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The Spectral Class of the Quantum-Mechanical Harmonic Oscillator

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Abstract. The purpose of this paper is to study the so-called *spectral class* **Q** of anharmonic oscillators $Q = -D^2 + q$ having the same spectrum $\lambda_n = 2n$ $(n \geqq 0)$ as the harmonic oscillator $Q^0 = -D^2 + x^2 - 1$. The *norming constants* $t_n = \lim_{x\uparrow\infty} \ell g\,[(-1)^n e_n(x)/e_n(-x)]$ of the eigenfunctions of Q form a complete set of coordinates in **Q** in terms of which the potential may be expressed as $q = x^2 - 1 - 2D^2\,\ell g\,\theta$ with

$$\theta = \det\left[\delta_{ij} + (e^{t_i} - 1)\int_x^\infty e_i^0 e_j^0 : 0 \leqq i, j, < \infty\right],$$

e_n^0 being the n^{th} eigenfunction Q^0. The spectrum and norming constants are canonically conjugate relative to the bracket $[F, G] = \int \nabla F D \nabla G dx$, *to wit*: $[\lambda_i, \lambda_j] = 0$, $[t_i, 2\lambda_j] = 1$ or 0 according to whether $i = j$ or not, and $[t_i, t_j] = 0$. This prompts an investigation of the symplectic geometry of **Q**. The function θ is related to the theta function of a singular algebraic curve. Numerical results are also presented.

1. Introduction

The spectrum of the quantum-mechanical harmonic oscillator[1] $Q^0 = -D^2 + x^2 - 1$ is $0, 2, 4, 6$, etc. The corresponding unit eigenfunctions are the Hermite functions:

$$e_n^0(x) = (\sqrt{\pi}2^n n!)^{-1/2} e^{x^2/2} D^n e^{-x^2} \qquad (n \geqq 0).$$

Let Δq belong to the class **S(R)** of real infinitely differentiable functions vanishing rapidly at $\pm\infty$[2]. The anharmonic oscillator $Q = -D^2 + q$ with potential $q = x^2 - 1 + \Delta q$ has a discrete spectrum of simple eigenvalues $\lambda_n = \lambda_n[q]$, increasing to $+\infty$ with n, and corresponding unit eigenfunctions e_n $(n \geqq 0)$ of class **S**. The

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1 D signifies differentiation with regard to x.

2 $x^i D^j \Delta q = o(1)$ for $x \to \pm\infty$ and every $i, j \geqq 0$.

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purpose of this paper is to study the *spectral class* $\mathbf{Q} = \mathbf{Q}[x^2 - 1]$ of such oscillators having the same spectrum $\lambda_n = \lambda_n^0 = 2n$ as Q^0, i.e., the aim is to explain *to what extent the quantum-mechanical oscillator is specified by its spectrum.* The principal results and their geometrical motivation will now be described.

Isospectral Flows

The flow of translation $\partial q/\partial t = \partial q/\partial x$ leads immediately out of the class $x^2 - 1 + \mathbf{S}$; more drastically, the KDV flow $\partial q/\partial t = 3q\,\partial q/\partial x - (1/2)\,\partial^3 q/\partial x^3$ does not even exist for $q \sim x^2$ as the individual terms cannot balance unless q is sublinear. Fortunately, a wide class of isospectral flows suggests itself by elementary geometrical reasoning: $\mathbf{Q}$ is defined by the relations $\lambda_n = 2n (n \geqq 0)$, so the *normal space* to $\mathbf{Q}$ at q is, or ought to be, the span of the gradients[3] $\nabla\lambda_n = e_n^2 (n \geqq 0)$. Now De_i^2 is perpendicular to e_j^2 for every i and j, as is plain for $i = j$, while for $i \neq j$, it follows from[4]

$$D[e_i, e_j] = e_i e_j'' - e_i'' e_j = (\lambda_i - \lambda_j) e_i e_j$$

and

$$\begin{aligned}\int e_i^2 De_j^2 &= -\int e_j^2 De_i^2 = \tfrac{1}{2}\int [e_i^2 De_j^2 - e_j^2 De_i^2] \\ &= \int e_i e_j [e_i, e_j] = (\lambda_i - \lambda_j)^{-1} \tfrac{1}{2} [e_i, e_j]^2 \big|_{-\infty}^{\infty} = 0.\end{aligned}$$

This suggests that the *tangent space* to $\mathbf{Q}$ at q ought to be the span of $De_n^2 (n \geqq 0)$ and that[5] $\partial q/\partial t = 2De_n^2 = \mathbf{X}_n q$ ought to be an isospectral flow. The perpendicularity of e_i^2 and De_j^2 can be phrased more elegantly in terms of the Poisson bracket,[6] $[F, G] = \int \nabla F D \nabla G\, dx$, as $[\lambda_i, \lambda_j] = 0$. This states that the eigenvalues are *involutive* and suggests that the flows $\partial q/\partial t = \mathbf{X} q$ ought to *commute*. It is a source of satisfaction that e_i^2 and De_j^2 span $\mathbf{L}^2(\mathbf{R})$ so that *no direction at q is left unclassified*; also, the gradients $\nabla\lambda_n = e_n^2$ of the relations $\lambda_n = 2n$ are highly independent in the sense that no direction in the span of some subclass lies in the span of the complementary class [see Sect. 4]. This indicates that $\mathbf{Q}$ *is a smooth submanifold of the ambient space* $x^2 - 1 + \mathbf{S}$, though the point is not pursued below.

The Exponential Map

It is a pleasant fact that the flows $\partial q/\partial t = 2De_n^2$ may be integrated in a simple and explicit manner, obviating any discussion of existence and the like. Let $\mathbf{X}_n$ be the vector field $q \to 2De_n^2$ and fix q^0 in $\mathbf{Q}[x^2 - 1]$, not necessarily at $q^0(x) = x^2 - 1$. Then the flow $\partial q/\partial t = \mathbf{X}_n q$ originating at $q = q^0$ is expressed by

$$q = q^0 - 2D^2 \ell g\, \theta,$$

3 ∇ is the gradient in function space. The evaluation is elementary from the simplicity of the eigenvalue, the variational equation $Qe_n^{\cdot} + q^{\cdot} e_n = \lambda_n^{\cdot} e_n + \lambda_n e_n^{\cdot}$, and the fact that $\int e_n e_n^{\cdot} = 0$ in view of $\int e_n^2 = 1$.

4 $[e_i, e_j] \equiv e_i e_j' - e_i' e_j$.

