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A Remarkable Connection Between the Representations of the Lie Superalgebras osp(1, 2n) and the Lie Algebras o(2n + 1)

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Abstract. We show that there is a one-to-one correspondence between the graded representations of $osp(1, 2n)$ and the non-spinorial representations of $o(2n+1)$. The Clebsch–Gordan series for $osp(1, 2n)$ reduce to the corresponding series for $o(2n+1)$ and the properly defined Casimir operators of order at least up to four have the same eigenvalues.

1. Introduction

In the present work we would like to draw the reader's attention to a fact which to us came somewhat as a surprise: there exists a rather close connection between the representations of the Lie superalgebra $osp(1, 2n)$ and those of the orthogonal Lie algebra $o(2n+1)$. More precisely, we show that there is a one-to-one correspondence between the graded representations of $osp(1, 2n)$ and the non-spinorial representations of $o(2n+1)$. The Clebsch–Gordan series for $osp(1, 2n)$ reduce to the corresponding series for $o(2n+1)$ and, most remarkably, the properly defined quadratic and quartic Casimir operators have the same eigenvalues. We conjecture that the latter also holds for the higher order Casimir operators. To appreciate these observations, recall that the Lie algebra contained in $osp(1, 2n)$ is isomorphic to the symplectic Lie algebra $sp(2n)$ and that $osp(1, 2n)$ and $o(2n+1)$ even have different dimensions. [For a detailed exposition of the theory of Lie superalgebras see [1] and [2].]

Let us describe some of the background which finally led to the conjecture that a relationship of this type might exist. Recall that the algebras $osp(1, 2n)$ [a subfamily of the orthosymplectic Lie superalgebras $osp(m, 2n)$] play a special role among the simple Lie superalgebras. They were among the first algebras to be discovered when the classification problem for simple Lie superalgebras was tackled

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[3] and soon turned out to have many properties in common with simple Lie algebras. For example, they are the only simple Lie superalgebras which have all their (graded finite-dimensional) representations completely reducible [4]. Moreover, a representation theory for these algebras has been developed which is completely analogous to that for semi-simple Lie algebras [5,6], see also [1,7]. The root system of osp$(1, 2n)$ is the unique non-reduced irreducible root system of rank n (i.e., is of the type BC_n) and it has been speculated that this fact might be the deeper reason for the similarities mentioned above.

The smallest among the algebras osp$(1, 2n)$ is the so-called di-spin algebra osp$(1, 2)$. Its Lie algebra is sp(2) and the corresponding representation of sp(2) in the odd subspace is just the elementary two-dimensional representation. This algebra has always been a "playground" for making educated guesses about the properties of simple Lie superalgebras and, in particular, of the algebras osp$(1, 2n)$. The representations of this algebra are easy to construct and are fully understood [3,8,9]. The irreducible representations are characterized by a sole number q which takes the values $0, 1/2, 1, 3/2, \ldots$. When restricted to sp$(2) \simeq$ sl(2), the q representation with $q \geqq 1/2$ splits into two sp(2)-multiplets corresponding to spin q and spin $q - 1/2$, respectively, and thus has dimension $4q + 1$. (Of course, the 0 representation is the trivial one-dimensional representation.) More precisely, the weights of the q representation are just the numbers $\pm q$, $\pm (q - 1/2)$, $\pm (q - 1), \ldots, 0$, and all these weights have multiplicity one. All this is very similar to the spin $2q$ representation of sl(2); the dimensions coincide, and apart from a rescaling by a factor of two, the weights and their multiplicities agree as well. But even more is true: the eigenvalue of the (suitably normalized) quadratic Casimir operator in the q representation of osp$(1, 2)$ is equal to $2q(2q + 1) \ldots$!

Once these well-known facts are considered to be sufficient evidence for a general relationship between the representations of the Lie superalgebra osp$(1, 2n)$ and those of some (semi-) simple Lie algebra, one has to find the candidate for the latter [recall that sl$(2) \simeq$ sp$(2) \simeq$ o(3)]. This is easily done: the weight system of the adjoint representation of osp$(1, 2n)$ (i.e., the root system of osp$(1, 2n)$ enlarged by the zero weight with multiplicity n) is just the weight system of the representation of o$(2n + 1)$ on the traceless symmetric tensors of rank two. Note that this coincidence is still another hint in favour of our conjecture.

