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On the Regularization of the Kepler Problem

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Abstract. An intimate relationship between Moser's regularization [1] and the KS-regularization [3] of the 3-dimensional Kepler problem is established. Explicit formulae linking Moser's and the KS-transformation are obtained in the case of negative as well as in the case of positive energies. As a side result it is shown that the KS-transformation owes its existence to the local isomorphism of SO(2,4) and SU(2,2).

1. Introduction

In [1] (see also [2]) Moser, starting from a stereographic projection in configuration space, constructs a diffeomorphism that carries the geodesic flow on the unit tangent bundle of the pointed n-sphere onto the flow of the n-dimensional Kepler problem on a surface of fixed negative energy. The missing point together with an $(n-1)$-sphere of directions correspond to the collision states of the Kepler problem. When the Kepler flow on a surface of fixed negative energy is replaced by the geodesic flow on the unit tangent bundle of the n-sphere, the collision states are "regularized", i.e. they loose their exceptional status and are indistinguishable from all the other states. This "regularization" has the fringe benefit of exposing the hidden $SO(n+1)$-symmetry of the Kepler problem. This symmetry in turn makes it obvious that besides the $\frac{1}{2}n(n-1)$ components of the angular momentum integral, the Kepler problem possesses n additional integrals which together make up the Lenz-Runge vector (see in particular [2]).

A seemingly quite different procedure which achieves a regularization of the Kepler problem was proposed by Kustaanheimo and Stiefel (KS) in [3]. Their procedure has been explained in great detail in the monograph [4]. It is based on the KS-transformation which generalizes the Levi Civita transformation from two to three dimensions. The KS-transformation replaces the 3-dimensional Kepler Hamiltonian (with fictitious time s) by a Hamiltonian of four harmonic oscillators in resonance-denoted by J in the sequel – whose energy surfaces are 7-spheres embedded $\mathbb{R}^8(=\mathbb{C}^4)$. However, only points that also lie on a certain seven-

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dimensional null-quadric $I^{-1}(0)$ (I = certain quadratic form) represent physical states. More precisely, the physical states are in one-to-one correspondence, not with the points of this quadric surface, but rather with the orbits induced on this surface by a certain action of the circular group U(1). In analogy to a similar situation in electrodynamics we shall refer to this group as "gauge group" (compare [5]). Stated differently, the phase space of the KS-regularized Kepler problem appears in the form of an orbit manifold of type $I^{-1}(0)/\mathrm{U}(1)$.

In the present note we shall establish an intimate relationship between Moser's and the KS-transformation. This relationship between the two transformations turns out to be of practical value when perturbation problems of the Kepler problem (such as the three dimensional lunar problem [8]) are studied. Instead of deciding from the outset for one of the two points of view inherent in the two regularization procedures, their close relationship allows us to switch from one point of view to the other, thereby enabling us to choose always the procedure that is best suited for the investigation of a particular aspect of our problem.

Apart from the introduction (Sect. 1) the present paper is broken into four sections. In Sect. 2 we present a review of Moser's transformation μ. In order to avoid a switch of position and momentum variables (which seems to be an ingredient of Moser's original version of his map), our point of departure is a homogenous version of the stereographic projection in momentum – rather than in configuration – space.

In Sect. 3 we review the KS-transformation which we write in terms of complex variables and Pauli matrices. (According to a personal communication of J. Waldvogel this was the way Kustaanheimo originally wrote his transformation (see also [5]).) After giving it a group-theoretical interpretation we link the KS-transformation to Penrose's twistor theory [9].

Whereas Sects. 2–3 of the present work contain essentially reformulations of old results, Sect. 4 contains our original contribution to the subject. We show that the "completed" phase space of the Kepler problem: $I^{-1}(0)/\mathrm{U}(1)$ is symplectically diffeomorphic to T^+S^3 [=(co-) tangent bundle of the 3-sphere from which the zero-section has been removed]. This is done with the help of an "extension $\hat{\pi}$ of the KS-map π" which explicitly reduces out the action of the gauge group U(1) on all of $I^{-1}(0)$. The relation between π, $\hat{\pi}$ and Moser's transformation μ is capsuled in the following diagram (see theorem of Sect. 4).

$$\begin{tikzcd}
(I^{-1}(0))' \arrow[r, "\mathfrak{C}"] \arrow[d, "\pi"] & I^{-1}(0) \arrow[d, "\hat{\pi}"] \\
(\mathbb{R}^3\backslash\{0\}) \times \mathbb{R}^3 \arrow[r, "\mu^{-1}"] & T^+S^3 .
\end{tikzcd}$$

Here $\mathfrak{C}$ [given in (4.2)] represents the injection of the circle bundle $(I^{-1}(0))'$ of non-collision states into the circle bundle $I^{-1}(0)$ encompassing all states.

The map $\hat{\pi}$ is closely tied to group theoretical concepts. It turns out that the group leaving the quadratic form I invariant is U(2, 2) acting linearly on $\mathbb{C}^4$. Moreover, this action is symplectic with respect to the very symplectic structure of $\mathbb{C}^4$ that in conjunction with the function J is the main ingredient in a Hamiltonian description of the system of four harmonic oscillators mentioned earlier.

Also, the map $\hat{\pi}$ is constructed in terms of certain "generators" of this group action. In fact, the relationship between Moser's and the KS-transformation can loosely be described as follows. If in Moser's transformation the momentum variables are replaced by certain generators of 1-parameter subgroups of $\mathrm{SU}(2,2)$ and the position variables by a quotient of such generators with denominator J, then the KS-transformation is obtained (see Corollary 1 to theorem of Sect. 4).

Now $\hat{\pi}$ not only reduces out the action of the gauge group $\mathrm{U}(1)$ on $I^{-1}(0)$ but also transfers the transitive action of $\mathrm{U}(2,2)=\mathrm{U}(1)\times\mathrm{SU}(2,2)$ on $I^{-1}(0)$ to T^+S^3 so that T^+S^3 appears in the form of a symplectic homogeneous space of $\mathrm{SU}(2,2)$ (see Corollary 2 to theorem of Sect. 4). In fact it turns out that the action of $\mathrm{SU}(2,2)$ on T^+S^3 coincides with the action of the identity component $\mathrm{SO}_0(2,4)$ of $\mathrm{SO}(2,4)$ that was previously described by Guillemin and Sternberg in [10]. [$\mathrm{SU}(2,2)$ doubly covers $\mathrm{SO}_0(2,4)$: see Appendix B.]

The fundamental role that the Lie algebra $\mathrm{so}(4,2)$ plays in the KS-regularization was also recognized by Baumgarte [11] who adapts some ideas presented by Barut in his study of the quantum mechanical Kepler problem [12] to classical mechanics.

Finally, in Sect. 5 we show how to attack our main problem under the assumption of positive (instead of negative) Kepler energies. Whereas the KS-transformation remains unchanged, Moser's transformation has to be modified in the sense of Belbruno [6] (see also [7]) at least if one still wants to linearize the Kepler flow. Accordingly, the relation between the two maps, although similar in nature as in the case of negative energies, is expressed explicitly by a different recipe.

The case of zero energy will not be dealt with here. In fact, our success in relating the two transformations in the case of non-zero energies is based on the fact that in our version of Moser's and the Moser-Belbruno map the transformation of the momentum variables is – like in the KS-transformation – homogenous of degree zero. However this property can no longer be salvaged in the case of zero energy. Therefore, if in this case there exists any relationship at all between the two regularization procedures it must be of a quite different nature than in the former two cases.

2. Review of Moser's Transformation

Before we turn to the proper subject of this section, namely a review of Moser's transformation, we shall make some general remarks.

We recall that a Hamiltonian system can be characterized as a triplet of objects (M,ω,H), where M is an even-dimensional smooth manifold, ω is a closed 2-form which is nondegenerate at each point of M, and H is a smooth real-valued function on M, called the Hamiltonian. Via the associated vector field X_H, which is defined by the formula

$$X_H \lrcorner\, \omega = -dH\,, \tag{2.1}$$

the Hamiltonian H induces (or generates) a flow (=action of the group $\mathbb{R}$) on M (at least if M is compact). Since H is an integral of (i.e. invariant under) this flow it carries each level (=energy-) surface into itself. More generally, $F\in C^\infty(M)$ is an integral of the flow if $\{H,F\}=0$, where $\{H,F\}=X_H(F)=\omega(X_H,X_F)$ is the so-called Poisson bracket associated with ω.

The standard example is the Hamiltonian system $(\mathbb{R}^{2n}, d\theta_0, H)$, where θ_0 is the 1-form

$$\theta_0=\mathbf{y}\cdot d\mathbf{x}. \tag{2.2}$$

[Here $x=(x_1,\dots,x_n)$, $y=(y_1,\dots,y_n)$ are coordinates on $\mathbb{R}^{2n}$ and the dot in (2.2) denotes the usual dot product of n-vectors.] In this example:

$$X_H=\frac{\partial H}{\partial \mathbf{y}}\cdot\frac{\partial}{\partial \mathbf{x}}-\frac{\partial H}{\partial \mathbf{x}}\cdot\frac{\partial}{\partial \mathbf{y}}, \tag{2.3}$$

and the flow of H is obtained by integration of the Hamiltonian equations

$$\frac{d\mathbf{x}}{dt}=\frac{\partial H}{\partial \mathbf{y}},\ \frac{d\mathbf{y}}{dt}=-\frac{\partial H}{\partial \mathbf{x}} \tag{2.4}$$

with general initial conditions.