5 The factor 2 is introduced with a view to the simplicity of subsequent formulas.

6 $[F, G]$ is skew-symmetric and satisfies Jacobi's identity; see, for example, McKean–Moerbeke [1975].

with

$$\theta = 1 + (e^t - 1)\int_x^\infty (e_n^0)^2,$$

e_n^0 being the initial eigenfunction $[Q^0 e_n^0 = \lambda_n e_n^0]$. More generally, let the numbers $t_n (n \geqq 0)$ vanish rapidly as $n \uparrow \infty$ and let $\mathbf{X} = \Sigma t_n \mathbf{X}_n$. Then $e^{\mathbf{X}} q^0$ belongs to $\mathbf{Q}$ and may be expressed by the same formula with[7]

$$\theta = \det\left[\delta_{ij} + (e^{t_i} - 1)\int_x^\infty e_i^0 e_j^0 : 0 \leqq i, j < \infty\right].$$

The numbers t_n have the interpretation of *norming constants*: $c_n^- = e_n/e_n^0(-\infty)$ and $c_n^+ = e_n/e_n^0(+\infty)$ exist and satisfy the *connection rule* $c_n^- c_n^+ = 1$, and $t_n = \ell g\, c_n^+/c_n^-$; in particular, if $q^0(x) = x^2 - 1$, as is mostly the preferred choice, then the map $t \to -t$ corresponds to the reflection of potentials $q(x) \to q(-x)$. The geometrical content of all this is that the formula $e^{\mathbf{X}} q^0 = q^0 - 2D^2\, \ell g\, \theta$ establishes an *exponential map* from the tangent space of $\mathbf{Q}$ at q^0 *into* $\mathbf{Q}$: it is 1:1 in view of the meaning of the parameters t_n, and in fact it is *onto* so that t is a *global coordinate* on $\mathbf{Q}$, relative, e.g., to the *origin* $q^0(x) = x^2 - 1$; in particular, $x^2 - 1$ *is the only even potential in its class*. The first fundamental form of $\mathbf{Q}$ at q is $g_{ij} = 4\int De_i^2 De_j^2$, i.e., $\|dq\|_2^2 = \sum g_{ij} dt_i dt_j$. This is a complicated animal even at the origin $q^0 = x^2 - 1$, and no attempt is made to deal with it beyond noting the formula

$$\sum g_{ij}^0 a^i b^j = 4(2\pi)^{-1/2}(1-a)^{1/2}(1-b)^{1/2}(1-ab)^{-1/2}\ (0 \leqq a, b < 1).$$

Symplectic Geometry

The quantities $\lambda_j (j \geqq 0)$ are involutive relative to the bracket $[F, G]$, as noted before. It turns out that so are quantities $t_i = \lim_{x \uparrow \infty} \ell g\,[(-1)^i e_i(x)/e_i(-x)]$, and the fact that the flow $\partial q/\partial t = \mathbf{X}_j q = [q, 2\lambda_j]$ advances t_i at speed 1 if $i = j$, and not at all if $i \neq j$, is to say that $t_i (i \geqq 0)$ and $2\lambda_j (j \geqq 0)$ are *canonically conjugate* relative to the bracket:

$$[t_i, 2\lambda_j] = 1 (i = j) \quad = 0 (i \neq j).$$

A more global viewpoint is now adopted. The involutive quantities t_n have an existence outside $\mathbf{Q}[x^2 - 1]$ and so produce commuting vector fields $\mathbf{Y}_n : q \to D\nabla t_n$ *transversal* to the isospectral fields envisaged before, leading off $\mathbf{Q}[x^2 - 1]$ into the ambient space; they fix the t_i's and move the λ_j's. It turns out that $x^2 - 1 + \mathbf{S}$ is too small an ambient space for the individual flows, but if $\mathbf{S}$ is appropriately enlarged things appear to proceed nicely. The ambient space is now cut up by two transverse foliations: one foliation has leaves $\mathbf{Q}$ defined by fixing $\lambda_n (n \geqq 0)$ not at the eigenvalues $\lambda_n[x^2 - 1] = 2n$, but at some other values with realistic comportment as $n \uparrow \infty$. The typical leaf $\mathbf{P}$ of the second foliation is obtained by fixing the numbers $t_n (n \geqq 0)$ in a similar fashion. Two leaves $\mathbf{Q}$ and $\mathbf{P}$ meet in a single point,

7 The formula is of Gelfand–Levitan-type; see, especially, Kay–Moses [1956] and Tanaka [1972/73].

and the meeting is *transversal* as expressed by the fact that the corresponding normal spaces, spanned by $\nabla\lambda_i$ and ∇t_j, meet only in the null function and fill up $\mathbf{L}^2$. The isospectral flows $\partial q/\partial t = \mathbf{X}_n q$ preserve the leaf $\mathbf{Q}$ and are integrated by the previous rule: $\Delta q = -2D^2 \ell g[1 + (e^t - 1)\int_x^\infty (e_n^0)^2]$; likewise, the transversal flows $\partial q/\partial t = \mathbf{Y}_n q$ preserve the leaf $\mathbf{P}$ and are integrated by a simple rule: $\Delta q = -2D^2 \ell g[e_n^0, f_n^0]$, f_n^0 being any independent eigenfunction of Q^0 with updated eigenvalue $\lambda_n = \lambda_n^0 + t$. The latter flows are of a different kind from the former in that their parameters $\lambda_n (n \geqq 0)$ lie on the ∞-dimensional simplex $\lambda_0 < \lambda_1 < \lambda_2 <$ etc. so that they have only a circumscribed existence; in this connection, it is amusing to note that the leaf $\mathbf{P}[x^2 - 1]$ appears to be precisely the class of even potentials and so is perfectly nice, only it is *incomplete* as regards these flows. This whole global picture is partly conjectural. The technical effort required to confirm it seems disproportionate to the result, so only the leaf $\mathbf{Q}[x^2 - 1]$ is treated below, though the proofs have a wider applicability [see Sect. 4–5]. To be candid, it is not even plain what the ambient space should be: for example, there exist potentials outside the present class $\mathbf{Q}[x^2 - 1]$ with spectrum $\lambda_n = 2n$ but exhibiting a *charge*: $\Delta q(\infty) - \Delta q(-\infty) = 4$. It is conjectured that this charge is always integral in any enlargement of $\mathbf{Q}[x^2 - 1]$ and labels its connected pieces.

