Commun. Math. Phys. 83, 77–82 (1982)

Communications in
Mathematical
Physics

The Perturbation Series for Φ_3^4 Field Theory is Divergent

C. de Calan and V. Rivasseau

Centre de Physique Théorique de l'Ecole Polytechnique, Plateau de Palaiseau, F-91128 Palaiseau Cedex, France

Abstract. We prove in a rigorous way the statement of the title.

I. Introduction

At least since a paper by Dyson [1], the perturbation expansion for field theories like Φ^4 or Q.E.D. is commonly believed to be a divergent series. However it has been periodically noticed [2] [3] [4] that this statement is rigorously proved only for the simplest models, i.e. $[P(\Phi)]_2$ [5].

In the case of euclidean Φ_3^4, we know the results of constructive field theory [6] and the Borel summability of the perturbation expansion [7]. Yet there is no proof that this series is not actually convergent. The difficulty which prevents Jaffe's method [5] from working for Φ_3^4 is the change in the signs of some amplitudes, due to the renormalization, which could produce cancellations at each order. We solve this problem by rewriting the usual perturbation expansion in terms of convergent positive amplitudes involving a "dressed" propagator. This method is an iteration of the procedure used by Hepp [3] for the regularized version of the model.

More generally, the control of signs in renormalized perturbation series could allow one to go beyond the recent results on the Borel transformed series for Φ^4 theories [8] [9]. Extensions of our method might then provide rigorous results on the presence—or absence—of singularities on the real axis of the Borel plane ("instantons" or "renormalons" [10]).

II. Proof

We consider massive scalar bosons (the mass-scale is fixed by taking $m=1$) self-interacting via the lagrangian density $\mathscr{L}_I = -\frac{\lambda}{4!} : \Phi^4 :$ in an euclidean space-time with three dimensions. The formal series in λ defining a given N-points connected Schwinger function S_N is given in terms of Feynman graphs G (with $N(G)$ external lines, $n(G)$ vertices) by:

$$S_N(p,\lambda) = \sum_{\substack{G \\ N(G)=N}} A_G^R(p,\lambda), \tag{1}$$

0010-3616/82/0083/0077/$01.20

where $A_G^R(p,\lambda) = (-\lambda)^{n(G)} I_G^R(p)$ is the renormalized amplitude corresponding to G, and $p = \{p_1, \ldots, p_N\}$ is the set of external euclidean momenta.

As in [7] we use Zimmermann's scheme to renormalize by subtracting at vanishing external momenta [11]. (Actually our proof does not depend on the subtraction point.) Now the only renormalization to be performed corresponds to the graph (or subgraph) in Fig. 1, which we call the "blob" in the rest of the paper. The bare propagator is:

$$P_0 = \frac{1}{p^2 + 1}, \tag{2}$$

and the blob renormalized amplitude is given by:

$$B_0(p,\lambda) = \frac{\lambda^2}{3} \int d^3k\, d^3k' \left[\frac{1}{(k^2+1)((k+k')^2+1)((p+k')^2+1)} - \frac{1}{(k^2+1)((k+k')^2+1)(k'^2+1)} \right]. \tag{3}$$

We introduce an operation $\mathscr{S}$ on the graphs $G : \mathscr{S}(G)$ is the graph obtained by replacing in G every maximal chain of blobs by a single line (see an example in Fig. 2). In order to add the amplitudes of the graphs G which have the same $\mathscr{S}(G)$, we define inductively dressed propagators P_i and amplitudes B_i by:

$$P_i(p,\lambda) = P_{i-1} + P_{i-1} B_{i-1} P_{i-1} + \ldots = \frac{1}{p^2 + 1 - \sum_{j=0}^{i-1} B_j(p,\lambda)}, \tag{4}$$

$$B_i(p,\lambda) = \frac{\lambda^2}{3} \int d^3k\, d^3k' \Big[P_i(k) P_i(k+k') P_i(p+k') - P_{i-1}(k) P_{i-1}(k+k') P_{i-1}(p+k') \Big]. \tag{5}$$

