$ is chosen such that $\|x-x'\|<\varepsilon$, for given $\varepsilon>0$, then one easily checks that $\|\pi(x_r)-\pi(x'_r)\|<\varepsilon$, and the strong convergence of $\{\pi(x_r)\}$ will follow by continuity. So assume $x=x^*\in\mathfrak{A}_l$, for some $l\in\mathbb{N}$.

We begin by showing that $\{\pi(x_r)\Omega\}$ is a Cauchy sequence. Let $r\geqq s$, then, since x_r, x_s are self-adjoint,

$$\begin{aligned}\|\pi(x_r)\Omega-\pi(x_s)\Omega\|^2&=\|\pi(x_r-x_s)\Omega\|^2\\&=\tau([x_r-x_s]^2)\\&\tau(x_r^2)-2\tau(x_rx_s)+\tau(x_s^2).\end{aligned}$$

Let $N(r;l)$ be the set of those $p\in S_{r,0}$ which permute all of the symbols of Λ_l into the set $\{l+1,\dots,r\}$. For such p, $pxp^{-1}\in\mathfrak{A}_{\{l+1,\dots,r\}}$, and therefore, since $x\in\mathfrak{A}_{\Lambda_l}$, $\tau(pxp^{-1}x)=\tau(pxp^{-1})\,\tau(x)=\tau(x)^2$. Furthermore, one may check by a counting argument that $\lim_{r\to\infty}[\#N(r;l)/r!]=1$. Then

$$\begin{aligned}\tau(x_r^2)&=(1/(r!)^2)\cdot\sum_{p,q\in S_{r,0}}\tau(pxp^{-1}qxq^{-1})\\&=(1/(r!)^2)\cdot\sum_{p,q\in S_{r,0}}\tau([q^{-1}pxp^{-1}q]x)\\&=(1/r!)\cdot\sum_{p\in S_{r,0}}\tau(pxp^{-1}x)\\&=(1/r!)\cdot\sum_{p\in N(r;l)}\tau(pxp^{-1}x)+(1/r!)\cdot\sum_{p\in S_{r,0}\setminus N(r;l)}\tau(pxp^{-1}x)\\&=(\#N(r;l)/r!)[\tau(x)]^2+(1/r!)\cdot\sum_{p\in S_{r,0}\setminus N(r;l)}\tau(pxp^{-1}x).\end{aligned}$$

The sum $(1/r!)\cdot\sum_{p\in S_{r,0}\setminus N(r;l)}\tau(pxp^{-1}x)$ is bounded in absolute value by $\|x\|^2\cdot[r!-\#N(r;l)]/r!$, hence it tends to 0 as $r\to\infty$, and therefore $\lim_{r\to\infty}\tau(x_r^2)=\tau(x)^2$. Similarly, $\lim_{s\to\infty}\tau(x_s^2)=\tau(x)^2=\lim_{r,s\to\infty}\tau(x_rx_s)$, thus $\lim_{r,s\to\infty}\|\pi(x_r)\Omega-\pi(x_s)\Omega\|=0$.

Let $y, z \in \mathfrak{A}_0$, then employing a convergence argument similar to the one above, one shows that the sequences $\{\pi(x_r)\pi(y)\pi(z)\Omega : r \in \mathbb{N}\}$ and $\{\pi(y)\pi(x_r)\pi(z)\Omega : r \in \mathbb{N}\}$ are Cauchy in H and that their limits coincide. Letting $y = I$ in the first sequence, one sees that the uniformly bounded (by $\|x\|$) sequence of operators $\{\pi(x_r)\}$ converges on all vectors in the dense subset $\pi(\mathfrak{A}_0)\Omega$ of H, and therefore has a strong limit in $\pi(\mathfrak{A})''$. Again using uniform boundedness, we have $\lim_{r\to\infty} \pi(y)\pi(x_r)\xi = \lim_{r\to\infty} \pi(x_r)\pi(y)\xi$, all $\xi \in H$, $y \in \mathfrak{A}_0$, hence

$$\text{st-}\lim_{r\to\infty} \pi(x_r) \in \pi(\mathfrak{A}_0)' \cap \pi(\mathfrak{A})'' = \pi(\mathfrak{A})' \cap \pi(\mathfrak{A})'' = \{\lambda\pi(I) : \lambda \in \mathbb{C}\}.$$