Theta Functions

The letter θ is used to point up the similarity between this determinant and the Riemann-theta function as it appears in the inverse theory of Hill's equation; see, for example, McKean–Trubowitz [1976]. A singular 2-sheeted curve of infinite genus, with singular points $\lambda_n = 2n (n \geqq 0)$ or whatever, lies in the background. The associated Jacobi variety splits up into an uncountable number of components indexed by the real number x, provided with the family of singular theta sums:

$$\theta = \theta_x(t_0, t_1, t_2, \ldots) = \det\left[\delta_{ij} + (e^{t_i} - 1)\int_x^\infty e_i^0 e_j^0 : 0 \leqq i, j < \infty\right].$$

The numbers $t_n (n \geqq 0)$ may even be expressed as sums, over the points of a certain divisor on the curve, of integrals of differentials of the first kind [see Sect. 6].

Numerical Results

The appendix contains pictures of several potentials from $\mathbf{Q}[x^2 - 1]$ displaying the effect of the isospectral flows. They were made by O. McBryan by numerical evaluation of θ.

Finite Interval

The whole situation is similar but technically simpler for operators Q acting, for example, on functions of $-1 \leqq x \leqq 1$ vanishing at $x = \pm 1$. This will be dealt with in detail in a forthcoming publication of E. Trubowitz.

2. The Exponential Map

Let $\mathbf{Q} = \mathbf{Q}[x^2 - 1]$ be the spectral class of the quantum-mechanical oscillator $Q^0 = -D^2 + x^2 - 1$ obtained by fixing the eigenvalues $\lambda_n = \lambda_n^0 = 2n(n \geqq 0)$ in the space $x^2 - 1 + \mathbf{S}$. It is proposed to make *an exponential map of the tangent vector*[8] $\mathbf{X} = \Sigma t_i \mathbf{X}_i$ *into* $\mathbf{Q}$ *via the rule* $e^X q^0 = q^0 - 2D^2 \ell g\, \theta = q$ with

$$\theta = \theta_x(t_0, t_1, t_2, \ldots) = \det\left[\delta_{ij} + (e^{t_i} - 1)\int_x^\infty e_i^0 e_j^0 : 0 \leqq i, j < \infty\right].$$

The initial potential q^0 is *any* point of $\mathbf{Q}$; it is specialized to the *origin* $x^2 - 1$ later. The numbers t_j vanish rapidly as $j \uparrow \infty$. They have the meaning of *norming constants*:

$$e_j / e_j^0(\pm\infty) = \exp(\pm t_j/2).$$

This makes plain that *the exponential map is* 1:1; the proof that it is also *onto* is postponed to Sect. 3.

Step 1. The discussion begins with a single parameter $t = t_n$ so that $\theta = 1 + (e^t - 1)\int_x^\infty (e_n^0)^2$. It is to be proved that $q = q^0 - 2D^2 \ell g \theta$ *is an integral curve of the vector field* $\mathbf{X}_n : q \to 2De_n^2$, i.e., $\partial q/\partial t = \mathbf{X}q$. The question of uniqueness is routine and may be left aside.

Proof. θ is positive by inspection, so the recipe makes sense; moreover, $\theta = 1$ at ∞ and e^t at $-\infty$, so $\Delta q = -2D^2 \ell g\, \theta$ is class $\mathbf{S}$. Now the function $\theta^{-1} e^{t/2} e_n^0 = f$ is an eigenfunction of $Q = -D^2 + q$ with eigenvalue $2n$, by direct computation, and as it has precisely n roots in common with e_n^0 and satisfies

$$\int f^2 = e^t \int (e_n^0)^2 \left[1 + (e^t - 1)\int_x^\infty (e_n^0)^2\right]^{-2} dx = e^t(e^t - 1)^{-1}\theta^{-1}\Big|_{-\infty}^{\infty} = 1,$$

it can only be the n^{th} eigenfunction e_n of Q. Besides,

$$\begin{aligned}\partial q/\partial t &= -2D^2 \ell g[\partial\theta/\partial t] = -2D^2\theta^{-1} e^t \int_x^\infty (e_n^0)^2 \\ &= 2D[\theta^{-1} e^{t/2} e_n^0]^2 = 2De_n^2 = \mathbf{X}_n q,\end{aligned}$$

by elementary computation, and since the flow preserves the spectrum of Q in view of $\lambda_m^{\cdot} = 2[\lambda_m, \lambda_n] = 0$, so q belongs to $\mathbf{Q}$ and may be identified as stated.

Amplification 1. *The other eigenfunctions of Q are*

$$e_m = e_m^0 - \theta^{-1}(e^t - 1)e_n^0 \int_x^\infty e_m^0 e_n^0 \qquad (m \neq n).$$

The computation is facilitated by the identity $\int_x^\infty e_m^0 e_n^0 = (\lambda_n^0 - \lambda_m^0)^{-1}[e_m^0, e_n^0]$.

8 $X_j : q \to De_j^2\ (j \geqq 0)$.

Amplification 2. *The preceding formula leads, after some tears, to the identity*

$$\int_x^\infty e_i e_j = \int_x^\infty e_i^0 e_j^0 - \theta^{-1}(e^t - 1)\int_x^\infty e_i^0 e_n^0 \int_x^\infty e_j^0 e_n^0 \qquad (i, j \neq n).$$

This will be used presently.

Step 2 is to prove that

$$q^0 - 2D^2 \ell g\, \theta$$

$$\theta = \det\left[\delta_{ij} + (e^{t_i} - 1)\int_x^\infty e_i^0 e_j^0 : 0 \leqq i, j \leqq n\right]$$

represents the action of $e^{\mathbf{X}}$ *upon* q^0 *for any tame combination* $\mathbf{X} = \sum_{i \leqq n} t_i \mathbf{X}_i$ *of the individual fields.*

Proof. The formula of Step 1 and the commutativity of the individual flows[9] imply that $q_n = e^{\mathbf{X}} q^0$ can be expressed inductively as

$$q_{n-1} - 2D^2 \ell g\, \theta_n,$$

with $q_{-1} = q^0$ and $\theta_n = 1 + (e^{t_n} - 1)\int_x^\infty (e_n^-)^2$, e_n^- being the n^{th} eigenfunction of the preceding potential q_{n-1}[10]. Thus, $\Delta q = q_n - q_{-1}$ is $(-2) \times$ the second logarithmic derivative of the product $\prod_{i \leqq n} \theta_i$, and it is required to identify the latter as the stated determinant. This is done by Gaussian elimination with the help of Amplification 2: elimination of the first row of the determinant leads to the product of θ_0 and

$$\det\left[\delta_{ij} + (e^{t_i} - 1)\left[\int_x^\infty e_i^0 e_j^0 - \theta_0^{-1}(e^{t_0} - 1)\int_x^\infty e_i^0 e_0^0 \int_x^\infty e_j^0 e_0^0\right] : 1 \leqq i, j \leqq n\right]$$

$$= \det\left[\delta_{ij} + (e^{t_i} - 1)\int_x^\infty e_i^+ e_j^+ : 1 \leqq i, j \leqq n\right],$$

in which $e_n^+ (n \geqq 0)$ are the eigenfunctions of $e^{t_0 \mathbf{X}_0} q_{-1} = q_0$. The rule $\theta = \prod_{i \leqq n} \theta_i$ follows by induction.