The n-dimensional Kepler problem is the Hamiltonian system $((\mathbb{R}^n\backslash\{0\})\times\mathbb{R}^n, d\theta_0, H_0)$, where

$$H_0=\tfrac{1}{2}\mathbf{y}^2-r^{-1},\ r=|\mathbf{x}|=(\mathbf{x}\cdot\mathbf{x})^{1/2}. \tag{2.5}$$

Notice that H_0 is singular at $r=0$. It is well known (see e.g. [4]) that the singularity can be removed by fixing the energy and introducing the "fictitious time" s via the recipe

$$\frac{ds}{dt}=r^{-1}, \tag{2.6}$$

so that X_{H_0} is multiplied by r. On the energy surface $H_0=-\frac{1}{2}$ the resulting vector field rX_{H_0} agrees with X_{K_0}, where K_0 is the following function on $\mathbb{R}^{2n}$:

$$K_0=\frac{r}{2}(\mathbf{y}^2+1). \tag{2.7}$$

Moreover, K_0 takes the value 1 there. Since the map that associates with each point $(\mathbf{e},\mathbf{y})$ $(|\mathbf{e}|=1)$ the point $(\mathbf{x}=2(\mathbf{y}^2+1)^{-1}\mathbf{e},\mathbf{y})$ can easily be shown to define a diffeomorphism from $S^{n-1}\times\mathbb{R}^n$ onto the energy surface $H_0=-\frac{1}{2}(K_0=1)$, this surface is not compact. Notice that it does not contain any collision states. Indeed, these states would correspond to $\{0\}\times S^{n-1}$, where $\{0\}$ denotes the origin in configuration (i.e. $\mathbf{x}$-) space and S^{n-1} respresents the "sphere of infinite radius" in momentum (i.e. $\mathbf{y}$-) space. As pointed out already in the introduction a "regularization" consists in "completing" the energy surface in such a way that it contains the collision states on the same footing with all other states. In order to explain how this is achieved in Moser's regularization, we let $q=(q_0,\mathbf{q})$, $p=(p_0,\mathbf{p})$ be vectors of $\mathbb{R}^{n+1}$. Their inner product is denoted by

$$\langle q,p\rangle=q_0p_0+\mathbf{q}\cdot\mathbf{p}. \tag{2.8}$$

We also use the notation $\|p\| = \langle p, p\rangle^{1/2}$ for the norm of a $(n+1)$-vector. In the following the manifold

$$T^+S^n = \{(q,p) \in \mathbb{R}^{2(n+1)}, \|q\| = 1, \langle p, q\rangle = 0, p \neq 0\},$$

i.e. the (co-) tangent bundle of the n-sphere from which the zero-section has been removed, will play a crucial role. T^+S^n is a symplectically embedded submanifold of $\mathbb{R}^{2(n+1)}$, i.e. if

$$\theta_1 = \langle p, dq\rangle, \tag{2.9}$$

then $d\theta_1|_{T^+S^n}$ (the bar means: "restriction to") is non-degenerate at every point. We introduce Moser's map μ as the restriction to T^+S^n of the surjection: $\mathbb{R}_*^{2(n+1)} = \{(q,p) \in \mathbb{R}^{2(n+1)}: p_0 + \|p\| \neq 0\} \to \mathbb{R}^{2n}$ defined by the formulae

$$\mathbf{x} = (\|p\| + p_0)\mathbf{q} - q_0\mathbf{p}, \mathbf{y} = (\|p\| + p_0)^{-1}\mathbf{p}. \tag{2.10}$$

μ has the following properties:

(i) μ is diffeomorphism of $(T^+S^n)' = T^+S^n \cap \mathbb{R}_*^{2(n+1)}$ onto $(\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}^n$,

(ii) $K_0 \circ \mu = \|p\|$,

(iii) $\mu^* d\theta_0 = d\theta_1|_{(T^+S^n)'}$.

Proof. (ii) and (iii) are established by straightforward computations. In order to guide the reader through these computations we present the following hints: Viewing $\mathbf{x}$ via (2.10) as a function on $(T^+S^n)'$ we replace in the expression for $\mathbf{x}^2$: $\mathbf{q}^2$ by $1 - q_0^2$, $\mathbf{q} \cdot \mathbf{p}$ by $-q_0 p_0$ and $\mathbf{p}^2$ by $\|p\|^2 - p_0^2$. We find $r = p_0 + \|p\|$ and (ii) follows at once. The same replacements supplemented by the additional one: $\mathbf{p} \cdot d\mathbf{p} \to \|p\| d\|p\| - p_0 dp_0$ in the formula for $\mathbf{y} \cdot d\mathbf{x}$ yields the relation $\mathbf{y} \cdot d\mathbf{x} = \langle p, dq\rangle - d(q_0\|p\|)$ from which (iii) immediately follows.

Finally, (i) is a consequence of the fact that μ possesses an inverse μ^{-1} described by the formulae

$$\begin{aligned} q &= [r(1+\mathbf{y}^2)]^{-1}(-2\mathbf{x}\cdot\mathbf{y}, (1+\mathbf{y}^2)\mathbf{x} - 2(\mathbf{x}\cdot\mathbf{y})\mathbf{y}), \\ p &= (\tfrac{1}{2}r(1-\mathbf{y}^2), r\mathbf{y}). \end{aligned} \tag{2.11}$$

In view of (iii) μ^{-1} can be interpreted as a symplectic injection of $(\mathbb{R}^n \backslash \{0\}) \times \mathbb{R}^n$ into T^+S^n.

This injection maps the energy surface $H_0 = -\frac{1}{2}(K_0 = 1)$ onto the manifold $(T^1S^n)' = T^1S^n \cap \mathbb{R}_*^{2(n+1)}$, where $T^1S^n = \{(q,p) \in T^+S^n: \|p\| = 1\}$ is the unit (co-) tangent bundle of the n-sphere. Points of T^1S^n outside the image of μ^{-1} have coordinates $q_0 = 0$, $p = (-1, \mathbf{0})$. Obviously they correspond to the collision states of the Kepler problem. By adding them in, the energy surface is replaced by the compact manifold T^1S^n.

Summarizing we see that in Moser's regularization, the flow on a surface of fixed negative energy of the Kepler problem is replaced by the flow that the Hamiltonian system

$$(T^+S^n, d\theta_1|_{T^+S^n}, K_1 = \|p\| |_{T^+S^n}) \tag{2.12}$$

induces on the surface $K_1 = 1 (= T^1S^n)$. In order to obtain a description of this flow we observe that quite generally, given any Hamiltonian f on $\mathbb{R}^{2(n+1)}$, the vector field $X_{\hat{f}}$ associated with the Hamiltonian system $(T^+S^n, d\theta_1|_{T^+S^n}, \hat{f}=f|_{T^+S^n})$ [see (2.1)] is the restriction to T^+S^n of the following vector field on $\mathbb{R}^{2(n+1)}$:

$$\tilde{X}_f=\left\langle \nabla_p f, \frac{\partial}{\partial q}\right\rangle - \left\langle \nabla_q f, \frac{\partial}{\partial p}\right\rangle + \left\langle \frac{\partial f}{\partial p}, \Gamma \frac{\partial}{\partial p}\right\rangle. \tag{2.13}$$

Here,

$$\nabla_p f=\frac{\partial f}{\partial p}-\left\langle \frac{\partial f}{\partial p}, q\right\rangle q,\ \nabla_q f=\frac{\partial f}{\partial q}-\left\langle \frac{\partial f}{\partial q}, q\right\rangle q$$

are the covariant derivatives (along S^n) and $\Gamma_{\mu\nu}=q_\mu p_\nu-q_\nu p_\mu$ is the "generator" of a rotation of the $\mu\nu$-plane in configuration space (see below). Specializing to the case $f=\|p\|$, i.e. $\hat{f}=K_1$, the recipe (2.12) yields a vector field that on T^1S^n gives rise to the differential equations

$$\dot{q}=p,\ \dot{p}=-q. \tag{2.14}$$

Hence we see that K_1 induces on T^1S^n the geodesic flow (compare [1, 2]).

Observe that as a consequence of the fact that our Hamiltonian system (2.12) is invariant under the obvious action of $SO(n+1)$ on $\mathbb{R}^{2(n+1)}$: $(q,p)\to(0q,0p)\,(0\in SO(n+1))$, all functions $\Gamma_{\mu\nu}$ are integrals. This is in particular true for the n-vector:

$$\mathbf{R}=q_0\mathbf{p}-p_0\mathbf{q}, \tag{2.15}$$

which if pulled back to the original phase space via μ^{-1} [see (2.11)] takes the form

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{y}^2-1)\mathbf{x}-(\mathbf{x}\cdot\mathbf{y})\mathbf{y}. \tag{2.16}$$

It follows that $\{K_0,\mathbf{R}\}=0$, where $\{\ ,\ \}$ is the Poisson bracket associated with the 2-form $d\theta_0$ [θ_0 defined in (2.2)]. Since $K_0=r(H_0+\tfrac{1}{2})+1$, we conclude $\{H_0,\mathbf{R}\}=r^{-1}(H_0+\tfrac{1}{2})\{\mathbf{R},r\}=-(H_0+\tfrac{1}{2})\mathbf{y}$ so that

$$\mathbf{R}+(H_0+\tfrac{1}{2})\mathbf{x}=-(\mathbf{x}\cdot\mathbf{y})\mathbf{y}+\mathbf{y}^2\mathbf{x}-r^{-1}\mathbf{x}$$

is an integral of H_0. Of course, this is the well known Runge-Lenz vector. (For more details see [2].)