Thus

$$\begin{aligned} \text{st-}\lim_{r\to\infty} \pi(x_r) &= \lim_{r\to\infty} \langle \pi(x_r)\Omega, \Omega\rangle \cdot \pi(I) \\ &= \lim_{r\to\infty} \tau(x_r) \cdot \pi(I) \\ &= \lim_{r\to\infty} (1/r!) \cdot \tau\Big(\sum_{p\in S_{r,0}} pxp^{-1}\Big) \cdot \pi(I) \\ &= \tau(x) \cdot \pi(I). \end{aligned}$$

This completes the proof of the lemma.

We describe a generalization of the "averaging map" defined in Lemma 2. Let $\mathfrak{A}_m^c$ be the commutant of $\mathfrak{A}_m$ relative to $\mathfrak{A}$ (i.e., $\mathfrak{A}_m^c = \{y \in \mathfrak{A} : xy = yx, \text{ all } x \in \mathfrak{A}_m\}$). In particular, if $t \in S_{r,m}$, then $txt^{-1} = x$, for all matrix units $x \in \mathfrak{A}_m$, by Eq. (1), so that $t \in \mathfrak{A}_m^c$. Hence $S_{r,m}$ lies in $\mathfrak{A}_m^c$. Let $y \in \mathfrak{A}_m^c$, and for $r > m$, form the operator

$$y_{r,m} = [1/(r-m)!] \cdot \sum_{p\in S_{r,m}} pyp^{-1}.$$

Then clearly $y_{r,m} \in \mathfrak{A}_m^c$, and the sequence $\{y_{r,m} : r > m\}$ is uniformly bounded in norm by $\|y\|$. Arguing as in Lemma 2, one shows that the sequence $\{\pi(y_{r,m}) : r > m\}$ converges strongly to an operator $\bar{y} \in \pi(\mathfrak{A})''$, and for all $z \in \mathfrak{A}_0 \cap \mathfrak{A}_m^c$, $\bar{y}\pi(z) = \pi(z)\bar{y}$, hence $\bar{y} \in \pi(\mathfrak{A}_0 \cap \mathfrak{A}_m^c)' = \pi(\mathfrak{A}_m^c)'$. Clearly, $\bar{y} \in \pi(\mathfrak{A}_m)'$ (since $y_{r,m} \in \mathfrak{A}_m^c$, all $r > m$), so that $\bar{y} \in \pi(\mathfrak{A}_m^c)' \cap \pi(\mathfrak{A}_m)' \cap \pi(\mathfrak{A})''$. Since $\mathfrak{A}$ is generated by $\mathfrak{A}_m^c$ and $\mathfrak{A}_m$, $\pi(\mathfrak{A}_m^c)' \cap \pi(\mathfrak{A}_m)' = \pi(\mathfrak{A})'$, thus $\bar{y} \in \pi(\mathfrak{A})' \cap \pi(\mathfrak{A})'' = \{\lambda\pi(I)\}$. Arguing as before, one now shows that $\bar{y} = \text{st-}\lim_{r\to\infty} \pi(y_{r,m}) = \tau(y) \cdot \pi(I)$.

Let $\{f_{ij} : 1 \leqq i, j \leqq n^m\}$ be matrix units for the $n^m \times n^m$-dimensional matrix algebra $\mathfrak{A}_m$. By [2], any $x \in \mathfrak{A}$ may be written uniquely in the form $x = \sum_{i,j=1}^{n^m} f_{ij}y_{ij}$, where the y_{ij} lie in $\mathfrak{A}_m^c$. For $r > m$ define $x_{r,m} = [1/(r-m)!] \cdot \sum_{p\in S_{r,m}} pxp^{-1}$. Then