We are thus led to compare the representations of the algebras osp$(1, 2n)$ and o$(2n + 1)$ with respect to their dimensions and weights. In view of the existing literature this is an easy task: we simply compare the character formulae and obtain a one-to-one correspondence between the representations of osp$(1, 2n)$ and the non-spinorial representations of o$(2n + 1)$ respecting the weight systems as well as the multiplicities of the weights. All this will be discussed in Sect. 2. An application of this result to the Clebsch–Gordan series for the tensor products of representations is also included.

There remains the problem of whether the correspondence extends to the eigenvalues of the Casimir operators. For the quadratic Casimir operators this is evident from the known formulae. We have shown that the agreement holds for the quartic Casimir operators as well. This will be the subject of Sect. 3.

Our paper closes with a short discussion in Sect. 4.

2. The Weights of the Representations of the Algebras osp(1, 2n) and o(2n + 1)

The base field for all algebras and representations will be any algebraically closed field of characteristic zero. All representations are supposed to be finite-dimensional and, in the case of Lie superalgebras, also graded.

Our discussion will be completely in terms of the roots and weights of the algebras and their representations (with respect to some Cartan subalgebra). As is well-known, we may therefore restrict our attention to the corresponding *real* root and weight systems.

Thus let V be an n-dimensional Euclidean vector space and let $(|)$ denote its (positive definite) scalar product. We choose an orthonormal basis $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ of V and construct the roots and weights of the algebras under consideration.

2.1. *Roots and Weights for* osp(1, 2n)

The root system Δ of the algebra osp(1, 2n) falls into the two subsets of even and odd roots. The set Δ_0 of even roots consists of the vectors

$$\begin{aligned} &\pm\varepsilon_i \pm \varepsilon_j; \quad 1 \leqq i < j \leqq n, \\ &\pm 2\varepsilon_i; \quad 1 \leqq i \leqq n. \end{aligned} \tag{2.1}$$

The set Δ_1 of odd roots has the elements

$$\pm\varepsilon_i; \quad 1 \leqq i \leqq n. \tag{2.2}$$

Note that Δ_0 is the root system of the Lie algebra sp(2n) and that Δ_1 is the weight system of the 2n-dimensional elementary representation of sp(2n).

We choose a basis $\alpha_1, \alpha_2, \ldots, \alpha_n$ of Δ by defining

$$\begin{aligned} &\alpha_i = \varepsilon_i - \varepsilon_{i+1}; \quad 1 \leqq i \leqq n-1, \\ &\alpha_n = \varepsilon_n. \end{aligned} \tag{2.3}$$

Then the positive even roots are

$$\begin{aligned} &\varepsilon_i \pm \varepsilon_j; \quad 1 \leqq i < j \leqq n, \\ &2\varepsilon_i; \quad 1 \leqq i \leqq n, \end{aligned} \tag{2.4}$$

and the positive odd roots are

$$\varepsilon_i; \quad 1 \leqq i \leqq n. \tag{2.5}$$

The corresponding fundamental weights $\omega_1, \omega_2, \ldots, \omega_n$ are defined through the equation

$$\frac{2(\omega_i|\alpha_j)}{(\alpha_j|\alpha_j)} = \delta_{i,j}; \quad 1 \leqq i, j \leqq n. \tag{2.6}$$

They are easily found to be

$$\begin{aligned} &\omega_i = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_i; \quad 1 \leqq i \leqq n-1, \\ &\omega_n = \tfrac{1}{2}(\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_n). \end{aligned} \tag{2.7}$$

Note that $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}, 2\alpha_n$ is a basis of Δ_0 and that $\omega_1, \omega_2, \ldots, \omega_{n-1}, 2\omega_n$ is the corresponding system of fundamental weights for sp(2n).