Another integral of H_0 [having its origin in the obvious $SO(n)$-symmetry] is the angular momentum. In the case $n=3$ it is also a 3-vector given by the expression

$$\mathbf{L}=\mathbf{q}\times\mathbf{p}=\mathbf{x}\times\mathbf{y}. \tag{2.17}$$

3. Review of the KS-Transformation

The canonical KS-transformation is a map $\pi:(\mathbb{R}^4\backslash\{0\})\times\mathbb{R}^4\to(\mathbb{R}^3\backslash\{0\})\times\mathbb{R}^3$. Here the target space is the phase space of the 3-dimensional Kepler problem [parametrized by $(\mathbf{x}, \mathbf{y})$]. Introducing variables (u_1, u_2, u_3, u_4), (v_1, v_2, v_3, v_4) in the domain space it is written in [4] in the following form

$$\begin{aligned} x_1 &= u_1^2 - u_2^2 - u_3^2 + u_4^2, \ y_1 = (2\|u\|^2)^{-1}(u_1v_1 - u_2v_2 - u_3v_3 + u_4v_4), \\ x_2 &= 2(u_1u_2 - u_3u_4), \ y_2 = (2\|u\|^2)^{-1}(u_1v_2 + u_2v_1 - u_3v_4 - u_4v_3), \\ x_3 &= 2(u_1u_3 + u_2u_4), \ y_3 = (2\|u\|^2)^{-1}(u_1v_3 + u_2v_4 + u_3v_1 + u_4v_2), \end{aligned} \tag{3.1}$$

where $\|u\|^2 = u_1^2 + u_2^2 + u_3^2 + u_4^2$.

Introducing complex variables in $(\mathbb{R}^4\backslash\{0\}) \times \mathbb{R}^4$ by means of the formulae

$$\begin{aligned} z &= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 2^{-1/2} \begin{pmatrix} -[u_2 + u_4 + i(u_1 + u_3)] \\ u_2 - u_4 - i(u_1 - u_3) \end{pmatrix}; \\ w &= \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 2^{-3/2} \begin{pmatrix} -(v_1 + v_3) + i(v_2 + v_4) \\ -[(v_1 - v_3) + i(v_2 - v_4)] \end{pmatrix}, \end{aligned} \tag{3.2}$$

it becomes a map $\pi : (\mathbb{C}^2\backslash\{0\}) \times \mathbb{C}^2 \to (\mathbb{R}^3\backslash\{0\}) \times \mathbb{R}^3$ which in terms of the usual inner product of $\mathbb{C}^2$:

$$\langle z, w \rangle = \bar{z}_1 w_1 + \bar{z}_2 w_2 \tag{3.3}$$

(bar means complex conjugation) and the vector of Pauli matrices

$$\boldsymbol{\sigma} = \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) \tag{3.4}$$

can be written in the following form

$$\mathbf{x} = \langle z, \boldsymbol{\sigma} z \rangle, \quad \mathbf{y} = \langle z, z \rangle^{-1} \operatorname{Im} \langle w, \boldsymbol{\sigma} z \rangle. \tag{3.5}$$

π is a canonical extension of the Hopf-map

$$\pi_0 : \mathbf{x} = \langle z, \boldsymbol{\sigma} z \rangle \tag{3.6}$$

in the following sense: Let $\mathbb{C}^4\backslash\{0\}$ be endowed with the symplectic structure that is canonically associated with the 1-form

$$\theta = 2 \operatorname{Im} \langle w, dz \rangle, \tag{3.7}$$

and let $I^{-1}(0)$ be the 7-dimensional quadric surface in $\mathbb{C}^4\backslash\{0\}$ on which the quadratic form

$$I = \operatorname{Re} \langle w, z \rangle \tag{3.8}$$

vanishes. Then π has the property

$$\pi^* \theta_0 = \theta|_{(I^{-1}(0))'}, \tag{3.9}$$

where θ_0 was defined in (2.1) and $\theta|_{(I^{-1}(0))'}$ is the restriction of θ to the manifold

$$(I^{-1}(0))' = I^{-1}(0) \cap \{(z, w) : z \neq 0\}. \tag{3.10}$$

In order to prove (3.9) we first note that the following formula holds for arbitrary elements $u, w, z \in \mathbb{C}^2$.

$$\langle u, \boldsymbol{\sigma} z \rangle \cdot \boldsymbol{\sigma} w = 2 \langle u, w \rangle z - \langle u, z \rangle w. \tag{3.11}$$

(The dot denotes the usual dot product of $\mathbb{R}^3$.)

Adding to (3.11) the relation obtained from it by interchanging u and z yields

$$\mathrm{Re}\langle u, \boldsymbol{\sigma} z\rangle \cdot \boldsymbol{\sigma} w = \langle u, w\rangle z + \langle z, w\rangle u - \mathrm{Re}\langle z, u\rangle w. \tag{3.12}$$

Multiplying both sides of (3.12) with $z^\dagger = (\bar{z}_1, \bar{z}_2)$ from the left and taking real parts we obtain

$$\mathrm{Re}\langle z, \boldsymbol{\sigma} u\rangle \cdot \mathrm{Re}\langle z, \boldsymbol{\sigma} w\rangle = \langle z, z\rangle\, \mathrm{Re}\langle u, w\rangle - \mathrm{Im}\langle z, u\rangle\, \mathrm{Im}\langle z, w\rangle. \tag{3.13}$$

Replacing in (3.13) u by dz and w by iw gives

$$\mathrm{Re}\langle z, \boldsymbol{\sigma} dz\rangle \cdot \mathrm{Im}\langle w, \boldsymbol{\sigma} z\rangle = \langle z, z\rangle\, \mathrm{Im}\langle w, dz\rangle - \mathrm{Re}\langle z, w\rangle\, \mathrm{Im}\langle z, dz\rangle.$$

On account of (3.5) this relation becomes for $(z, w) \in (I^{-1}(0))'$:

$$\mathbf{y} \cdot d\mathbf{x} = 2\, \mathrm{Im}\langle w, dz\rangle.$$

However, this is precisely the content of formula (3.9).

The physical states in the KS-regularization are the orbits induced on the manifold $(I^{-1}(0))'$ by the action: $e^{is}(z, w) \to (e^{is}z, e^{is}w)$ of the "gauge group" $U(1)$ whose "infinitesimal generator" is the Hamiltonian $2I$. In fact, the KS-map π establishes a diffeomorphism between the orbit space $(I^{-1}(0))'/U(1)$ and the phase space $(\mathbb{R}^3 \backslash \{0\}) \times \mathbb{R}^3$ of the Kepler problem and formula (3.9) can be interpreted as saying that this diffeomorphism is symplectic. In accordance with this interpretation we seek a Hamiltonian $J = J(z, w)$ such that $J = K_0 \circ \pi$ on $(I^{-1}(0))'$, where K_0 was defined in (2.7). In order to construct this Hamiltonian we first express r and $\mathbf{y}^2$ in terms of the variables (z, w). Replacing u and w in (3.13) first by z, then by iw we find on $I^{-1}(0)$:

$$|\mathbf{x}| = r = \langle z, z\rangle, \ \mathbf{y}^2 = \langle z, z\rangle^{-1}\langle w, w\rangle, \tag{3.14}$$

so that

$$J = \tfrac{1}{2}[\langle z, z\rangle + \langle w, w\rangle]. \tag{3.15}$$

J is not only defined on $(I^{-1}(0))'$ but on the entire surface $I^{-1}(0)$ which encompasses the collision states. In fact, in the KS-regularization the collision states with energy $-\frac{1}{2}$ are represented by the orbits

$$(z = 0, e^{is}w)_{s \in \mathbb{R}} (\langle w, w\rangle = 2 \ \text{ since } \ J = 1) \text{ of the gauge group } U(1).$$

Accordingly, the "completed" phase space of the KS-regularized Kepler problem appears in the form of the orbit space $I^{-1}(0)/U(1)$. In Sect. 4 we shall prove that $I^{-1}(0)/\mathrm{U}(1)$ is symplectically diffeomorphic to T^+S^3. Moreover, the explicit map $\hat{\pi}: I^{-1}(0) \to T^+S^3$ which accomplishes the reduction of the group U(1) can be viewed as an extension of the KS-map from $(I^{-1}(0))'$ to $I^{-1}(0)$ in the sense of the diagram of Sect. 1.

Points of $I^{-1}(0)$ are called "null-twistors" by Penrose [9] who uses them in order to compactify the manifold of null-lines N of Minkowski space. Like a surface of fixed negative energy of the Kepler problem this manifold possesses the topological character $S^2 \times \mathbb{R}^3$. Indeed, selecting a fixed space-like hyperplane $\mathbb{R}^3$ in Minkowski space a null-line is determined by the following data:

(i) a directional vector = point of S^2,

(ii) a point of intersection with the hyperplane $\mathbb{R}^3$.

Thus, the manifold N can be identified with a surface of fixed negative energy of the Kepler problem which in turn we have identified with the orbit manifold

$$[(I^{-1}(0))' \cap J^{-1}(1)]/\mathrm{U}(1) \tag{3.16}$$

via the KS-transformation. Its compactification is achieved by dropping the prime in (3.16). This process which corresponds to filling in the collision states in the Kepler problem is interpreted as "attaching a null-cone at infinity" in the case of the manifold N.