$$\begin{aligned} \text{st-}\lim_{r\to\infty} \pi(x_{r,m}) &= \text{st-}\lim_{r\to\infty} [1/(r-m)!] \cdot \sum_{p\in S_{r,m}} \sum_{i,j=1}^{n^m} \pi(pf_{ij}y_{ij}p^{-1}) \\ &= \text{st-}\lim_{r\to\infty} [1/(r-m)!] \cdot \sum_{i,j=1}^{n^m} \Big[\pi(f_{ij}) \sum_{p\in S_{r,m}} \pi(py_{ij}p^{-1})\Big] \\ &= \sum_{i,j=1}^{n^m} \pi(f_{ij})\tau(y_{ij}) \\ &= \pi\Big\{\sum_{i,j=1}^{n^m} f_{ij}\tau(y_{ij})\Big\}. \end{aligned}$$

By [2, Lemma 2], $\sum_{i,j=1}^{n^m} f_{ij}\tau(y_{ij})=\phi_m(x)$, where ϕ_m is the conditional expectation of the trace τ onto $\mathfrak{A}_m$. Hence $\text{st-}\lim_{r\to\infty} \pi(x_{r,m})=\pi(\phi_m(x))$. Thus we have

Lemma 3. *Let $x\in\mathfrak{A}$, and for fixed m define $x_{r,m}$ as above. Then the sequence $\{\pi(x_{r,m}):r>m\}$ has a strong limit in $\pi(\mathfrak{A})''$, and there exists a unique element $\phi_m(x)\in\mathfrak{A}_m$ such that*

$$\pi(\phi_m(x))=\text{st-}\lim_{r\to\infty} \pi(x_{r,m}).$$

The mapping $\phi_m:\mathfrak{A}\to\mathfrak{A}_m$ is the conditional expectation of the trace onto $\mathfrak{A}_m$.

Proof. The above argument shows that the conditional expectation ϕ_m has the required properties. Uniqueness follows from the faithfulness of π.

Lemma 4. *Let Δ be a dense linear subset of $\mathfrak{A}$. Then ϕ_m maps Δ onto $\mathfrak{A}_m$.*

Proof. Let $x\in\mathfrak{A}_m$, and for given $\varepsilon>0$, choose $y\in\Delta$ such that $\|x-y\|<\varepsilon$. Since $\|\phi_m\|=1$, by [2, Lemma 2], $\|x-\phi_m(y)\|=\|\phi_m(x)-\phi_m(y)\|\leqq\|x-y\|$. Hence $\phi_m(\Delta)$ is dense in $\mathfrak{A}_m$. But since ϕ_m is linear and $\mathfrak{A}_m$ is finite-dimensional, $\phi_m(\Delta)=\mathfrak{A}_m$.

Lemma 5. *Let δ be a *-derivation with dense domain $D(\delta)\subset\mathfrak{A}$ which satisfies $\tau\circ\delta\equiv 0$. Let $\mathscr{D}$ be the *-subalgebra of $\mathfrak{A}$ consisting of all elements $A\in\mathfrak{A}$ such that there exists a sequence $\{A_n:n\in\mathbb{N}\}\subseteqq D(\delta)$ satisfying:*

(i) *$\{A_n\}$ and $\{\delta A_n\}$ are uniformly bounded sequences in $\mathfrak{A}$.*

(ii) *$\{\pi(A_n)\}$ and $\{\pi(\delta A_n)\}$ are strongly convergent sequences in $\pi(\mathfrak{A})''$.*

(iii) *$\pi(A)=\text{st-}\lim_{n\to\infty}\pi(A_n)$, and there exists an $A'\in\mathfrak{A}$ such that $\pi(A')=\text{st-}\lim_{n\to\infty}\pi(\delta A_n)$.*

*Define a linear operator $\delta':\mathscr{D}\to\mathfrak{A}$ by $\delta'A=A'$, then δ' is a well-defined *-derivation on $\mathfrak{A}$ extending δ and satisfying $\tau\circ\delta'=0$.*