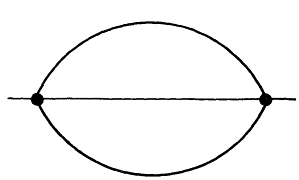

Fig. 1

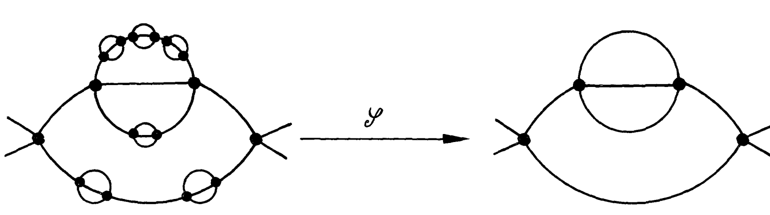

Fig. 2

$B_i(p,\lambda)$ is the sum of the amplitudes built from a blob with P_{i-1} propagators by inserting an arbitrary number of B_{i-1} on its three internal lines, the total number of insertions being not zero.

Next we associate to any graph G an amplitude $A_G^i(p,\lambda)$ in the following way. Let $\tilde{G}$ be the graph obtained by reducing every blob in G to a two-line vertex. Then $A_G^i(p,\lambda)$ is the amplitude corresponding to $\tilde{G}$ with a propagator P_i for each line, a factor B_i for each reduction vertex and a factor $(-\lambda)$ for each ordinary four-line vertex (in particular $A_G^0(p,\lambda)=A_G^R(p,\lambda)$). With these definitions we obtain:

$$\sum_{\substack{G' \\ \mathscr{S}(G')=G}} A_{G'}^i(p,\lambda)=A_G^{i+1}(p,\lambda). \tag{6}$$

Equation (6) is trivial from definition (4) if G contains no blob ($\mathscr{S}(G)=G$), and takes into account definition (5) if G contains blobs ($\mathscr{S}(G)\neq G$).

Lemma 1. *There exist three positive constants M_1,ρ_1 and $\varepsilon(\varepsilon<1)$ such that, for $|\lambda|<\rho_1$, $B_i(p,\lambda)$ is analytic in λ, and:*

$$\forall i \geqq 0, \quad |B_i(p,\lambda)|<(p^2+1)^{\varepsilon}\cdot(\tfrac{1}{3})^{i+1}\cdot 2M_1|\lambda^2|. \tag{7}$$

Proof. Inequality (7) is well-known for B_0. Let us assume it is true for B_j, $\forall j \leqq i-1$. Then:

$$\forall k \leqq i-1, \quad \left|\sum_{j=0}^{k} B_j(p,\lambda)\right| \leqq (p^2+1)^{\varepsilon} M_1|\lambda^2| \leqq \frac{p^2+1}{2}, \tag{8}$$

if we take

$$\rho_1^2 \leqq \frac{1}{2M_1}. \tag{9}$$

Therefore $\forall k \leqq i$:

$$\frac{2/3}{p^2+1} \leqq \frac{1}{(p^2+1)(1+M_1|\lambda^2|)} \leqq |P_k(p,\lambda)| \leqq \frac{1}{(p^2+1)(1-M_1|\lambda^2|)} \leqq \frac{2}{p^2+1} \tag{10}$$

and

$$|P_i-P_{i-1}|=|B_{i-1}P_iP_{i-1}| \leqq \frac{8M_1|\lambda^2|(\frac{1}{3})^i}{(p^2+1)^{2-\varepsilon}} \leqq 2|P_{i-1}|. \tag{11}$$

We write equation (5) as

$$B_i=\frac{\lambda^2}{3}\int[(P_i-P_{i-1})(P_i-P_{i-1})(P_i-P_{i-1})-3P_iP_{i-1}(P_i-P_{i-1})], \tag{12}$$

and we find

$$|B_i| \leqq 8M_1|\lambda^2|\frac{28|\lambda^2|}{3}\left(\frac{1}{3}\right)^i \int \frac{d^3k\,d^3k'}{(k^2+1)^{2-\varepsilon}((k+k')^2+1)((p+k')^2+1)}. \tag{13}$$

The integral in (13) is known to be bounded by a constant M_2. Choosing

$\rho_1^2 < \mathrm{Inf}\left(\frac{1}{2M_1}, \frac{1}{4.28\,M_2}\right)$ proves the lemma.