Before we close this section we want to express the angular momentum in the variables (z, w). To this end we recall the relation

$$(\mathbf{a}\cdot\boldsymbol{\sigma})\boldsymbol{\sigma} = \mathbf{a}\sigma_0 + i(\boldsymbol{\sigma}\times\mathbf{a}), \tag{3.17}$$

which is valid for all 3-vectors $\mathbf{a}$. Here σ_0 is the 2 by 2 unit matrix. We apply both sides of (3.17) to $v \in \mathbb{C}^2$ and simultaneously set $\mathbf{a} = \langle u, \boldsymbol{\sigma}z\rangle$. We obtain

$$(\langle u, \boldsymbol{\sigma}z\rangle\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}v = \langle u, \boldsymbol{\sigma}z\rangle v + i(\boldsymbol{\sigma}v \times \langle u, \boldsymbol{\sigma}z\rangle). \tag{3.18}$$

Replacing w by $\boldsymbol{\sigma}v$ in (3.11) yields a relation which allows us to replace the left side of (3.18) by $2\langle u, \boldsymbol{\sigma}v\rangle z - \langle u, z\rangle\boldsymbol{\sigma}v$. If the resulting identity is multiplied from the left by $w^\dagger$ the following relation is obtained

$$\langle w, v\rangle\langle u, \boldsymbol{\sigma}z\rangle + i\langle w, \boldsymbol{\sigma}v\rangle \times \langle u, \boldsymbol{\sigma}z\rangle = 2\langle w, z\rangle\langle u, \boldsymbol{\sigma}v\rangle - \langle u, z\rangle\langle w, \boldsymbol{\sigma}v\rangle. \tag{3.19}$$

Setting $u = v = z$ in (3.19) and simultaneously replacing w by iw yields

$$\langle w, \boldsymbol{\sigma}z\rangle \times \langle z, \boldsymbol{\sigma}z\rangle = -i\langle w, z\rangle\langle z, \boldsymbol{\sigma}z\rangle + i\langle z, z\rangle\langle w, \boldsymbol{\sigma}z\rangle.$$

Comparing imaginary parts on both sides of the last identity we obtain

$$\langle z, \boldsymbol{\sigma}z\rangle \times \mathrm{Im}\langle w, \boldsymbol{\sigma}z\rangle = \mathrm{Re}\langle w, z\rangle\langle z, \boldsymbol{\sigma}z\rangle - \langle z, z\rangle\,\mathrm{Re}\langle w, \boldsymbol{\sigma}z\rangle. \tag{3.20}$$

On account of (3.5) and (2.17) this relation reduces on $I^{-1}(0)$ to the simple form

$$\mathbf{L} = -\mathrm{Re}\langle w, \boldsymbol{\sigma}z\rangle. \tag{3.21}$$

Actually, a much more elegant derivation of this formula based on group theory can be presented: Observe that the action of SU(2) on $\mathbb{C}^4\backslash\{0\}$: $U:(z, w)\to(Uz, Uw)$ $(U\in\mathrm{SU}(2))$ is exact symplectic [i.e. leaves θ (defined in (3.7)) invariant] and also that $U_{\mathbf{e}}(s) = \exp\left(-\frac{is}{2}\mathbf{e}\cdot\boldsymbol{\sigma}\right) \in \mathrm{SU}(2)(|\mathbf{e}|=1)$ induces via π a simultaneous rotation of $\mathbf{x}$ and $\mathbf{y}$ about $\mathbf{e}$ through the angle s. It follows that the Hamiltonian inducing the flow $s\to U_{\mathbf{e}}(s)$, i.e. $-\mathrm{Re}\langle w, \boldsymbol{\sigma}z\rangle\cdot\mathbf{e}$ and the Hamiltonian $\mathbf{L}\cdot\mathbf{e}$ inducing the corresponding rotation must be π-related. Since this is true for all unit vectors $\mathbf{e}$ relation (3.21) follows.

4. The Relationship between Moser's and the KS-Regularization

In this section we carry through the program announced in the last section. Our point of departure is the recognition that the group U(2, 2) acts symplectically on the space $\mathbb{C}^4\backslash\{0\}$. In order to see this we subject (z, w) to the transformation

$$\begin{pmatrix}\eta\\ \zeta\end{pmatrix}=\mathfrak{C}\begin{pmatrix}z\\ w\end{pmatrix} \quad \text{with inverse} \quad \begin{pmatrix}z\\ w\end{pmatrix}=\mathfrak{C}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}, \tag{4.1}$$

where $\mathfrak{C}=\mathfrak{C}^{-1}$ is the matrix

$$\mathfrak{C}=2^{-1/2}\begin{pmatrix}\sigma_0 & \sigma_0\\ \sigma_0 & -\sigma_0\end{pmatrix}. \tag{4.2}$$

(Remember σ_0 is the 2 by 2 unit matrix.)

Expressing the 1-form (3.7) in the new variables it becomes cohomologous (i.e. equal up to an exact form; in symbols $\cong$) to one of the following forms

$$\begin{aligned}\theta_{\eta,\zeta}&\cong \operatorname{Im}\langle\eta, d\eta\rangle-\operatorname{Im}\langle\zeta, d\zeta\rangle\\ &\cong\frac{1}{i}[\langle\eta, d\eta\rangle-\langle\zeta, d\zeta\rangle]=\frac{1}{i}(\eta^{\dagger},\zeta^{\dagger})\mathfrak{J}\begin{pmatrix}d\eta\\ d\zeta\end{pmatrix}.\end{aligned} \tag{4.3}$$

Here $\mathfrak{J}$ is the 4 by 4 matrix

$$\mathfrak{J}=\begin{pmatrix}\sigma_0 & 0\\ 0 & -\sigma_0\end{pmatrix}. \tag{4.4}$$

The right side of (4.3) is manifestly invariant under the obvious action $U\in \mathrm{U}(2,2)$: $\begin{pmatrix}\eta\\ \zeta\end{pmatrix}\to U\begin{pmatrix}\eta\\ \zeta\end{pmatrix}$ of $\mathrm{U}(2,2)$. Since the expression (3.8) in the new variables becomes

$$I=\tfrac{1}{2}[\langle\eta,\eta\rangle-\langle\zeta,\zeta\rangle] \tag{4.5}$$

the same holds true for I and the null-quadric $I^{-1}(0)$. The gauge group $\mathrm{U}(1)$ (generated by I) appears now as the center of the group $\mathrm{U}(2,2)=\mathrm{U}(1)\times \mathrm{SU}(2,2)$.

Our goal is the construction of the map $\hat{\pi}$ entering the diagram of Sect. 1 in terms of generators of 1-parameter subgroups of $\mathrm{SU}(2,2)$ which in turn are labeled by members of the Lie algebra $\mathrm{su}(2,2)$. The Lie algebra

$$\mathrm{u}(2,2)=\mathrm{u}(1)\oplus\mathrm{su}(2,2), \tag{4.6}$$

as well as its dual $\mathrm{u}(2,2)^*$ will be identified with the Hilbert space of all complex 4 by 4 matrices $\mathfrak{A}$ for which $\mathfrak{J}\mathfrak{A}$ is Hermitian, equipped with the inner product $(\mathfrak{A},\mathfrak{B}\in \mathrm{u}(2,2))$

$$\langle\mathfrak{A},\mathfrak{B}\rangle=\operatorname{tr}(\mathfrak{J}\mathfrak{A}\mathfrak{J}\mathfrak{B}). \tag{4.7}$$

Since $\mathrm{u}(1)$ in (4.6) is spanned by the 4 by 4 unit matrix $\mathbb{1}$ and the members of $\mathrm{su}(2,2)$ are characterized by zero trace, the decomposition (4.6) is orthogonal. The appropriate bracket for the Lie algebra $\mathrm{u}(2,2)$ is $(\mathfrak{A},\mathfrak{B}\in \mathrm{u}(2,2))$:

$$[\mathfrak{A},\mathfrak{B}]=\frac{1}{i}(\mathfrak{A}\mathfrak{B}-\mathfrak{B}\mathfrak{A}), \tag{4.8}$$

and the 1-parameter subgroup corresponding to $\mathfrak{A}\in \mathrm{u}(2,2)$ is $\{\exp(is\,\mathfrak{A})\}_{s\in\mathbb{R}}$. Its action is generated by the Hamiltonian

$$\psi_{\mathfrak{A}}(\eta,\zeta)=(\eta^{\dagger},\zeta^{\dagger})\mathfrak{J}\mathfrak{A}\begin{pmatrix}\eta\\ \zeta\end{pmatrix}=\langle\psi(\eta,\zeta),\mathfrak{A}\rangle, \tag{4.9}$$

where

$$\psi(\eta,\zeta)=\mathfrak{J}\binom{\eta}{\zeta}(\eta^{\dagger}\zeta^{\dagger})=\begin{pmatrix}\eta\eta^{\dagger} & \eta\zeta^{\dagger}\\ -\zeta\eta^{\dagger} & -\zeta\zeta^{\dagger}\end{pmatrix} \tag{4.10}$$

is the so called "moment-map": $\mathbb{C}^4\backslash\{0\}\to \mathrm{u}^*(2,2)$ associated with our action of $\mathrm{U}(2,2)$. (For this notion see [13–17].)