Amplification 3. *The product rule for* θ *leads to a proof of its existence and non-vanishing for general rapidly vanishing parameters:*

$$\theta = \prod_{n \geqq 0}\left[1 + (e^{t_n} - 1)\int_x^\infty (e_n^-)^2\right]$$

and

$$\sum |e^{t_n} - 1| \int_x^\infty (e_n^-)^2 \leqq \sum |e^{t_n} - 1| < \infty,$$

independently of x.

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9 $[\lambda_i, \lambda_j] = 0$ expresses this fact.

10 The abuse of notation is only momentary.

Amplification 4. *The evaluation* $\Delta q = -2D^2 \ell g\, \theta$ *and the rule* $\partial q/\partial t_j = X_j q = 2De_j^2$ *lead to the identity* $e_j^2 = -D[\partial \ell g\, \theta/\partial t_j]\,(j \geqq 0)$.

Amplification 5. *The formula for the individual flows of Step* 1 *lead inductively to the evaluations*

$$c_k^- = (e_k/e_k^0)(-\infty) = e^{-t_k/2}, \qquad c_k^+ = (e_k/e_k^0)(+\infty) = e^{t_k/2},$$

whence the connection rule $c_k^- c_k^+ = 1$ *and the interpretation of* $t_k = \ell g\, c_k^+/c_k^-$ *as a norming constant. The fact that the (tame) exponential map is* 1 : 1 *is now plain.*

Step 3 is to carry Step 2 over from tame to general rapidly vanishing $t_n (n \geqq 0)$. Routine estimates based upon the expansion[11]

$$\begin{aligned}\theta &= 1 + \sum_{n=1}^{\infty} \mathrm{sp} \wedge^n \left[(e^{t_i} - 1) \int_x^{\infty} e_i^0 e_j^0 \right] \\ &= 1 + \sum_i (e^{t_i} - 1)\Delta[e_i^0] + \sum_{i<j} (e^{t_i} - 1)(e^{t_j} - 1)\Delta[e_i^0, e_j^0] + \text{etc.}\end{aligned}$$

show that $\Delta q = -2D^2 \ell g\, \theta$ is of class **S**. The isospectrality of $q = q^0 + \Delta q$ and its identification as $e^{\mathbf{X}} q^0$ will be plain, the only moot point being the interpretation of the parameters t_n as norming constants, as in Amplification 5. But that is clear from the estimate

$$\sum_{i \neq j} \left[(e^{t_i} - 1) \int_x^{\infty} e_i^0 e_j^0 \right]^2 \leqq \sum_i (e^{t_i} - 1)^2 \int_x^{\infty} (e_i^0)^2 = o(1) \quad (x \uparrow \infty)$$

and Amplification 4[12]:

$$\begin{aligned} e_n^2 &= -D[\partial \ell g\, \theta/\partial t_n] \\ &= -D \sum_{i=0}^{\infty} \left[\delta_{in} + (e^{t_i} - 1) \int_x^{\infty} e_i^0 e_n^0 \right]^{-1} e^{t_n} \int_x^{\infty} e_n^0 e_i^0 \\ &= e^{t_n} (e_n^0)^2 \times [1 + o(1)] \qquad (x \uparrow \infty). \end{aligned}$$

Amplification 6. *Choose* $q^0(x) = x^2 - 1$ *as the origin of* **Q**. *Then the alternately even and odd parity of the Hermite functions* e_n^0 *implies*

$$\exp\left(\sum t_n\right)\theta_x(-t_0, -t_1, -t_2, \text{etc.}) = \theta_{-x}(t_0, t_1, t_2, \text{etc.}),$$

i.e., the map $\mathbf{X} \to -\mathbf{X}$ *expressing reflection in the tangent space is mirrored in the reflection* $q(x) \to q(-x)$ *of the potential* $q = e^{\mathbf{X}} q^0$. *The present choice of origin is to be understood until further notice.*

3. Surjection

The fact that the exponential map *is onto the whole of* **Q** is harder to prove. The n^{th} eigenfunction e_n of a fixed potential q of class **Q** is proportional to $x^n e^{-x^2/2}$

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11 $\Delta[f_1, \ldots, f_n] = \det\left[\int_x^{\infty} f_i f_j : 1 \leqq i, j \leqq n \right]$.

12 The exponent -1 in the sum signifies the inverse matrix. The fact that $e_n(x) = 0$ has precisely n roots must also be used.

at $\pm\infty$, so the constants

$$c_n^- = (e_n/e_n^0)(-\infty), \qquad c_n^+ = (e_n/e_n^0)(+\infty)$$

exist. It is required to prove the *connection rule* $c_n^- c_n^+ = 1$ and the *rapid vanishing* of $t_n = \ell g\, c_n^+/c_n^-$, and to verify $q = e^{\mathbf{X}} q^0$ with $\mathbf{X} = \sum t_n \mathbf{X}_n$.

Connection Rule

The first 2 steps are preparatory.

Step 1. Introduce the more-or-less standard Hermite–Weber function:

$$w_+^0(x,\lambda) = \Gamma(p)\frac{e^{-x^2/2}}{2\pi\sqrt{-1}}\int e^{2xy-y^2} y^{-p}\, dy$$

with $p = 1 + \lambda/2$. The integral is performed about the contour of Fig. 1 and the fractional power y^{-p} is the principal branch in the plane cut along $(-\infty, 0]$. The allied function $w_-^0(x,\lambda) = w_+^0(-x,\lambda)$ is also introduced. The following information comes from Bateman [1953: 116 et seq.]:

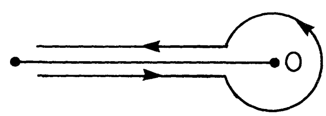

Fig. 1.

a) $Q^0 w = \lambda w$ for $w = w_-^0$ or w_+^0.
b) w_+^0 is comparable to $x^{p-1}e^{-x^2/2}$ at $x = \infty$ and to $x^{-p}e^{x^2/2}$ at $x = -\infty$.
c) w_+^0 is an integral function of λ, of order 1 and maximal type for fixed x.
d) $\Delta^0(\lambda) = [w_-^0, w_+^0]$ is independent of x; it is an integral function of the same class with simple roots at $\lambda = 0, 2, 4, 6$ etc. and no others; in particular, for $\lambda = 2n \geqq 0$, both w_-^0 and w_+^0 are proportional to the Hermite function e_n^0.