Proof. Clearly, $\mathscr{D}$ is a linear set containing $D(\delta)$. Suppose A and B are elements of $\mathfrak{A}$ with corresponding sequences $\{A_n\}$, $\{B_n\}$ satisfying the conditions of the lemma. Then by (iii) and the faithfulness of π there exist unique elements A', B' of $\mathfrak{A}$ such that $\pi(A')=\text{st-}\lim_{n\to\infty}\pi(\delta A_n)\left[\text{respectively, } \pi(B')=\text{st-}\lim_{n\to\infty}\pi(\delta B_n)\right]$. Using (i) one verifies easily that the sequences $\{A_nB_n\}$, $\{A_n\delta B_n\}$, $\{(\delta A_n)B_n\}$ are uniformly bounded, hence so is $\{\delta(A_nB_n)\}$, since $\delta(A_nB_n)=(\delta A_n)B_n+A_n(\delta B_n)$. Let $M=\sup_n\{\|A_n\|\}$, and suppose that $f\in H_\tau$. Then applying the strong convergence of the sequences $\{\pi(A_n)\}$, $\{\pi(B_n)\}$, one has

$$\begin{aligned}\lim_{n\to\infty}\|[\pi(AB)-\pi(A_nB_n)]f\| &\leqq \lim_{n\to\infty}\{\|[\pi(AB)-\pi(A_nB)]f\|+\|[\pi(A_nB)-\pi(A_nB_n)f\|\}\\ &\leqq \lim_{n\to\infty}\{\|[\pi(A)-\pi(A_n)](\pi(B)f)\|+M\|[\pi(B)-\pi(B_n)]f\|\}\\ &=0,\end{aligned}$$

so that $\text{st-}\lim_{n\to\infty}\pi(A_nB_n)=\pi(AB)$. Similarly, one verifies that the sequence $\{\pi(\delta A_n\cdot B_n)\}$ [respectively, $\{\pi(A_n\cdot\delta B_n)\}$] converges strongly to $\pi(A'B)$ [respectively, $\pi(AB')$] and therefore the sequence $\{\pi(\delta(A_nB_n))\}=\{\pi(\delta A_n\cdot B_n)+\pi(A_n\cdot\delta B_n)\}$ converges strongly to $\pi(A'B+AB')$. Thus $AB\in\mathscr{D}$.

Now suppose $A \in \mathcal{D}$ with corresponding sequence $\{A_n\} \subseteq D(\delta)$. Then the sequences $\{A_n^*\}$ and $\{\delta(A_n^*)\}$ $(=\{(\delta A_n)^*\})$ are uniformly bounded. To see that $\{\pi(A_n^*)\}$ converges strongly to $\pi(A^*)$ it suffices to check, by the uniform boundedness of $\{A_n^*\}$, that $\lim_{n\to\infty} \pi(A_n^*)f = \pi(A^*)f$ for all f in the dense subspace $\pi(\mathfrak{A})\Omega_\tau$ of H. Let $f = \pi(z)\Omega_\tau$, $z \in \mathfrak{A}$; then

$$\begin{aligned}
\lim_{n\to\infty} \| [\pi(A^*) - \pi(A_n^*)]f \|^2 &= \lim_{n\to\infty} \langle \pi(A^* - A_n^*)\pi(z)\Omega_\tau, \pi(A^* - A_n^*)\pi(z)\Omega_\tau \rangle \\
&= \lim_{n\to\infty} \langle \pi(z^*)\pi(A - A_n)\pi(A^* - A_n^*)\pi(z)\Omega_\tau, \Omega_\tau \rangle \\
&= \lim_{n\to\infty} \tau(z^*(A - A_n)(A^* - A_n^*)z) \\
&= \lim_{n\to\infty} \tau([A^* - A_n^*]zz^*[A - A_n]) \\
&\leqq \lim_{n\to\infty} \|zz^*\| \cdot \tau([A^* - A_n^*][A - A_n]) \\
&= \lim_{n\to\infty} \|zz^*\| \cdot \|\pi(A - A_n)\Omega_\tau\|^2 = 0 .
\end{aligned}$$

Similarly, one verifies that $\text{st-}\lim_{n\to\infty} \pi(\delta A_n^*) = \text{st-}\lim_{n\to\infty} \pi((\delta A_n)^*) = \pi(A')^*$.