In particular, denoting the generators associated with the following (orthogonal) basis of $\mathrm{su}(2,2)$

$$\tfrac{1}{2}\mathfrak{J},\ \mathfrak{M}=\tfrac{1}{2}\begin{pmatrix}-\boldsymbol{\sigma} & 0\\ 0 & 0\end{pmatrix},\ \mathfrak{N}=\tfrac{1}{2}\begin{pmatrix}0 & 0\\ 0 & -\boldsymbol{\sigma}\end{pmatrix} \tag{4.11}$$

$$\mathfrak{Q}=\tfrac{1}{2}\left(i\begin{pmatrix}0 & \sigma_0\\ \sigma_0 & 0\end{pmatrix},\begin{pmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{pmatrix}\right),\ \mathfrak{P}=\tfrac{1}{2}\left(\begin{pmatrix}0 & \sigma_0\\ -\sigma_0 & 0\end{pmatrix},\frac{1}{i}\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix}\right)$$

by the corresponding Roman letters, we find

$$J=\psi_{\frac{1}{2}\mathfrak{J}}=\tfrac{1}{2}[\langle\eta,\eta\rangle+\langle\zeta,\zeta\rangle], \tag{4.12}$$

$$\mathbf{M}=\psi_{\mathfrak{M}}=-\tfrac{1}{2}\langle\eta,\boldsymbol{\sigma}\eta\rangle,\ \mathbf{N}=\psi_{\mathfrak{N}}=\tfrac{1}{2}\langle\zeta,\boldsymbol{\sigma}\zeta\rangle, \tag{4.13}$$

$$\begin{aligned}Q&=\psi_{\mathfrak{Q}}=(-\operatorname{Im}\langle\eta,\zeta\rangle,\operatorname{Re}\langle\eta,\boldsymbol{\sigma}\zeta\rangle),\\ P&=\psi_{\mathfrak{P}}=(\operatorname{Re}\langle\eta,\zeta\rangle,\operatorname{Im}\langle\eta,\boldsymbol{\sigma}\zeta\rangle).\end{aligned} \tag{4.14}$$

Remark. Taking the transformation (4.1) into account the expressions for J in (3.15) and (4.12) agree.

Our map $\hat{\pi}$ will now be defined as the restriction to $I^{-1}(0)$ of a map: $\mathbb{C}^4\backslash\{0\}\to\mathbb{R}^8$ whose coordinate expression has the form

$$q=J^{-1}Q,\ p=P. \tag{4.15}$$

The significance of the map $\hat{\pi}$ is summarized in the following theorem and its first corollary.

Theorem. *The map $\hat{\pi}:I^{-1}(0)\to\mathbb{R}^8$ obtained by restricting the map defined in* (4.15) *to $I^{-1}(0)$ has range T^+S^3. $\hat{\pi}$ induces a symplectic diffeomorphism between the "completed" phase space $I^{-1}(0)/\mathrm{U}(1)$ and T^+S^3. In fact, the relation*

$$\hat{\pi}^*\theta_1=\theta|_{I^{-1}(0)} \tag{4.16}$$

holds.

Remark. θ in (4.16) denotes the last expression in (4.3) [which differs from the expression in (3.7) by an exact form]. θ_1 was defined in (2.9).

Corollary 1. *The* KS-*transformation π in the form* (3.5) *is the composition of the map $\hat{\pi}\circ\mathfrak{C}|_{(I^{-1}(0))'}$ and Moser's transformation $\mu:(T^+S^3)'=\mathbb{R}^8_*\cap T^+S^3\to(\mathbb{R}^3\backslash 0)\times\mathbb{R}^3$ defined in* (2.10). *In other words, the diagram of Sect.* 1 *holds.*

More informally, this means: If in Moser's transformation (2.10) p is replaced by P and q by $J^{-1}Q$ and the variables η,ζ are replaced by $2^{-1/2}(z+w)$ and $2^{-1/2}(z-w)$, respectively, then the transformation (3.5) results.

The proof of the theorem and its first corollary is based on the

Lemma. *The following relations exist between the generators* (4.5) *and* (4.12)–(4.14):

$$\mathbf{M}^2=\tfrac{1}{4}(J+I)^2,\ \mathbf{N}^2=\tfrac{1}{4}(J-I)^2, \tag{4.17}$$

$$\|Q\|^2=\|P\|^2=J^2-I^2, \tag{4.18}$$

$$\begin{aligned} -\tfrac{1}{2}(J+I)\mathbf{Q}&=P_0\mathbf{M}+\mathbf{M}\times\mathbf{P},\\ \tfrac{1}{2}(J+I)\mathbf{P}&=Q_0\mathbf{M}+\mathbf{M}\times\mathbf{Q}, \end{aligned} \tag{4.19}$$

$$\langle P,Q\rangle=0 \quad \text{for} \quad I+J\neq 0, \tag{4.20}$$

$$\begin{aligned} 2(J-I)\mathbf{M}&=Q_0\mathbf{P}-P_0\mathbf{Q}+\mathbf{Q}\times\mathbf{P},\\ 2(J+I)\mathbf{N}&=-(Q_0\mathbf{P}-P_0\mathbf{Q})+\mathbf{Q}\times\mathbf{P}. \end{aligned} \tag{4.21}$$

Remark. In (4.18) and (4.20) we used the notation of Sect. 2 according to which $\langle P,Q\rangle=P_0Q_0+\mathbf{P}\cdot\mathbf{Q}$, $\|Q\|=\langle Q,Q\rangle^{1/2}$.

Proof of the Lemma. Setting all variables in (3.13) equal to η or ζ, respectively and taking (4.5), (4.12) into account, the relations (4.17) result. Replacing in (3.13) and (3.20) z by η and $u=w$ first by ζ, then by $i\zeta$ the relations (4.18) and (4.19) are obtained. Forming the dot product of the first of the Eqs. (4.19) with $\mathbf{P}$ and of the second with $\mathbf{M}$ thereby taking (4.17) into account provides us with two equations from which (4.20) is deduced by elimination of the term $\mathbf{P}\cdot\mathbf{M}$. Setting in (3.19) $w=z=\zeta$, $v=u=\eta$ yields the first of the relations (4.21). The second is obtained from the first by an interchange of η and ζ. The proof of the Lemma is complete.

Proof of Corollary 1. Substituting $p=P$, $q=J^{-1}Q$ into the right side of (2.10) and taking (4.18), (4.21) into account yields on $I^{-1}(0)$:

$$\begin{aligned} \mathbf{x}&=J^{-1}[(\|P\|+P_0)\mathbf{Q}-Q_0\mathbf{P}]=\mathbf{Q}-J^{-1}(Q_0\mathbf{P}-P_0\mathbf{Q})=\mathbf{Q}-(\mathbf{M}-\mathbf{N})\\ &=\tfrac{1}{2}[2\,\mathrm{Re}\langle\eta,\boldsymbol{\sigma}\zeta\rangle+\langle\eta,\boldsymbol{\sigma}\eta\rangle+\langle\zeta,\boldsymbol{\sigma}\zeta\rangle]=\langle z,\boldsymbol{\sigma}z\rangle. \end{aligned}$$

Furthermore, since

$$J+P_0=\tfrac{1}{2}[\langle\zeta,\zeta\rangle+\langle\eta,\eta\rangle+2\,\mathrm{Re}\langle\eta,\zeta\rangle]=\langle z,z\rangle,$$

we also have

$$\mathbf{y}=(J+P_0)^{-1}\mathbf{P}=\langle z,z\rangle^{-1}\,\mathrm{Im}\langle w,\boldsymbol{\sigma}z\rangle.$$

Comparing these expressions with (3.5) verifies the statement of Corollary 1.

Remark. Setting $I=0$ in (4.21) and adding and subtracting the two relations yields

$$J(\mathbf{M}+\mathbf{N})=\mathbf{Q}\times\mathbf{P}, \tag{4.22}$$

$$J(\mathbf{M}-\mathbf{N})=Q_0\mathbf{P}-P_0\mathbf{Q}. \tag{4.23}$$

A comparison of these relations with (2.15) and (2.17) yields the following KS-representation of the angular momentum – and the Lenz-Runge-vector

$$\mathbf{L}=\mathbf{M}+\mathbf{N}, \tag{4.24}$$

$$\mathbf{R}=\mathbf{M}-\mathbf{N}. \tag{4.25}$$

Of course, (4.24) can also be obtained by subjecting the expression (3.21) to the transformation (4.1).

Proof of the Theorem. In view of (4.18) and (4.20) it is clear that the image of $\hat{\pi}$ is contained in T^+S^3. It remains to prove

(i) $\hat{\pi}$ actually maps $I^{-1}(0)$ onto T^+S^3.

(ii) The inverse image of each point of T^+S^3 is an orbit of the gauge group U(1).

(iii) Formula (4.16) holds.

In order to prove these statements we first write $\hat{\pi}$ as a composition of three maps $\hat{\pi}=j\circ(\mathrm{id}\times\pi_0)\circ F$. Here, F is the map which associates with each point $(\eta,\zeta)\in I^{-1}(0)$ the point $(J^{-1}Q,\eta)$ of $S^3\times(\mathbb{C}^2\backslash\{0\})$, π_0 is the Hopf map defined in (3.6) applied to η so that $(\mathrm{id}\times\pi_0)$ maps $S^3\times(\mathbb{C}^2\backslash\{0\})$ onto $S^3\times(\mathbb{R}^3\backslash 0)$, taking the point (q,η) into $(q,-2\mathbf{M})$ [see (4.13)] and finally the map $j: S^3\times(\mathbb{R}^3\backslash\{0\})\to T^+S^3$ is defined by the formula: $j(q,\mathbf{m})=(q,(\mathbf{q}\cdot\mathbf{m},\ -q_0\mathbf{m}+\mathbf{q}\times\mathbf{m}))$. Indeed, we have on $I^{-1}(0)$

$$\begin{aligned}(j\circ(\mathrm{id}\times\pi_0)\circ F)(\eta,\zeta)&=j(J^{-1}Q,-2\mathbf{M})\\&=J^{-1}(Q,(-2\mathbf{M}\cdot\mathbf{Q},2Q_0\mathbf{M}-2\mathbf{Q}\times\mathbf{M})=(J^{-1}Q,P)=\hat{\pi}(\eta,\zeta),\end{aligned}$$