Step 2. The Hermite–Weber functions of Step 1 may be imitated for general Q with Δq vanishing at $\pm\infty$. These new functions are designated $w_\pm(x,\lambda)$ and together with $\Delta(\lambda) = [w_-, w_+]$ have the same properties a)–d) as their prototypes. The routine discussion is based upon the recipe

$$w_+(x) = w_+^0(x) + \int_x^\infty [w_+^0(x)w_-^0(y) - w_-^0(x)w_+^0(y)]\, w_+(y)\Delta q(y)\, dy$$

and may be omitted. The comparison $w_+ = w_+^0[1 + 0(\lambda^{-1/2})]$, valid for fixed x and $\lambda \downarrow -\infty$, is noted for future use; it may be differentiated by x.

Step 3 is the proof of the connection rule: Off spectrum, the Green's function $(Q-\lambda)_{xy}^{-1}$ may be expressed either as $-\Delta^{-1} w_-(x) w_+(y)\,(x < y)$ or as the sum

$\Sigma(\lambda_n - x)^{-1} e_n(x) e_n(y)$, whence[13]

$$[\Delta(\lambda_n)]^{-1} w_-(x, \lambda_n) w_+(y, \lambda_n) = e_n(x) e(y) \qquad (x < y),$$

by matching residues at $\lambda = \lambda_n$, and $c_n^- c_n^+ = \Delta^{0\cdot}/\Delta^0$ at $\lambda = \lambda_n$. But Δ and its prototype Δ^0 are integral functions of order 1 vanishing simply at $\lambda_n = 2n (n \geqq 0)$, and $\Delta(\lambda) \sim \Delta^0(\lambda)$ for $\lambda \downarrow -\infty$ by the comparison of Step 2. $\Delta = \Delta^0$ and $c_n^- c_n^+ = 1$ follow: such a function Δ differs from its Hadamard product only by a factor $e^{a + b\lambda}$ and two such factors coincide if their ratio tends to 1 along a ray.

Uniqueness

It will be necessary to know that $c_n^+/c_n^- = 1 (n \geqq 0)$ *only at the origin of* $\mathbf{Q}$, i.e., *only if* $q(x) = x^2 - 1$. It is a corollary that $x^2 - 1$ *is the only even potential of class* $\mathbf{Q}$; if $q(x) = q(-x)$, then its eigenfunctions, like those of $q^0(x) = x^2 - 1$, are alternately even or odd, so that $c_n^+/c_n^- = 1$. The proof of uniqueness is modeled upon Levinson [1949] and is the analogue of a theorem of Borg [1945].

Step 1 consists of preliminary estimates. Fix a number x and let $q^x(y)$ be $y^2 - 1$ or $q(y)$ according as $y < x$ or $y \geqq x$. The eigenvalues λ_n^x and the eigenfunctions $e_n^x(y)$ of the corresponding operator $Q^x = -d^2/dy^2 + q^x$ move with x, and it is easy to see that

$$\frac{\partial \lambda_n^x}{\partial x} = [e_n^x(x)]^2 \Delta q(x).$$

Now[14] for $|y| \leqq L = n^{1/6-}$,

$$e_n^x(y) = a_n [\sin(\lambda_n^x)^{1/2}(y + b_n) + O(n^{-1/2} L^3)]$$

by an elementary appraisal, and from the inequality $1 = \int (e_n^x)^2 \geqq a_n^2 [1/2 - o(1)] 2L$, the evaluation $\lambda_n^\infty = 2n$, and the rapid vanishing of Δq, it develops that $e_n^x(y) = O(n^{-1/2+})$ and

$$\lambda_n^x = 2n + O(n^{-1/6+}) \qquad (n \uparrow \infty).$$

The same trick applies to the eigenfunction:

$$\frac{\partial e_n^x(y)}{\partial x} = -e_n^x(x) \Delta q(x) \sum_{k \neq n} (\lambda_k^x - \lambda_n^x)^{-1} e_k^x(x) e_k^x(y),$$

so

$$|e_n^x(y) - e_n^\infty(y)| \leqq \sum_{k \neq n} |k - n|^{-1} O(k^{-1/6+}) \times O(n^{-1/12+}) = O(n^{-1/4+}).$$

But the Hermite function $e_n^\infty(y) = O(n^{-1/4})$ by routine appraisal, so $e_n^x(y) = O(n^{-1/4+})$, and this is improved to $O(n^{-1/4})$ *plain* by substituting back; in particular, $e_n(y) = e_n^{-\infty}(y) = O(n^{-1/4})$. The improvement also applies to the eigenvalue: $\lambda_n^x = 2n + O(n^{-1/2})$, independently of x.

13 $^\cdot$ signifies differentiation with regard to λ.

14 $c-$ [$c+$] means any number slightly smaller [larger] than c.

Step 2 is to explain the plan of the proof: Q and Q^0 have common spectrum $\lambda_n = 2n$ and if also $c_n^+/c_n^- = 1$, then their eigenfunctions e_n and e_n^0 are *common* multiples of $w_\pm$ and $w_\pm^0$, respectively, for $\lambda = 2n$. This permits you to write [15]

$$-\sum \frac{[\Delta e_n(x)]^2}{\lambda_n - \lambda_{-1}} + \frac{\Delta w_-(x,\lambda)\Delta w_+(x,\lambda)}{\Delta(\lambda)} \text{evaluated at } \lambda = \lambda_{-1}$$

$$= \frac{1}{2\pi\sqrt{-1}} \int \frac{\Delta w_-(x,\lambda)\Delta w_+(x,\lambda)}{(\lambda - \lambda_{-1})\Delta(\lambda)}\, d\lambda \equiv I$$

for fixed x and $\lambda_{-1} < 0$, the sum being taken over $n \leqq N$ and the integral about a circle of odd radius $2N+1$ enclosing λ_{-1}. The plan is to prove that $I = o(1)$ as $N \uparrow \infty$. Then

$$\frac{\Delta w_-(x,\lambda)\Delta w_+(x,\lambda)}{\Delta(\lambda)} \quad \text{at} \quad \lambda = \lambda_{-1} = \sum \frac{[\Delta e_n(x)]^2}{\lambda_n - \lambda_{-1}}.$$

The estimate $w = w^0[1 + O(\lambda^{-1/2})]$ for $\lambda \downarrow -\infty$ permits the left-hand side to be replaced by the free Green's function $(Q^0 - \lambda_{-1})^{-1}_{xx} \times O(1/\lambda_{-1})$, all of which is $o(1/\lambda_{-1})$ for $\lambda_{-1} \downarrow -\infty$, and comparison with the sum implies $[\Delta e_n(x)]^2 = 0$ for every $n \geqq 0$. $Q = Q^0$ is immediate from that.