To see that δ' is well-defined, suppose $\text{st-}\lim_{n\to\infty} \pi(A_n) = 0$ and $\text{st-}\lim_{n\to\infty} \pi(\delta A_n) = B$. In particular, $\pi(\delta A_n)$ converges weakly to B, hence for all f, g in the dense subspace $\pi(D(\delta))\Omega_\tau$ of H_τ we have, letting $f = \pi(z)\Omega_\tau$, [respectively, $g = \pi(y^*)\Omega_\tau$], $z, y \in D(\delta)$,

$$\begin{aligned}
\langle Bf, g \rangle &= \lim_{n\to\infty} \langle \pi(\delta A_n)\pi(z)\Omega_\tau, \pi(y^*)\Omega_\tau \rangle \\
&= \lim_{n\to\infty} \tau(y[\delta A_n]z) = \lim_{n\to\infty} \tau(zy[\delta A_n]) = \lim_{n\to\infty} -(\tau([\delta(zy)]A_n) \\
&= \lim_{n\to\infty} -\langle \pi(A_n)\Omega_\tau, \pi(\delta[zy])^*\Omega_\tau \rangle = 0 .
\end{aligned}$$

Thus $B = 0$, by continuity, and δ' is well-defined. Clearly, δ' extends δ.

Again let $A, B \in \mathcal{D}$, with corresponding sequences $\{A_n\}$, $\{B_n\}$. Then AB^* has corresponding sequence $\{A_n B_n^*\}$, and

$$\begin{aligned}
\pi(\delta'[AB^*]) &= \text{st-}\lim_{n\to\infty} \pi(\delta[A_n B_n^*]) \\
&= \text{st-}\lim_{n\to\infty} \{\pi(\delta A_n)\pi(B_n^*) + \pi(A_n)\pi(\delta[B_n^*])\} \\
&= \text{st-}\lim_{n\to\infty} \{\pi(\delta A_n)\pi(B_n)^* + \pi(A_n)\pi([\delta B_n]^*)\} \\
&= \pi([A'(B^*) + A(B')^*]) \\
&= \pi((\delta' A)B^* + A(\delta' B)^*) ,
\end{aligned}$$

hence $\delta'(AB^*) = (\delta' A)B^* + A(\delta' B)^*$, by the faithfulness of π, and therefore δ' is a $*$-derivation. Finally, note that for $A \in \mathcal{D}$,

$$\begin{aligned}
\tau(\delta' A) &= \langle \pi(\delta' A)\Omega_\tau, \Omega_\tau \rangle \\
&= \lim_{n\to\infty} \langle \pi(\delta A_n)\Omega_\tau, \Omega_\tau \rangle \\
&= \lim_{n\to\infty} (\tau \circ \delta)(A_n) = 0 ,
\end{aligned}$$

so that $\tau \circ \delta' = 0$. This completes the proof of the lemma.

Corollary. *Let δ be a *-derivation on $\mathfrak{A}$ vanishing on $S(\infty)$ and satisfying $\tau(\delta x)=0$, all $x\in D(\delta)$. Then there exists a generator $\hat{\delta}$ which extends δ,* i.e., *$D(\delta)\subset D(\hat{\delta})$, and $\hat{\delta}|_{D(\delta)}=\delta$.*

Proof. Let δ' be the extension of δ given in the lemma above. We show $\mathfrak{A}_0\subset\mathscr{D}$ $[=D(\delta')]$. To see this, let $x\in D(\delta)$, let m be a positive integer, and form the sequence of operators $\{x_{r,m}:r>m\}$, where $x_{r,m}$ is defined as in Lemma 3. Clearly, $\{x_{r,m}:r>m\}$ is a uniformly bounded sequence contained in $D(\delta)$; moreover,

$$\begin{aligned}\delta(x_{r,m})&=[1/(r-m)!]\sum_{p\in S_{r,m}}\delta(pxp^{-1})\\&=[1/(r-m)!]\sum_{p\in S_{r,m}}p(\delta x)p^{-1}\\&=(\delta x)_{r,m},\end{aligned}$$