where in the second to last equality the relations (4.19) and the first of the relations (4.17) have been used. Now F and j are actually diffeomorphisms onto their target spaces. Indeed, one easily checks that the map $G:\mathbb{R}^4\times(\mathbb{C}^2\backslash\{0\})\to\mathbb{C}^4\backslash\{0\}$ defined by $G(q,\eta)=(\eta,(\mathbf{q}\cdot\boldsymbol{\sigma})\eta-iq_0\eta)$ restricted to $S^3\times(\mathbb{C}^2\backslash\{0\})$ is an inverse of F. Similarly, the map $\varrho:\mathbb{R}^8\to\mathbb{R}^4\times(\mathbb{R}^3\backslash\{0\})$ defined by $\varrho(q,p)=(q,p_0\mathbf{q}-q_0\mathbf{p}+\mathbf{p}\times\mathbf{q})$, if restricted to T^+S^3, provides us with an inverse of j. Since F and j are diffeomorphisms and $\mathrm{id}\times\pi_0$ is a surjection statement (i) is now obvious. We turn to a proof of statement (ii): Given $(q,p)\in T^+S^3$ we compute $\hat{\pi}^{-1}(q,p)=[G\circ(\mathrm{id}\times\pi_0^{-1})\circ\varrho](q,p)=(\eta,\ (\mathbf{q}\cdot\boldsymbol{\sigma})\eta-iq_0\eta)$, where $\eta\in\mathbb{C}^2\backslash\{0\}$ is the general solution of the equation $\langle\eta,\boldsymbol{\sigma}\eta\rangle=p_0\mathbf{q}-q_0\mathbf{p}+\mathbf{p}\times\mathbf{q}$. Abbreviating the right side of this equation by $\mathbf{m}$ and setting $m=|\mathbf{m}|$ we find for the general solution

$$\eta=\begin{cases}2^{-1/2}(m+m_3)^{-1/2}(m+m_3,m_1+im_2)e^{i\alpha}, & \mathbf{m}\notin\mathrm{neg}\,m_3\text{-axis}\\ 2^{-1/2}(m-m_3)^{-1/2}(m_1-im_2,m-m_3)e^{i\alpha}, & \mathbf{m}\notin\mathrm{pos}\,m_3\text{-axis}\end{cases}$$

(α arbitrary real number), i.e. $\hat{\pi}^{-1}(q,p)$ is an orbit of the gauge group U(1). It remains to prove (iii), i.e. formula (4.16): On account of (4.20) we first compute:

$$\pi^*\theta_1=\hat{\pi}^*\langle p,dq\rangle=\langle P,d(J^{-1}Q)\rangle=J^{-1}\langle P,dQ\rangle.$$

Replacing in (3.13) u by $-i\zeta$, z by η and w by $d\zeta$ yields:

$$\mathrm{Im}\langle\eta,\boldsymbol{\sigma}\zeta\rangle\,\mathrm{Re}\langle\eta,\boldsymbol{\sigma}d\zeta\rangle=-\langle\eta,\eta\rangle\,\mathrm{Im}\langle\zeta,d\zeta\rangle+\mathrm{Re}\langle\eta,\zeta\rangle\,\mathrm{Im}\langle\eta,d\zeta\rangle.$$

Since $\mathbf{P}\cdot d\mathbf{Q}$ is obtained from the left side of the last relation by antisymmetrization in (η,ζ) we find on $I^{-1}(0)$:

$$\begin{aligned}\mathbf{P}\cdot d\mathbf{Q}=&-J(\mathrm{Im}\langle\zeta,d\zeta\rangle-\mathrm{Im}\langle\eta,d\eta\rangle)\\&+\mathrm{Re}\langle\eta,\zeta\rangle(\mathrm{Im}\langle\eta,d\zeta\rangle-\mathrm{Im}\langle\zeta,d\eta\rangle),\end{aligned}$$

or in view of (4.14)

$$\pi^*\theta_1 = J^{-1}\langle P, dQ\rangle = \mathrm{Im}\langle \eta, d\eta\rangle - \mathrm{Im}\langle \zeta, d\zeta\rangle .$$

The Theorem is proved.

It is clear that we can use the map $\hat{\pi}$ to transfer the action of SU(2, 2) on $I^{-1}(0)$ to T^+S^3. Corollary 2 tells us more about the transferred action.

Corollary 2. *T^+S^3 is a symplectic homogeneous space of the group* SU(2, 2). *More precisely, the fundamental linear action of* SU(2, 2) *on* $\mathbb{C}^4$ *is transferred by* $\hat{\pi}$ *to* T^+S^3, *where it can be regarded as a (non-linear) transitive and symplectic action of the identity component* $\mathrm{SO}_0(2,4)$ *of* SO(2, 4).

Proof. By Witt's theorem U(2, 2) acts transitively on $I^{-1}(0)$. Hence the action of SU(2, 2) on $I^{-1}(0)/\mathrm{U}(1)$ is transitive and the transferred action on T^+S^3 must have the same property. Formula (4.16) also guarantees that this action is symplectic. Since $U \in \mathrm{SU}(2,2)$ and $-U$ induce the same symplectic automorphism of T^+S^3 and since $\mathrm{SU}(2,2)/(\mathbb{1}, -\mathbb{1})$ is isomorphic to $\mathrm{SO}_0(2,4)$ (see Appendix B) the second statement of the Corollary follows and its proof is complete.

Remark. Starting from the formula (4.10) for the moment map it is not difficult to see that $I^{-1}(0)/\mathrm{U}(1)$ can also be realized as the orbit $\{\mathfrak{A} \in \mathrm{su}(2,2)^*: \mathfrak{J}\mathfrak{A} = \text{orthogonal projection onto a line of } \mathbb{C}^4\}$ of SU(2, 2) in su(2, 2)*, equipped with the symplectic structure that was discovered for such orbits by Kirillov [18]. (Here we think of $\mathbb{C}^4$ as being equipped with the inner product associated with the norm $2J$.) Combining this result with the insight expressed in Corollary 2 we recover the result of Guillemin and Sternberg [10] according to which T^+S^3 can be realized as a certain orbit of $\mathrm{SO}_0(2,4)$ in so(2, 4)*. Actually, the constructions of the last named authors generalize to arbitrary dimensions $n \geqq 2$, i.e. T^+S^n can be regarded as a symplectic homogeneous space of $\mathrm{SO}_0(2, n+1)$. From the point of view of Lie group theory the existence of the KS-transformation in the case $n=3$ is due to the local isomorphism of $\mathrm{SO}_0(2,4)$ and SU(2, 2): The action of $\mathrm{SO}_0(2,4)$ on T^+S^3 is implemented by the fundamental linear action of SU(2, 2) on $\mathbb{C}^4$ via the existence $\hat{\pi}$ of the KS-map. [Similarily, the existence of the Levi Civita transformation in the case $n=2$ can be understood as being due to the local isomorphism of the groups $\mathrm{SO}_0(2,3)$ and $\mathrm{Sp}(2,\mathbb{R})$.]

5. The Case of Positive Energies

So far we have concerned ourselves with the regularization of the Kepler problem on a surface of fixed negative energy. (Actually, we only treated the case with energy $-\frac{1}{2}$. However, the general case can be reduced to this case by an appropriate scaling; see [1, 2].) In this section we address the question of the relationship between the two regularization procedures in the case of positive energies. Again it suffices to study a special case, e.g. $H_0 = \frac{1}{2}$. The Hamiltonian (2.7) is now replaced by

$$K_0 = \frac{r}{2}(\mathbf{y}^2 - 1) \tag{5.1}$$

and the surface $H_0=\frac{1}{2}$, i.e. $K_0=1$ is diffeomorphic to $S^{n-1}\times\mathbb{R}^n_*$, where $\mathbb{R}^n_*=\{\mathbf{y}:|\mathbf{y}|>1\}$ is the outside of the unit ball in momentum space. Subjecting the Hamiltonian (5.1) to Moser's transformation (2.10) and to the KS-transformation (3.5) we obtain $-p_0$ and $-P_0$ respectively. Applying the recipe (2.13) to $f=-p_0$ we find the following equations of motion

$$\dot{q}_0=q_0^2-1, \dot{\mathbf{q}}=q_0\mathbf{q}, \dot{p}_0=0, \dot{\mathbf{p}}=p_0\mathbf{q}-q_0\mathbf{p}. \tag{5.2}$$

On the other hand the Hamiltonian $-P_0$ gives rise to the linear flow $V(-s)$ defined in (A7) (see Appendix A). It follows that the integration of (5.2) yields the non-linear flow described in (A9) and (A10) (with s replaced by $-s$). Whereas the KS-transformation still linearizes the Kepler flow it is necessary to modify Moser's transformation in order to achieve this property for positive energies. How this should be done has been explained by Belbruno in [6]. However, for our purposes we need a modification of his procedure that imitates the procedure for negative energies as closely as possible. All computations and proofs are left to the reader.