Step 3 is to carry out the appraisal of I. Let $\Delta^x(\lambda)$ be the value of $[w_-^0, w_+]$ at x and G_λ^x the Green's function for the operator Q^x based upon the potential $q^x(y) = y^2 - 1$ or $q(y)$ according as $y < x$ or not. Then $G_\lambda^x(x,y) = -[\Delta^x(\lambda)]^{-1} w^0(x,\lambda) w_+(y,\lambda)$ for $y > x$, and the integral I of Step 2 is recognized as a sum of four pieces, of which

$$I_1 = \int [G_\lambda^x(x,x) - G_\lambda^\infty(x,x)] \frac{d\lambda}{\lambda - \lambda_{-1}}$$

and

$$I_2 = \int G_\lambda^x(x,x) \left[\frac{\Delta^x(\lambda)}{\Delta^\infty(\lambda)} - 1\right] \frac{d\lambda}{\lambda - \lambda_{-1}}$$

are typical. The bracketed part of the first integrand is

$$G_\lambda^x[q^x - q^\infty] G_\lambda^\infty(x,x) = \int_x^\infty G_\lambda^x(x,y)\Delta q(y) G_\lambda^\infty(y,x)\, dy,$$

and so is controlled by

$$\left[\sum \frac{[e_n^x(x)]^2}{|\lambda_n^x - \lambda|^2} \sum \frac{[e_n^\infty(x)]^2}{|\lambda_n^\infty - \lambda|^2}\right]^{1/2} \leqq \sum_{n \neq N} \frac{O(n^{-1/2})}{|n - N|^2} = O(N^{-1/2})$$

by the preliminary estimates $e_n^x = O(n^{-1/4})$ and $\lambda_n^x = 2n + O(n^{-1/2})$ of Step 1. This disposes of $I_1 = O(N^{-1/2})$. The discussion of I_2 is similar: The preliminary estimates

15 $\Delta e_n = e_n - e_n^0$, etc.

provide the appraisals:

$$\frac{\Delta^{x}(\lambda)}{\Delta^{\infty}(\lambda)} = \prod_{n=0}^{\infty} \frac{\lambda - \lambda_n^x}{\lambda - \lambda_n^{\infty}} = 1 + O(N^{-1/2+})^{16}$$

and

$$|G_\lambda^x(x, x)| \leqq \sum_{n \neq N} \frac{O(n^{-1/2})}{|n - N|} = O(N^{-1/2+}),$$

so that $I_2 = O(N^{-1+})$. The proof is finished.

Behaviour of Norming Constants

The next item of business is to confirm that c_n^+/c_n^- *tends rapidly to* 1 *as* $n \uparrow \infty$. This is the most difficult part.

Step 1. Let f_n^0 be any solution of $Q^0 f = \lambda_n f$ with $[f_n^0, e_n^0] = 1$. Then $D[f_n^0, e_n] = e_n f_n^0 \Delta q$, so $I_n = \int e_n f_n^0 \Delta q = c_n^+ - c_n^-$ and, by the connection rule, it suffices to prove that this integral vanishes rapidly as $n \uparrow \infty$. Now you may take

$$f_n^0(x) = n^{-1/2} 2^{n/2} (n!)^{-1/2} e^{x^2/2} \int_0^{\infty} {\sin \atop \cos} (2xy) e^{-y^2} y^n \, dy,$$

the upper [lower] trigonometrical function being employed for even [odd] n, and it is easy to see that

$$f_n^0(x) = O(n^{-3/4}) e^{x^2/2}; \quad \text{also,} \quad e_n^2(x) = O[x^{2n+c} e^{-x^2/2}],$$

independently of $|x| \geqq 1$ and n, if $c \geqq \Delta q$.[17] Let $L = n^{1/6-}$ as before. The estimates imply the rapid vanishing of the contribution to $I_n = \int e_n f_n^0 \Delta q$ from $|x| \geqq L$. The contribution from $|x| < L$ is more subtle. To begin with, for $|x| < L$, $Qh = \lambda h$ has two independent solutions of the form

$$h_1(x, \lambda) = A \cos\sqrt{\lambda}(x + L) + \lambda^{-1/2} B \sin\sqrt{\lambda}(x + L)$$
$$h_2(x, \lambda) = A \sin\sqrt{\lambda}(x + L) - \lambda^{-1/2} B \cos\sqrt{\lambda}(x + L),$$

A and B being formal power series in λ^{-1} determined by $-2\lambda A' + B'' = qB$, $2B' + A'' = qA$, and the values $A = 1$, $A' = 0$, $B = 0$, $B' = 0$ at $x = -L$. The error involved in breaking off such a series is of the naively apparent order *after multiplication by* $\Delta q(x)$. Now for $|x| < L$, e_n is a superposition of h_1 and h_2 evaluated at $\lambda = 2n$, of amplitude $n^{-1/4}$, while f_n^0 is a similar superposition of the corresponding functions h^0 for Q^0, of amplitude $n^{-3/4}$, so that, up to rapidly vanishing stuff, I_n is a sum of integrals $\int h_i h_j^0 \Delta q$ over $|x| < L$, with coefficients of magnitude n^{-1} or better. Terms involving sin cos, $\sin^2 - 1/2$, or $\cos^2 - 1/2$ are negligible as $n \uparrow \infty$

16 The possibility of an exponential factor $e^{a+b\lambda}$ is obviated by the fact that $\Delta^x = \Delta^\infty[1 + o(1)]$ as $\lambda \downarrow \infty$.
17 $\Sigma e^{-2nt} e_n^2(x)$ represents the elementary solution of $\partial p/\partial t = -Qp$ on diagonal which is overestimated by the product of e^{tc} and the free elementary solution on diagonal. The latter is $[2\pi(1 - \varepsilon^2)]^{-1/2} \exp[-x^2(1 - \varepsilon)(1 + \varepsilon)^{-1}]$ with $\varepsilon = e^{-2t}$. Now take $\varepsilon = 1/2x^2$.