Let ρ_0 and ρ_1 denote half the sum of the positive even, respectively odd, roots. It is well-known that ρ_0 is also the sum of the fundamental weights for sp(2n). Thus we have

$$\rho_0 = \omega_1 + \omega_2 + \ldots + \omega_{n-1} + 2\omega_n, \qquad \rho_1 = \omega_n. \tag{2.8}$$

We set

$$\rho = \rho_0 - \rho_1 \tag{2.9}$$

and obtain

$$\rho = \omega_1 + \omega_2 + \ldots + \omega_n. \tag{2.10}$$

It is known [1, 5, 6] that the irreducible (finite-dimensional) representations of osp(1, 2n) are uniquely fixed by their highest weight, the latter taking the form

$$\Lambda = p_1\omega_1 + p_2\omega_2 + \ldots + p_n\omega_n, \tag{2.11}$$

with integers $p_i \geqq 0$ and p_n *even*. We remark that the last condition simply means that Λ is the highest weight of a finite-dimensional irreducible representation of sp(2n), a condition which is obviously necessary.

Now let P be the weight lattice of sp(2n). By definition, P is the additive subgroup of V generated by the weights $\omega_1, \omega_2, \ldots, \omega_{n-1}, 2\omega_n$, hence also the additive subgroup of V generated by $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. The elements of P are the possible weights of the finite-dimensional representations of sp(2n). In particular, all the weights of a finite-dimensional representation of osp(1, 2n) are in P.

Let $(e^\lambda)_{\lambda \in P}$ be the canonical basis of the group algebra of P. Suppose we are given a finite-dimensional representation g of osp(1, 2n). For every $\lambda \in P$, let $m_g(\lambda)$ denote the multiplicity of the weight λ in g. Then the *character* of g is defined to be

$$ch_g = \sum_{\lambda \in P} m_g(\lambda) e^\lambda. \tag{2.12}$$

The crucial point is now that the character can be calculated in the following manner. Let $\mathscr{W}$ be the Weyl group of osp(1, 2n), i.e., the Weyl group of sp(2n). By definition, $\mathscr{W}$ is the (finite) group of orthogonal transformations in V generated by the reflections on the hyperplanes orthogonal to the roots of sp(2n). Now let Λ be the highest weight of an irreducible representation of osp(1, 2n) and let ch_Λ denote its character. Then we have [5, 7]

$$ch_\Lambda = \left(\sum_{w \in \mathscr{W}} \det(w) e^{w(\Lambda + \rho)} \right) \left(\sum_{w \in \mathscr{W}} \det(w) e^{w\rho} \right)^{-1}, \tag{2.13}$$

with ρ given by Eq. (2.10). The meaning of the formula is that, in the group algebra of P, the nominator on the right-hand side is divisible by the denominator and that the quotient is equal to ch_Λ.

2.2. *Roots and Weights for* o(2n + 1)

The discussion for o(2n + 1) is completely analogous to the one for osp(1, 2n) and can be given in terms of the definitions introduced above. The root system $\tilde{\Delta}$ of $o(2n+1)$ consists of the vectors

$$\begin{aligned} &\pm\varepsilon_i \pm \varepsilon_j; \quad 1 \leqq i < j \leqq n, \\ &\pm\varepsilon_i; \quad 1 \leqq i \leqq n. \end{aligned} \tag{2.14}$$

The vectors α_i, $1 \leqq i \leqq n$ [see Eq. (2.3)] form a basis of $\tilde{\Delta}$, the corresponding fundamental weights of $o(2n+1)$ are just the vectors ω_i, $1 \leqq i \leqq n$, as given in Eq. (2.7), and the half-sum of the positive roots [with respect to the basis (α_i)] is equal to ρ [see Eq. (2.10)]. The weight lattice of $o(2n+1)$ is equal to $P \cup (\omega_n + P)$ and the highest weights of the irreducible representations of $o(2n+1)$ take the form (2.11) with arbitrary integers $p_i \geqq 0$. Recall that the weights of an irreducible representation are either all contained in P or else all contained in $\omega_n + P$; in the former, respectively latter, case the representation is said to be *non-spinorial*, respectively *spinorial.*

The characters are again defined by Eq. (2.12) with P replaced by $P \cup (\omega_n + P)$. Obviously, the Weyl groups of $sp(2n)$ and $o(2n+1)$ coincide. Finally, formula (2.13) is still valid (this is the famous character formula by H. Weyl). Comparing the above results for $osp(1,2n)$ and $o(2n+1)$, we obtain the following theorem.