Let $\mathbb{R}^{n+1}$ be equipped with the Lorentz metric $\langle q,p\rangle=q_0p_0-\mathbf{q}\cdot\mathbf{p}$ and consider the submanifold $T^+H^n=\{(q,p):\langle q,q\rangle=-1,\langle q,p\rangle=0,p_0>|\mathbf{p}|\}$ of $\mathbb{R}^{2(n+1)}$. Notice that for $(q,p)\in T^+H^n$ p is "time-like" so that the definitions $\|p\|=\langle p,p\rangle^{1/2}$, $\mathbb{R}^{2(n+1)}_*=\{(q,p):\langle p,p\rangle>0\, p_0\neq\|p\|\}$ make sense. Now define the Moser-Belbruno map $\beta:(TH^n)'=\mathbb{R}^{2(n+1)}_*\cap T^+H^n\to(\mathbb{R}^n\backslash\{0\})\times\mathbb{R}^n_*$ by means of the formulae

$$\mathbf{x}=(p_0-\|p\|)\mathbf{q}-q_0\mathbf{p}, \mathbf{y}=(p_0-\|p\|)^{-1}\mathbf{p}. \tag{5.3}$$

This map has the properties:

(i) β is a diffeomorphism of $(T^+H^n)'$ onto $(\mathbb{R}^n\backslash\{0\})\times\mathbb{R}^n_*$ whose inverse is described by the formulae:

$$\begin{aligned} q&=r(\mathbf{y}^2-1)^{-1}(-2\mathbf{x}\cdot\mathbf{y},(\mathbf{y}^2-1)\mathbf{x}-2(\mathbf{x}\cdot\mathbf{y})\mathbf{y}),\\ p&=\left(\frac{r}{2}(1+\mathbf{y}^2), r\mathbf{y}\right), \end{aligned} \tag{5.4}$$

(ii) $K_0\circ\beta=\|p\|$ [K_0 given in (5.1)],

(iii) $\beta^*d\theta_0=d\theta_1|_{(T^+H)'}$,

where

$$\theta_0=\mathbf{y}\cdot d\mathbf{x},\ \theta_1=-\langle p,dq\rangle. \tag{5.5}$$

Moreover, setting

$$\begin{aligned}(I^{-1}(0))_*&=I^{-1}(0)\cap\{(z,w):\langle w,w\rangle>\langle z,z\rangle\}\\ &=I^{-1}(0)\cap\{(\eta,\zeta):P_0<0, P_0 \text{ defined in (4.14)}\}\end{aligned}$$

and $(I^{-1}(0))'_*=(I^{-1}(0))_*\cap\{(z,w):z\neq0\}$ the diagram of Sect. 1 is replaced by the following diagram

$$\begin{array}{ccc} (I^{-1}(0))'_* & \xrightarrow{\mathfrak{C}} & (I^{-1}(0))_* \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\hat{\pi}} \\ (\mathbb{R}^3\backslash0)\times\mathbb{R}^3_* & \xrightarrow[\beta^{-1}]{} & T^+H^3. \end{array}$$

The extension $\hat{\pi}$ of the KS-map is now described explicitly by the following formulae

$$q = -P_0^{-1}(Q_0, \mathbf{R}),\ p = (J, \mathbf{P}), \tag{5.6}$$

where Q_0, P_0, $\mathbf{P}$ were defined in (4.14) and the $\eta-\zeta$-expressions of J and $\mathbf{R}$ are found in (4.12) and (4.25), respectively. The map $\hat{\pi}$ has the property

$$\hat{\pi}^*\theta_1 = \theta|_{(I^{-1}(0))_*},$$

[with θ, θ_1 defined in (4.3) and (5.5), respectively] and therefore it induces a symplectic diffeomorphism between $(I^{-1}(0))_*/\mathrm{U}(1)$ and T^+H^3. Finally, the relation between the MB-map β and the KS-map π can be described as follows: If in (5.3) the substitutions (5.6) are made (keeping in mind that $P_0<0$) then the KS-transformation (3.5) is obtained.

Appendix A

In this appendix we explore the action of SU(2, 2) on T^+S^3 in greater detail. First we show that the subgroup SU(2) × SU(2) of SU(2, 2) acting on η, ζ separately

$$\eta' = U_1\eta,\ \zeta' = U_2\zeta\,;\ U_1, U_2 \in \mathrm{SU}(2)$$

induces on T^+S^3 the fundamental action of SO(4) that was described in Sect. 2. [See the subsection following formula: (2.14).] To that end observe that a pair of elements U_1, $U_2 \in \mathrm{SU}(2)$ induces a linear norm-preserving correspondence $a' = aO$ ($a = (a_0, \mathbf{a})$ is thought of as a row with 4 entries) of $\mathbb{R}^4$ onto itself by means of the formula

$$U_1^\dagger(a_0\sigma_0 - i\mathbf{a}\cdot\boldsymbol{\sigma})U_2 = a_0'\sigma_0 - i\mathbf{a}'\cdot\boldsymbol{\sigma}\,. \tag{A1}$$

Since SU(2) is connected O must belong to the identity component SO(4) of O(4) (compare [19]). Now sandwiching both sides of (A1) between η and ζ yields

$$a_0\langle\eta', \zeta'\rangle - i\mathbf{a}\cdot\langle\eta', \boldsymbol{\sigma}\zeta'\rangle = a_0'\langle\eta, \zeta\rangle - i\mathbf{a}'\cdot\langle\eta, \boldsymbol{\sigma}\zeta\rangle\,.$$

Equating real and imaginary parts of this relation we find in view of the definition (4.15) of $\hat{\pi}$ (in obvious notation)

$$\langle a, q'\rangle = \langle a', q\rangle = \langle aO, q\rangle = \langle a, Oq\rangle$$

(q is thought of as a column) and on analogous identity with q replaced by p. Since the two identities are valid for all rows $a \in \mathbb{R}^4$ we conclude $q' = Oq$, $p' = Op$, where $O = O(U_1, U_2) \in \mathrm{SO}(4)$. Hence, SU(2) × SU(2) acts on T^+S^3 via the fundamental action of SO(4) that was described in Sect. 2. In order to study this action on an infinitesimal level we first observe that quite generally (see [13–18]) the association $\mathfrak{A} \in \mathrm{su}(2,2) \to \psi_{\mathfrak{A}} \in C^\infty(\mathbb{C}^4\backslash\{0\})$ defined in (4.9) is a homomorphism of Lie algebras in the sense that

$$\{\psi_{\mathfrak{A}}, \psi_{\mathfrak{B}}\} = -i\psi_{[\mathfrak{A},\mathfrak{B}]}\,. \tag{A2}$$

Here $[\mathfrak{A}, \mathfrak{B}]$ was defined in (4.8) and { , } is the Poission bracket with respect to the 2-form $\mathrm{i}d\theta$, i.e. for $f, g \in C^\infty(\mathbb{C}^4\backslash\{0\})$ we have:

$$\{f, g\} = (\nabla f)^\dagger\mathfrak{J}\nabla g - (\nabla g)^\dagger\mathfrak{J}\nabla f = \mathrm{tr}\{\mathfrak{J}[\nabla g(\nabla f)^\dagger - (\nabla f)(\nabla g)^\dagger]\}\,, \tag{A3}$$

where

$$\nabla f = \left(\frac{\partial f}{\partial \eta_1}, \frac{\partial f}{\partial \eta_2}, \frac{\partial f}{\partial \zeta_1}, \frac{\partial f}{\partial \zeta_2}\right)$$

is viewed as a column and

$$(\nabla f)^\dagger = \left(\frac{\partial f}{\partial \bar\eta_1}, \frac{\partial f}{\partial \bar\eta_2}, \frac{\partial f}{\partial \bar\zeta_1}, \frac{\partial f}{\partial \bar\zeta_2}\right)$$

is viewed as a row.

Returning to our $SU(2)\times SU(2)$-action we observe that it is obviously generated by **M** and **N** as defined in (4.13). In view of (3.17) an application of the recipe (A2) provides us with the following Poission brackets between these generators

$$\{M_K, M_l\} = i\varepsilon_{klj}M_j,\ \{N_k, N_l\} = i\varepsilon_{klj}N_j,\ \{M_k, N_l\} = 0\,. \tag{A4}$$

Here and in the following k, l vary freely over $1, 2, 3$ whereas over j a sum is extended from 1 to 3. $\Bigg($Also, $\varepsilon_{klj}=0$ unless klj is a permutation of 123, in which case $\varepsilon_{klj} = \pm 1$ depending on whether klj is an $\begin{Bmatrix}\text{even}\\ \text{odd}\end{Bmatrix}$ permutation of 123.$\Bigg)$ Another distinguished subgroup of $SU(2,2)$ is the group $SU(1,1)$ consisting of matrices $U = \begin{pmatrix} a & b \\ \bar b & \bar a\end{pmatrix}$, where $|a|^2 - |b|^2 = 1$. (Here we use slightly abusive notation: all four entries should actually be multiplied by σ_0 so that U is indeed a 4 by 4 matrix.) A basis of the corresponding Lie algebra $su(1,1)$ is $\frac{1}{2}\mathfrak{J}$, $\mathfrak{P}_0$, $\mathfrak{Q}_0$. Its members obey the bracket relations

$$[\tfrac{1}{2}\mathfrak{J}, \mathfrak{P}_0] = -\mathfrak{Q}_0,\ [\mathfrak{Q}_0, \mathfrak{P}_0] = -\tfrac{1}{2}\mathfrak{J},\ [\tfrac{1}{2}\mathfrak{J}, \mathfrak{Q}_0] = \mathfrak{P}_0\,. \tag{A5}$$

Applying the recipe (A2) to these relations we find the corresponding Poisson brackets

$$\{J, P_0\} = iQ_0,\ \{Q_0, P_0\} = iJ,\ \{Q_0, J\} = iP_0\,. \tag{A6}$$

The 1-parameter subgroups of $SU(1,1)$ generated by J, P_0, Q_0 are

$$U(s) = \begin{pmatrix} \exp i\dfrac{s}{2} & 0 \\ 0 & \exp\left(-i\dfrac{s}{2}\right)\end{pmatrix}$$

$$V(s) = \begin{pmatrix} \cosh\dfrac{s}{2} & i\sinh\dfrac{s}{2} \\ -i\sinh\dfrac{s}{2} & \cosh\dfrac{s}{2}\end{pmatrix}$$

$$W(s) = \begin{pmatrix} \cosh\dfrac{s}{2} & -\sinh\dfrac{s}{2} \\ -\sinh\dfrac{s}{2} & \cosh\dfrac{s}{2}\end{pmatrix}. \tag{A7}$$