From Lemma 1, the infinite sum in (6) is absolutely convergent for $|\lambda| < \rho_1$. Furthermore:

$$P_i(p,\lambda) \xrightarrow[i\to\infty]{} P_\infty(p,\lambda). \tag{14}$$

For $|\lambda| < \rho_1, P_\infty(p,\lambda)$ is analytic in λ, real positive for λ real, and satisfies:

$$\frac{1}{1+M_1|\lambda^2|}P_0(p) \leqq |P_\infty(p,\lambda)| \leqq \frac{1}{1-M_1|\lambda^2|}P_0(p). \tag{15}$$

Let G be a graph without blob ($\mathscr{S}(G) = G$). We call $\Gamma_r(G)$ the set of graphs G' such that $\mathscr{S}^r(G') = G, r$ being the smallest such integer. Then we have from (6):

$$\sum_{G'\in\Gamma_r(G)} A^0_{G'} = A^r_G - A^{r-1}_G, \quad \forall r \geqq 1 \tag{16}$$

and from (14) and (16), with $\Gamma(G) = \bigcup\limits_{r=0}^{\infty} \Gamma_r(G)$:

$$\sum_{G'\in\Gamma(G)} A^0_{G'}(p,\lambda) = A^\infty_G(p,\lambda), \tag{17}$$

where A^∞_G is defined by replacing each propagator in G by P_∞. Again the sum (17) is absolutely convergent.

Remark. The same work can be done with the modulus of the amplitudes. Definitions (4) and (5) can be replaced by:

$$Q_i = \frac{1}{p^2 + 1 + \sum\limits_{j=0}^{i-1} C_j(p,\lambda)}, \tag{18}$$

$$C_0 = |B_0|, \quad C_i = \frac{|\lambda^2|}{3}\int [Q_iQ_iQ_i - Q_{i-1}Q_{i-1}Q_{i-1}], \quad \forall i \geqq 1. \tag{19}$$

Inequality (7) is also valid for the C_i's and we find similarly:

$$\sum_{G'\in\Gamma(G)} |A^0_{G'}(p,\lambda)| = F^\infty_G(p,\lambda), \tag{20}$$

where F^∞_G is defined by using a factor $|\lambda|$ for each vertex, and a propagator Q_∞ for each line of G, which satisfies for $|\lambda| < \rho_1$:

$$P_0(p) \leqq Q_\infty(p,\lambda) \leqq \frac{1}{1-M_1|\lambda^2|}P_0(p). \tag{21}$$

Lemma 2. *There exists a positive* K, *depending only on the external momenta* $p_1, \ldots, p_N$, *such that for any graph* G *without blob and any negative value of* λ, *with* $-\rho_1 < \lambda < 0$, *we have*:

$$A^0_G(p,\lambda) > (-\lambda K)^{n(G)}. \tag{22}$$

Proof. The proof follows the same lines as in [5]. In the Schwinger parametric representation (with the notations of [9]):

$$A_G^0 = (-\lambda)^n \int_0^\infty \pi d\alpha U^{-3/2} e^{-\Sigma\alpha - V/U} > (-\lambda)^n \int_0^1 \pi d\alpha U^{-3/2} e^{-\Sigma\alpha - V/U}. \tag{23}$$

U (respectively V) is a sum of monomials in the α's corresponding to the one-trees (respectively two-trees) of G. The number of one-trees is bounded by $2^{\ell(G)}$ where the number $\ell(G)$ of internal lines of G is:

$$\ell(G) = 2n(G) - \frac{N(G)}{2}. \tag{24}$$

Any two-tree is obtained by removing a line from some one-tree. Therefore it is easy to see that for $0 \leqq \alpha_i \leqq 1, \forall i$, we have $U^{-3/2} > 2^{-3/2\,\ell}$ and $\frac{V}{U} > (n-1)M^2$, where M^2 is a bound on the invariants built from the external momenta. Lemma 2 follows, with $K = \frac{1}{8}\exp(-2 - M^2)$.