Theorem. *The highest weights of both the graded irreducible representations of* $osp(1,2n)$ *and the non-spinorial irreducible representations of* $o(2n+1)$ *are exactly the vectors* $\sum_{i=1}^{n} p_i\omega_i$ *with integers* $p_i \geqq 0$ *and* p_n *even. If a graded irreducible representation of* $osp(1,2n)$ *and a non-spinorial irreducible representation of* $o(2n+1)$ *have the same highest weight, then the multiplicity of any weight is the same for both representations, in particular, the dimensions of both representations coincide.*

Let us give a simple application of the theorem. For any weight Λ of the type (2.11) (with $p_i \geqq 0$ and p_n even) let M_Λ (respectively M'_Λ) denote the irreducible $osp(1,2n)$-module [respectively $o(2n+1)$-module] with highest weight Λ. Suppose we are given three such weights $\Lambda_1, \Lambda_2, \Lambda$. Then the multiplicity of M_Λ in $M_{\Lambda_1} \otimes M_{\Lambda_2}$ is the same as the multiplicity of M'_Λ in $M'_{\Lambda_1} \otimes M'_{\Lambda_2}$, i.e., the (generalized) Clebsch–Gordan series for $M_{\Lambda_1} \otimes M_{\Lambda_2}$ and $M'_{\Lambda_1} \otimes M'_{\Lambda_2}$ "coincide." In fact, since the modules $M_{\Lambda_1} \otimes M_{\Lambda_2}$ and $M'_{\Lambda_1} \otimes M'_{\Lambda_2}$ are completely reducible [4], their Clebsch–Gordan series can be obtained by counting the multiplicities of their weights.

3. Eigenvalues of the Casimir Operators

Let L be a Lie superalgebra and let $U(L)$ denote its enveloping algebra. Any even element C of $U(L)$ which commutes with all elements of L [hence with all elements of $U(L)$] will be called a (generalized) *Casimir element of* L. There exist various standard procedures to construct such elements [see, for example, [2, 10 and 11]].

Now let g be a representation of L. For simplicity, the canonical extension of g to $U(L)$ will also be denoted by g. Then the *Casimir operator* $g(C)$ commutes with $g(X)$, for all $X \in L$. Suppose now that g is irreducible. According to Schur's lemma $g(C)$ is then a scalar multiple of the identity. It is these scalar factors that we are interested in.

3.1. The Quadratic Casimir Operators

Suppose that the Killing form ϕ of L is non-degenerate. Then the standard procedure to construct a quadratic Casimir element goes as follows. Choose a basis $(E_i)_{1 \leqq i \leqq s}$ of L whose elements are homogeneous (i.e., even or odd). Define a second basis $(F_j)_{1 \leqq j \leqq s}$ of L through the condition

$$\phi(F_j, E_i) = \delta_{i,j}; \quad 1 \leqq i, j \leqq s. \tag{3.1}$$

Then

$$C_K = \sum_{i=1}^{s} E_i F_i \tag{3.2}$$

is an even Casimir element, which does not depend on the choice of the basis (E_i).

In the following we shall restrict our attention to the cases where L is equal to osp$(1, 2n)$ or o$(2n+1)$. Note that the construction above applies to both of these algebras; we denote the corresponding Casimir elements by C_K and C'_K, respectively. Let g be the irreducible representation of L with highest weight Λ. Then the eigenvalue $C_K(\Lambda)$ of $g(C_K)$ [respectively $C'_K(\Lambda)$ of $g(C'_K)$] is equal to $(\Lambda|\Lambda + 2\rho)_K$, where ρ is the vector specified in Eq. (2.10) and $(|)_K$ is the scalar product on V induced from the Killing form of osp$(1, 2n)$ [respectively o$(2n+1)$] [1]. It is well-known that $(|)_K$ is proportional to $(|)$; we obtain

$$C_K(\Lambda) = \frac{1}{4n+2}(\Lambda|\Lambda + 2\rho), \tag{3.3}$$

$$C'_K(\Lambda) = \frac{1}{4n-2}(\Lambda|\Lambda + 2\rho). \tag{3.3$'$}$$

Apart from normalization these expressions coincide for all Λ.

3.2. The Quartic Casimir Operators

For the construction of higher order Casimir elements we prefer to use a different procedure [10]. The following exposition is taken from a forthcoming paper by one of the authors [11]. We use the terminology introduced in [2] and consider all orthosymplectic Lie superalgebras (including the orthogonal and symplectic Lie algebras) simultaneously.