Using the definitions (4.14), (4.15) we find that $U(s)$ induces on T^+S^3(via $\hat{\pi}$) the flow

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}. \tag{A8}$$

If restricted to T^1S^3 this is the geodesic flow which in turn-via Moser's transformation-corresponds to the Kepler flow (compare Sect. 2). Similarily, we find that the flow $V(s)$ after transferring it to T^+S^3 via $\hat{\pi}$ becomes

$$q_0' = [\cosh s + q_0 \sinh s]^{-1}(\sinh s + q_0 \cosh s)\,; \mathbf{q}' = [\cosh s + q_0 \sinh s]^{-1}\mathbf{q}\,, \tag{A9}$$

$$p_0' = p_0, \mathbf{p}' = (q_0\mathbf{p} - p_0\mathbf{q})\sinh s + \mathbf{p}\cosh s\,. \tag{A10}$$

One checks by direct computation that (A9) represents a conformal map of S^3 onto itself whereas (A10) makes the combined transformation a symplectic automorphism of T^+S^3. In fact, we easily recognize that the transformation (A9) together with SO(4) generate a conformal action of $\mathrm{SO}_0(1,4)$ on S^3 (compare [10, p. 177]). $\mathfrak{M}$, $\mathfrak{N}$, $\mathfrak{P}$ form a basis of the corresponding Lie algebra so(1, 4). The associated generators **M**, **N**, P obey the Poisson brackets (A4) supplemented by the following bracket relations [see also (4.24) and (4.25)].

$$\begin{aligned} &\{P_0, \mathbf{L}\} = 0,\ \{\mathbf{R}, P_0\} = i\mathbf{P},\ \{\mathbf{P}, P_0\} = i\mathbf{R}, \\ &\{R_k, P_l\} = 0,\ \{L_k, P_l\} = i\varepsilon_{klj}P_j,\ \{P_k, P_l\} = -i\varepsilon_{klj}L_j. \end{aligned} \tag{A11}$$

The transformation $W(s)$ [see (A7)] together with SU(2) × SU(2) give (via $\hat{\pi}$) rise to another action of $\mathrm{SO}_0(1,4)$ on T^+S^3, a "complete set of generators" of this group action being **M**, **N**, Q. These generators obey bracket relations that are obtained from (A4) and (A11) by a systematic replacement of P's by Q's. Finally, for the sake of completeness we write down the remaining 27 of the total 105 Poisson brackets involving the generators (4.12)–(4.14) associated with our basis (4.11) of su(2, 2) (= so(2, 4)):

$$\begin{aligned} &\{J, \mathbf{M}\} = 0,\ \{J, \mathbf{N}\} = 0,\ \{\mathbf{Q}, J\} = i\mathbf{P},\ \{J, \mathbf{P}\} = i\mathbf{Q} \\ &\{P_0, \mathbf{Q}\} = \{Q_0, \mathbf{P}\} = 0,\ \{Q_k, P_l\} = iJ\delta_{kl}\,. \end{aligned}$$

Appendix B

In this appendix we study the relationship between SU(2, 2) and $\mathrm{SO}_0(2,4)$. For this purpose we first define an action of SU(2, 2) on the space of complex anti-symmetric 4 × 4 matrices so(4, $\mathbb{C}$) by means of the formula

$$\phi_U(\mathfrak{A}) = U\mathfrak{A}U^t,\ U \in \mathrm{SU}(2,2), \mathfrak{A} \in \mathrm{so}(4, \mathbb{C})\,. \tag{B1}$$

Now each matrix $\mathfrak{A} \in \mathrm{so}(4, \mathbb{C})$ has a representation

$$\mathfrak{A} = \begin{pmatrix} a\varepsilon & A \\ -A^t & d\varepsilon \end{pmatrix}, \tag{B2}$$

where $A \in \mathrm{gl}(2, \mathbb{C})$; $a, d \in \mathbb{C}$, $\varepsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. On $\mathrm{so}(4, \mathbb{C})$ we define a conjugation $*$ and a complex-valued function λ by setting

$$*\mathfrak{A} = \begin{pmatrix} \bar{d}\varepsilon & -\varepsilon\bar{A}\varepsilon \\ \varepsilon A^{\dagger}\varepsilon & \bar{a}\varepsilon \end{pmatrix},$$

and $\lambda(\mathfrak{A}) = \det A - ad$ for $\mathfrak{A}$ given in (B2). $\mathfrak{A}$ in (B2) is real, i.e. $*\mathfrak{A} = \mathfrak{A}$ if and only if $d = \bar{a}$ and A has the form $A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ $(\alpha, \beta \in \mathbb{C})$. The function λ on such an element takes the value $\lambda(\mathfrak{A}) = -|a|^2 + |\alpha|^2 + |\beta|^2$. Breaking down a, α, β into real and imaginary parts we recognize that the space of real elements $\mathrm{so}(4, \mathbb{C})_{\mathbb{R}} = \{\mathfrak{A} \in \mathrm{so}(4, \mathbb{C}) : *\mathfrak{A} = \mathfrak{A}\}$ can be identified with $\mathbb{R}^{2,4}$. Our goal is to show that the action ϕ of $\mathrm{SU}(2,2)$ defined in (B1) induces transformations of $\mathrm{SO}(2,4)$ in $\mathrm{so}(4, \mathbb{C})_{\mathbb{R}}$. For this purpose it is sufficient to prove that the following two statements hold for all matrices $\mathfrak{A} \in \mathrm{so}(4, \mathbb{C})$

(i) $\lambda(\phi_U(\mathfrak{A})) = \lambda(\mathfrak{A})$,

(ii) $*\phi_U(\mathfrak{A}) = \phi_U(*\mathfrak{A})$.

(i) is an immediate consequence of the formula $\det \mathfrak{A} = (\lambda(\mathfrak{A}))^2$, the proof of which we leave to the reader: Since obviously ϕ preserves $\det \mathfrak{A}$ we have $\lambda(\phi_U(\mathfrak{A})) = \pm\lambda(\mathfrak{A})$. However for $U = \mathbb{1}$ ($\mathbb{1}$ = unit matrix) the plus sign holds. Since $\mathrm{SU}(2,2)$ is connected, the plus sign must hold for any $U \in \mathrm{SU}(2,2)$. We prove (ii) for invertible $\mathfrak{A} \in \mathrm{so}(4, \mathbb{C})$ for which the formula

$$*\mathfrak{A} = \lambda(\bar{\mathfrak{A}})\mathfrak{J}\bar{\mathfrak{A}}^{-1}\mathfrak{J} \tag{B3}$$

is valid. Indeed, in view of (B3) and statement (i) we compute

$$*\phi_U(\mathfrak{A}) = \lambda(\bar{\mathfrak{A}})\mathfrak{J}(U^{\dagger})^{-1}\bar{\mathfrak{A}}^{-1}\bar{U}^{-1}\mathfrak{J} = \lambda(\bar{\mathfrak{A}})\,U\mathfrak{J}\bar{\mathfrak{A}}^{-1}\mathfrak{J}U^{t} = \phi_U(*\mathfrak{A}).$$

Now statement (ii) follows for any $\mathfrak{A} \in \mathrm{so}(4, \mathbb{C})$ by continuity. Formula (B3) is an immediate consequence of the following two formulae whose proof is left to the reader:

$$*\mathfrak{A} = \mathfrak{J}\hat{\bar{A}}\mathfrak{J}, \quad \mathfrak{A}^{-1} = \lambda(\mathfrak{A})^{-1}\hat{\mathfrak{A}}. \tag{B4}$$

Here,

$$\hat{\mathfrak{A}} = \begin{pmatrix} d\varepsilon & \varepsilon A\varepsilon \\ -\varepsilon A^{t}\varepsilon & a\varepsilon \end{pmatrix}$$

if $\mathfrak{A}$ has the representation (B2). Whereas the first of the formulae (B4) holds for any $\mathfrak{A} \in \mathrm{so}(4, \mathbb{C})$ the second requires that $\mathfrak{A}$ be invertible.

The relation between $\mathrm{SU}(2,2)$ and $\mathrm{SO}(2,4)$ is most succinctly expressed in the

Proposition. $\mathrm{SU}(2,2)/(\mathbb{1}, -\mathbb{1}) \cong \mathrm{SO}_0(2,4)$.

Here $\cong$ means (analytically) isomorphic. As a consequence $\mathrm{SU}(2,2)$ doubly covers $\mathrm{SO}_0(2,4)$. Sketch of a proof: Let $\tilde{\phi}$ be the induced-action of $\mathrm{SU}(2,2)$ on $\mathbb{R}^{2,4}(\mathrm{so}(4, \mathbb{C})_{\mathbb{R}})$ that was constructed above. $h: U \to \tilde{\phi}_U$ is a homomorphism of $\mathrm{SU}(2,2)$ into $\mathrm{SO}(2,4)$ with kernel containing $\mathbb{1}, -\mathbb{1}$. By a straightforward computation one shows that the kernel contains no other elements, i.e. $h(U)=\mathbb{1}$ implies $U=\pm\mathbb{1}$. In particular, h is a local isomorphism so that $h(\mathrm{SU}(2,2))$ is an open subgroup of $\mathrm{SO}(2,4)$. Since it is the complement of the union of its cosets it is also closed. Hence $h(\mathrm{SU}(2,2))=\mathrm{SO}_0(2,4)$ and the statement of our Proposition follows.

Addendum. After this manuscript was completed the paper [20] appeared in print. Using different methods from ours this paper anticipates some minor results of the present paper. However, it neither touches upon the role which the group SU(2, 2) plays in the KS-regularization nor on the main topic of the present work, namely the (explicit) relationship between Moser's and the KS-transformation.

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