Theorem. *The expansion* (1), *as a power series in* λ, *has a vanishing radius of convergence.*

Proof. If the expansion (1) has a non-vanishing radius of convergence ρ_2, then there exists a positive constant M_3 such that, for $|\lambda| \leqq \frac{1}{2}\rho_2$, the partial sums of the expansion are uniformly bounded by M_3, that is:

$$\forall q \geqq 1, \quad \forall |\lambda| \leqq \tfrac{1}{2}\rho_2, \quad |S_{N_q}| = \Bigg| \sum_{\substack{G \\ N(G)=N \\ n(G)\leqq q}} A_G^0 \Bigg| \leqq M_3. \tag{25}$$

However for λ real negative, the A_G^∞'s are real positive. From (17) we have for $-\operatorname{Inf}(\rho_1, \frac{1}{2}\rho_2) \leqq \lambda \leqq 0$:

$$|S_{N_q}| = \Bigg| \sum_{\substack{G \\ \mathscr{S}(G)=G \\ n(G)\leqq q}} A_G^\infty - \sum_{\substack{G' \\ G'\in\Gamma(G) \\ n(G)\leqq q \\ n(G')>q}} A_{G'}^0 \Bigg| \geqq \sum_{\substack{G \\ \mathscr{S}(G)=G \\ n(G)\leqq q}} A_G^\infty - \sum_{\substack{G' \\ G'\in\Gamma(G) \\ n(G)\leqq q \\ G'\neq G}} |A_{G'}^0|. \tag{26}$$

By (15) we have:

$$A_G^\infty \geqq \left(\frac{1}{1+M_1\lambda^2}\right)^{\ell(G)} A_G^0, \tag{27}$$

and by (20) and (21):

$$\sum_{\substack{G'\in\Gamma(G) \\ G'\neq G}} |A_{G'}^0| \leqq \left[\left(\frac{1}{1-M_1\lambda^2}\right)^{\ell(G)} - 1\right] A_G^0. \tag{28}$$

Since by (24) $\ell(G) \leqq 2n(G)$, we find:

$$|S_{N_q}| \geqq \sum_{\substack{G \\ \mathscr{S}(G)=G \\ n(G)\leqq q}} A_G^0 \left\{\left(\frac{1}{1+M_1\lambda^2}\right)^{2n(G)} - \left[\left(\frac{1}{1-M_1\lambda^2}\right)^{2n(G)} - 1\right]\right\} \tag{29}$$

It is easy to verify that there exists a constant $M_4, 0 < M_4 < \mathrm{Inf}(\rho_1, \frac{1}{2}\rho_2)$ such that, for $\lambda = -M_4 q^{-1/2}$, we have

$$\left(\frac{1}{1+M_1\lambda^2}\right)^{2n(G)} - \left[\left(\frac{1}{1-M_1\lambda^2}\right)^{2n(G)} - 1\right] \geqq \frac{1}{2}, \quad \forall n(G) \leqq q. \tag{30}$$

Now by an easy adaptation of the argument in [5] there are at least $\left(n-\frac{N}{2}\right)!$ graphs with n vertices and $\mathscr{S}(G) = G$. Therefore we get, using Lemma 2:

$$|S_{N_q}| \geqq \sum_{\substack{G \\ \mathscr{S}(G)=G \\ n(G)\leqq q}} \frac{A_G^0}{2} \geqq \sum_{\substack{G \\ \mathscr{S}(G)=G \\ n(G)=q}} \frac{A_G^0}{2} \geqq \frac{\left(q-\frac{N}{2}\right)!(KM_4)^q}{2q^{9/2}} \tag{31}$$

which obviously contradicts (25) for q large enough.

References

1. Dyson, F.: Phys. Rev. **85**, 631 (1952)
2. Reed, M., Simon, B.: Methods of modern mathematical physics, Vol. IV, Chapter XII, New York: Academic Press 1978
3. Hepp, K.: Théorie de la renormalisation, p. 194, Berlin: Springer-Verlag, 1969
4. Simon, B.: Large Orders and summability of eigenvalue perturbation theory. A mathematical overview, Proc. of the Sanibel Workshop (1981)
5. Jaffe, A.: Commun. Math. Phys. **1**, 127 (1965)
6. Glimm, J., Jaffe, A.: Fortsch. Phys. **21**, 327 (1973)
7. Magnen, J., Sénéor, R.: Commun. Math. Phys. **56**, 237 (1977)
8. Rivasseau, V., Speer, E.: Commun. Math. Phys. **72**, 293 (1980)
9. de Calan, C., Rivasseau, V.: Local existence of the Borel transform in euclidean Φ_4^4, Commun. Math. Phys. **82**, 69–100 (1981)
10. 't Hooft, G.: Lecture given at the Ettore Majorana School, Erice, Sicily (1977)
11. Zimmermann, W.: Commun. Math. Phys. **11**, 1 (1968) and **15**, 208 (1969)

Communicated by A. Jaffe

Received August 3, 1981