Let $W = W_{\bar{0}} \oplus W_{\bar{1}}$ be a Z_2-graded vector space with

$$\dim W_{\bar{0}} = m, \quad \dim W_{\bar{1}} = 2n, \tag{3.4}$$

where m, n are non-negative integers. Choose an even supersymmetric non-degenerate bilinear form b on W. The Lie superalgebra of all linear mappings of W into itself is denoted by $pl(W)$. Let osp(b) be the (graded) subalgebra of $pl(W)$ consisting of those elements which leave b invariant; the algebra osp(b) is isomorphic to osp$(m, 2n)$.

Define a bilinear map

$$t: W \times W \to \text{osp}(b) \tag{3.5a}$$

through the equation

$$t(x,y)z = b(y,z)x - (-1)^{\xi\eta} b(x,z)y \tag{3.5b}$$

for all $x \in W_\xi, y \in W_\eta, z \in W; \xi, \eta \in Z_2$. The map t is osp(b)-invariant and its image generates osp(b) considered as a vector space. We remark that

$$t(x,y) = -(-1)^{\xi\eta} t(y,x) \quad \text{for all } x \in W_\xi, y \in W_\eta, \tag{3.6}$$

Choose any basis $e_1, e_2, \ldots, e_{m+2n}$ of W consisting of homogeneous elements and let $\eta_i \in Z_2$ be the degree of e_i. Introduce the corresponding dual basis $f_1, f_2, \ldots, f_{m+2n}$ through the equation

$$b(e_i, f_j) = \delta_{i,j}; \quad 1 \leqq i, j \leqq m+2n \tag{3.7}$$

(the element f_j is homogeneous of degree η_j). Now define for any integer $r \geqq 1$

$$C_r = \sum_{j_1 \ldots j_r} \sigma(j_1)\sigma(j_2)\ldots\sigma(j_{r-1}) t(f_{j_r}, e_{j_1})\, t(f_{j_1}, e_{j_2}) \ldots t(f_{j_{r-1}}, e_{j_r}), \tag{3.8a}$$

with

$$\sigma(j) = (-1)^{\eta_j} \quad \text{for } 1 \leqq j \leqq m+2n. \tag{3.8b}$$

Then C_r is an (even) r^{th} order Casimir element of osp(b) and does not depend on the choice of the basis (e_i). Note that in Eq. (3.8a) there is no sign factor $\sigma(j_r)$.

Obviously, we have $C_1 = 0$. More generally, it can be shown [10] that, for any integer $p \geqq 0$, the element C_{2p+1} can be written as a linear combination of products of the form $C_{2q_1} C_{2q_2} \ldots C_{2q_t}$ where the q_i are positive integers with $\sum_t q_t \leqq p$. In particular, we have

$$C_3 = \tfrac{1}{2}(m - 2n - 2)C_2. \tag{3.9}$$

Consequently, it is sufficient to consider the C_r with r even.

After these general remarks we restrict our attention to the algebras osp(1, 2n) and o(2n + 1) and denote the Casimir element (3.8) by C_r and C'_r, respectively. Again let g be the irreducible representation of osp(1, 2n) [respectively o(2n + 1)] with highest weight Λ and let $C_r(\Lambda)$ [respectively $C'_r(\Lambda)$] denote the eigenvalue of $g(C_r)$ [respectively $g(C'_r)$]. We write Λ in the form

$$\Lambda = \sum_{i=1}^{n} m_i \varepsilon_i, \tag{3.10}$$

and set

$$r_i = n + \tfrac{1}{2} - i \quad \text{for } 1 \leqq i \leqq n. \tag{3.11}$$

Using techniques to be described in [11] we obtain

$$C_2(\Lambda) = -2 \sum_{i=1}^{n} ((m_i + r_i)^2 - r_i^2), \tag{3.12}$$

$$C'_2(\Lambda) = 2 \sum_{i=1}^{n} ((m_i + r_i)^2 - r_i^2). \tag{3.12$'$}$$

These results agree with [10], where the eigenvalues of C_2 are given for all orthosymplectic algebras. The formula for $C_2'(\Lambda)$ is also contained in [12]. Actually, the Eqs. (3.12) and (3.12)′ can be derived from subsection 3.1, for it is not difficult to see that