in view of the fact that Δq is of class **S**, and the upshot is that I_n is controlled, up to rapidly vanishing stuff, by

$$\int_{-L}^{L} [AA^0 + \lambda^{-1} BB^0]\Delta q = 2(A^0 B - AB^0) - [A, A^0] - \lambda^{-1}[B, B^0]$$

and

$$\int_{-L}^{L} [AB^0 - A^0 B]\Delta q = 2BB^0 - [A, B^0] - [A^0, B] + 2\lambda(AA^0 - 1),$$

the right-hand sides being evaluated at $\lambda = 2n$ and $x = L$. Now the key to the proof is the

Lemma. *The formal power series $A - A^0$ and $B - B^0$ evaluated at $x = L$ vanish to all orders in λ^{-1}.*

This will be proved in Steps 2 and 3 below. The rapid vanishing of I_n is easily confirmed with its help. The first integral is plainly negligible, while the second reduces to

$$C = 2B^2 - 2[A, B] + 2\lambda(A^2 - 1),$$

evaluated at $x = L$. But C is independent of x $[C' = 0]$ and vanishes at $x = -L$ so $C(L) = 0$, too, completing the proof.

Step 2 is preparatory to the proof of the lemma. Let $H_0 = \int q$, $H_1 = \int [(1/2)q^2]$, $H_2 = \int [(1/2)q^3 + (1/4)(q')^2], \ldots, H_n = \int I_n[q]$, etc. be the usual KDV invariants.[18] They have no meaning in the class **Q**, but the *relative invariants*[19]

$$J_n = \int [I_n - I_n^0]$$

do, and since the H's represent spectral information when they make sense, it is not surprising that the J's *vanish identically in* **Q**. The proof employs the fact that if $p(t, x, y)$ is the elementary solution of $\partial p/\partial t = -Qp$, then $\Sigma e^{-\lambda_n t} = \int p(t, x, x)$, so that $\int [p(t, x, x) - p^0(t, x, x)]$ vanishes identically. But this *relative trace* admits the development[20] $(4\pi t)^{-1/2}[J_0 t + J_1 t^2 + \text{etc.}]$ for $t \downarrow 0$. The vanishing of $J_n (n \geqq 0)$ is now plain.

Step 3 is the proof of the lemma. Introduce the *discriminant* of Q in the interval $|x| < L: D(\lambda) = h_1 + \lambda^{-1/2}h_2'$ evaluated at $x = L$. The logarithm of $D(\lambda)$ admits a development[21] in powers $1/2 - n(n \geqq 0)$ of $\lambda \downarrow -\infty$, in which the coefficients may be expressed as *short* KDV invariants $\int_{-L}^{L} I_n$ plus gradients of *long* KDV invariants H_j evaluated at $x = \pm L$; similarly, the logarithm of $D(\lambda)/D^0(\lambda)$ admits a development with (*vanishing*) relative invariants in place of short invariants, plus rapidly vanishing contributions from $x = \pm L$. The upshot is that $\ell g[D(\lambda)/D^0(\lambda)]$ vanishes rapidly as $\lambda \downarrow -\infty$. The proof of the lemma is finished by comparison with the development

$$D(\lambda) = e^{2L\sqrt{-\lambda}} \times [A - (2\lambda)^{-1}B + (-\lambda)^{-1/2}(B + A'/2)] \text{ evaluated at } x = L$$

18 McKean–Moerbeke [1975], for instance.

19 $I_n^0 = I_n[x^2 - 1]$.

20 McKean–Moerbeke [1975] can serve as a model for the proof.

21 McKean–Moerbeke [1975].

and its counterpart for $D^0(\lambda)$.

Proof of the surjection. Choose a potential q of class $\mathbf{Q}$. The numbers $t_n = \ell g\, c_n^+/c_n^-$ vanish rapidly as $n \uparrow \infty$ and so may be used to form the tangent vector $\mathbf{X} = \Sigma t_n \mathbf{X}_n$. The flow $e^{-t\mathbf{X}}: 0 < t \leqq 1$ moves q to some point $q^0 \in \mathbf{Q}$, reducing all the numbers c_n^+/c_n^- to unity. But q^0 can only be the origin $q^0(x) = x^2 - 1$, by uniqueness, and surjectivity follows by reversing the flow: $q = e^{\mathbf{X}} q^0$. The proof is finished.

4. Canonically Conjugate Variables

The region of definition of the parameters $t_n (n \geqq 0)$ is extended from $\mathbf{Q}$ into the ambient space by the rule

$$t_n = \lim_{x \uparrow \infty} \ell g \left[(-1)^n e_n(x)/e_n(-x) \right].$$

It is to be proved that *the variables $2\lambda_i$ and t_j are canonically conjugate relative to the bracket $[F, G] = \int \nabla F D \nabla G$.* The vanishing of $[\lambda_i, \lambda_j]$ was already noted; also $[t_i, 2\lambda_j] = 1$ or 0 according to whether $i = j$ or not since the flow $\partial q/\partial t = \mathbf{X}_j q = [q, 2\lambda_j]$ advances t_i at speed 1 if $i = j$ and not at all if $i \neq j$. The only point still at issue is the vanishing of $[t_i, t_j]$.

Proof. The gradient of t_n is found to be[22] $\nabla t_n = (\Delta^\cdot)^{-1} [w_+^\cdot w_- - w_-^\cdot w_+]$ evaluated at $\lambda = \lambda_n$. This expression is of the form $e_n f_n$ with $Q f_n = \lambda_n f_n$ since e_n is proportional to $e^{-t_n/2} w_- = e^{t_n/2} w_+$, while $h = e^{t_n/2} w_+ - e^{-t_n/2} w_-$ satisfies $Qh^\cdot = \lambda h^\cdot + h$ and $h = 0$ for $\lambda = \lambda_n$; also, $[f_n, e_n] = 1$ is easily proved; finally, $\nabla t_n = O(x^{-1})$ at $\pm \infty$ since $e_n(x)$ behaves like $x^n e^{-x^2/2}$ and $f_n(x)$ like $x^{-n-1} e^{x^2/2}$. Now it is plain that $[t_i, t_j]$ exists. It vanishes automatically if $i = j$, while if $i \neq j$, the same follows from the identity $Kh = 2\lambda Dh$ for $h = e_n f_n$, $\lambda = \lambda_n$, and $K = qD + Dq - (1/2)D^3$: in detail, $2\lambda_j [t_i, t_j] = 2\lambda_j \int h_i D h_j = \int h_i K h_j = -\int h_j K h_i = -2\lambda_i \int h_j D h_i = 2\lambda_i \int h_i D h_j = 2\lambda_i [t_i, t_j]$. The only step needing clarification is that from $\int h_i K h_j$ to $-\int h_j K h_i$. This is dealt with by noting $1 = [f_n, e_n] \sim -2x e_n f_n$, so that $2q h_i h_j \sim 2x^2 h_i h_j \sim 1/2$ and the contributions from $\pm \infty$ in the necessary partial integration cancel out.