and it is immediate that the sequence $\{(\delta x)_{r,m}:r>m\}$ is also uniformly bounded. By Lemma 3, $\pi(\phi_m(x))=\text{st-}\lim_{r\to\infty}\pi(x_{r,m})$ $\left[\text{respectively, } \pi(\phi_m(\delta x))=\text{st-}\lim_{r\to\infty}\pi((\delta x)_{r,m})\right]$, hence by the preceding lemma, $\phi_m(x)\in D(\delta')$ and $\delta'(\phi_m(x))=\phi_m(\delta x)$. Since $\phi_m:D(\delta)\to\mathfrak{A}_m$ is onto, by Lemma 4, the preceding equation implies $\delta':\mathfrak{A}_m\to\mathfrak{A}_m$, for all m. Thus $\mathfrak{A}_0$ is a dense set of analytic elements for δ'.

Since $\tau\circ\delta'=0$, δ' is closable, by [1, Theorem 6]: denote its closure by $\hat{\delta}$. Then $\delta\subset\delta'\subset\hat{\delta}$, and $\hat{\delta}$ is a closed *-derivation with a dense set of analytic elements, hence [1, Theorem 6], $\hat{\delta}$ is a generator.

Finally we can prove

Theorem 6. *Let δ be a symmetric *-derivation on $\mathfrak{A}$ which vanishes on $S(\infty)$ and satisfies $\tau\circ\delta=0$. Then δ has an extension $\hat{\delta}$ which is a generator of a strongly continuous one-parameter group $\{\beta_t:t\in\mathbb{R}\}$ of product automorphisms of the form $\beta_t=\bigotimes_{k\geq 1}\beta_t'$.*

Proof. By the corollary to Lemma 5, δ has an extension to a generator $\hat{\delta}$. We have only to show that the associated one-parameter group $\{\beta_t\}$ has the desired form.

First note that $\hat{\delta}:B_1\to B_1$ (since $\mathfrak{A}_1=B_1$ and $\hat{\delta}:\mathfrak{A}_m\to\mathfrak{A}_m$ for all m), so that B_1 consists of analytic elements for $\hat{\delta}$. Let $p\in S(\infty)$, then $\hat{\delta}p=\delta p=0$. Hence for $x\in B_1$, $p\in S(\infty)$, pxp^{-1} is entire analytic for $\hat{\delta}$ and

$$\begin{aligned}\beta_t(pxp^{-1})&=\sum_{n\geq 0}(t^n/n!)[(\hat{\delta})^n(pxp^{-1})]\\&=\sum_{n\geq 0}(t^n/n!)p[(\hat{\delta})^n x]p^{-1}\\&=p\left\{\sum_{n\geq 0}(t^n/n!)[(\hat{\delta})^n x]\right\}p^{-1}\\&=p\beta_t(x)p^{-1}.\end{aligned}\tag{3}$$

Letting $p=I\ [=\varrho(e)]$, Eq. (3) gives $\beta_t:B_1\to B_1$. Now suppose $x=e^1_{ij}\in B_1$ and $\beta_t(e^1_{ij}) = \sum_{r,s=1}^{n} \alpha_{ijrs}(t)e^1_{rs}$. Letting $p=(1k)\in S(\infty)$ and applying both Eqs. (1) and (3), we have

$$\begin{aligned}\beta_t(e^k_{ij}) &= \beta_t(pe^1_{ij}p^{-1})\\ &= p\beta_t(e^1_{ij})p^{-1}\\ &= \sum_{r,s=1}^{n} \alpha_{ijrs}(t)e^k_{rs}.\end{aligned}$$

Hence $\beta_t:B_k\to B_k$, all k, and under the obvious identification $B_1=B_2=\dots$, we have $\beta_{t|B_1}=\beta_{t|B_2}=\dots$. Thus $\beta_t=\bigotimes_{k\geqq 1}\beta'_t$, where $\beta'_t=\beta_{t|B_1}$, and the proof is complete.

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