$$C_2 = -(8n+4)C_K, \tag{3.13}$$

$$C_2' = (8n-4)C_K'. \tag{3.13$'$}$$

Thus the case $r=4$ is the first which is really new. We find

$$C_4(\Lambda) - (n+\tfrac{1}{2})C_2(\Lambda) = -2\sum_{i=1}^{n} ((m_i+r_i)^4 - r_i^4), \tag{3.14}$$

which is to be compared with the formula taken from [12]

$$C_4'(\Lambda) + (n-\tfrac{1}{2})C_2'(\Lambda) = 2\sum_{i=1}^{n} ((m_i+r_i)^4 - r_i^4). \tag{3.14$'$}$$

Visibly, there is no normalization of C_4 and C_4' which would make their eigenvalues coincide for all Λ. However, we recall that the quartic Casimir elements are not uniquely (up to normalization) defined anyhow. Quite generally, if two r^{th} order Casimir elements differ only by a linear combination of (strictly) lower order Casimir elements, none of the two is *a priori* "better" than the other. (Of course, there might exist special additional conventions to make the choice unique.) Consequently, the Eqs. (3.14) and (3.14)′ show that the fourth order Casimir elements $-C_4 + (n+\frac{1}{2})C_2$ and $C_4' + (n-\frac{1}{2})C_2'$ of osp(1, 2n) and o(2n + 1) respectively, have the same eigenvalues in corresponding irreducible representations. We conjecture that an analogous result holds for the higher order Casimir elements as well.

4. Discussion

In the preceding sections we have established a one-to-one connection between the graded representations of the Lie superalgebra osp(1, 2n) and the non-spinorial representations of the Lie algebra o(2n + 1). Corresponding representations have the same weights with the same multiplicities. Consequently, the Clebsch–Gordan series are also the same in both cases. Quite remarkably, the correspondence extends even to the eigenvalues of the Casimir operators in corresponding irreducible representations, at least in second and fourth order.

At present we do not know of a deeper reason why all this should happen. What is missing is some link between the algebras osp(1, 2n) and o(2n + 1) which would enable us to predict *a priori* that our results should hold and which, in particular, would yield the correspondence between the eigenvalues of the Casimir operators without having to calculate them explicitly.

We close with the remark that osp(1, 4) is among the algebras to which our correspondence applies. As is well-known, this algebra is via contraction related to the algebra of ordinary supersymmetry. It would be interesting to see how much of the above correspondence shows up in physical applications.

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Note added in proof. The general correspondence between the Casimir elements of $\mathrm{osp}(1, 2n)$ and $\mathrm{o}(2n+1)$, as conjectured in subsection 3.2, can be established as follows. Let L denote one of the algebras $\mathrm{osp}(1, 2n)$ or $\mathrm{o}(2n+1)$ and let $\mathfrak{h}$ be a Cartan subalgebra of L, furthermore, let $U(L)$ be the enveloping algebra of L and let $S(\mathfrak{h})$ be the symmetric algebra of the vector space $\mathfrak{h}$. Then there exists an isomorphism (the so-called Harish-Chandra isomorphism) of the center $Z(L)$ of $U(L)$ onto the subalgebra $S(\mathfrak{h})^W$ of $S(\mathfrak{h})$ consisting of the elements which are invariant under the Weyl group. This is classical for $\mathrm{o}(2n+1)$ and follows from [7] for $\mathrm{osp}(1, 2n)$. We have seen in Sect. 2 that the Cartan subalgebras of $\mathrm{osp}(1, 2n)$ and $\mathrm{o}(2n+1)$ can be identified such that both the Weyl groups and the vectors ρ coincide. This implies that the above isomorphisms yield an isomorphism of the algebra $Z(\mathrm{osp}(1, 2n))$ onto the algebra $Z(\mathrm{o}(2n+1))$ such that corresponding elements have the same eigenvalue under irreducible representations with the same highest weight. We are grateful to V. G. Kac for a comment on this point.

Regrettably, the above correspondence between the two centers is far from explicit. In the meantime, we have extended the results of Sect. 3 up to sixth order (see [11]).