Amplification 1. *The formula $\nabla t_n = e_n f_n$ leads to a direct evaluation of $[t_i, 2\lambda_j]$: if $i \neq j$, the bracket is proportional to*

$$2\lambda_j \int e_i f_i D e_j^2 = \int e_i f_i K e_j^2 = -\int e_j^2 K e_i f_i = 2\lambda_i \int e_i f_i D e_j^2$$

and so vanishes, while if $i = j$, it is

$$2 \int e_i f_i D e_i^2 = -2 \int e_i^2 D e_i f_i = \int e_i^2 [f_i, e_i] = \int e_i^2 = 1.$$

A More Global View

Now the spectrum $\lambda_n (n \geqq 0)$ and the logarithmic norming constants $t_n (n \geqq 0)$ should provide a complete set of canonically conjugate coordinates not just

22 $w_- = e^{t_n} w_+$ at $\lambda = \lambda_n$. The gradient of this relation is now taken and combined with the following facts: a) $\partial w_\pm(x)/\partial q(y)$ vanishes at $x = y$, b) $\nabla \lambda_n = e_n^2$, c) $e_n = c e^{-t_n/2} w_- = c e^{t_n/2} w_+$, d) $c^2 = -1/\Delta^\cdot(\lambda_n)$. The spot signifies differentiation with regard to λ.

in $\mathbf{Q} = \mathbf{Q}[x^2 - 1]$ but in an appropriate ambient space, $x^2 - 1 + \mathbf{S}$, say, or wider, i.e., the general spectral class $\mathbf{Q}$ obtained by fixing $\lambda_n (n \geqq 0)$ not at $\lambda_n[x^2 - 1] = 2n$, but at some other values with realistic comportment as $n \uparrow \infty$, should meet the class $\mathbf{P}$ obtained by a similar fixing of $t_n (n \geqq 0)$ *in a single point*; moreover, these $\mathbf{Q}$'s and $\mathbf{P}$'s should be the leaves of transversal foliations of the ambient space. The purpose of the present section is to examine the geometry of this picture in the special leaf $\mathbf{Q} = \mathbf{Q}[x^2 - 1]$. The situation is outlined in the next three articles.

Completeness

The directions $\nabla\lambda_i = e_i^2$ are normal to the leaf $\mathbf{Q}$ at q, while the directions $\partial q/\partial t_j = De_j^2$ are tangent to it, as expressed by the first canonical identity $[\lambda_i, \lambda_j] = 0$, and it is a source of satisfaction that *these directions together span* $\mathbf{L}^2$ *so that no direction is left unclassified*. This is termed *completeness*. The same holds for the leaf $\mathbf{P}$ at $q: \nabla t_i = e_i f_i$ is normal, $\partial q/\partial\lambda_j = De_j f_j$ is tangent in view of the third canonical identity $[t_i, t_j] = 0$, and *the same type of completeness obtains*. Contrariwise, $\nabla\lambda_i$ and $D\nabla t_j\,[D\nabla\lambda_i$ and $\nabla t_j]$ *need not span*: for example, at the origin $q^0(x) = x^2 - 1$, they span only the even [odd] functions.

Transversality

The second canonical identity

$$[t_i, 2\lambda_j] = 2\int e_i f_i De_j^2 = 1(i = j) \;= 0(i \neq j)$$

implies that the normal spaces to $\mathbf{Q}$ and $\mathbf{P}$ meet only in the null function in view of what went before, and it is a further source of satisfaction to find that these two spaces together span $\mathbf{L}^2$, expressing a *transversality* of $\mathbf{Q}$ and $\mathbf{P}$; the same holds for the tangent spaces, confirming the picture of Fig. 2.

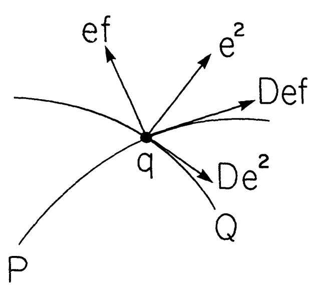

Fig. 2.

Independence

The second canonical identity also implies a stringent independence of normal and tangent directions. For example, *if the normal directions* $\nabla\lambda_n = e_n^2$ *are divided into two classes, then nothing in the span of the first class lies in the span of the second*, the point being that anything in the intersection is simultaneously perpendicular to

De_i^2 and to $De_i f_i$ and is null, by completeness and transversality. This type of independence indicates that **Q** *is a smooth submanifold of the ambient space*; naturally, *the same holds for* **P**.

Proof. The rest of this section is occupied by the proof of completeness and transversality.

Step 1. Fix a complex number λ outside the spectrum, let $G = G_\lambda(x, y)$ be the Green function $(Q-\lambda)^{-1}_{xy}$, and introduce the operator

$$H: f \to \int (\pm 1) \times [G^2(x,y) - G(x,x)G(y,y)] f(y)\,dy,$$

the $+$ sign being employed if $y > x$ and the $-$ sign otherwise. The present step contains appraisals of H required below. $G(x,x) \leqq c_1(x^2-\lambda)^{-1/2}$ for $\lambda < 0$ by routine estimation[23]. This provides the *preliminary bound*:

$$\begin{aligned}&|G^2(x,y) - G(x,x)G(y,y)|\\ &\leqq 2\Sigma|\lambda_n - \lambda|^{-1} e_n^2(x)\Sigma|\lambda_n-\lambda|^{-1}e_n^2(y)\\ &\leqq 2\max_{n\geqq 0}\left|\frac{\lambda_n+1}{\lambda_n-\lambda}\right|^2 G_{-1}(x,x)G_{-1}(y,y)\\ &\leqq c_2(\lambda)[(x^2+1)(y^2+1)]^{-1/2},\end{aligned}$$

which implies

$$\|Hf\|_2^2 \leqq c_2^2 \int \frac{dx}{x^2+1}\int\frac{dy}{y^2+1}\|f\|_2^2 = c_2^2\pi^2\|f\|_2^2.$$

Thus, *H is a bounded operator on* $\mathbf{L}^2$: in particular, $c_2(\lambda) = O(N^2)$ on circles of odd radius $2N+1$, so that $\|H\|_2 = O(N^2)$ there. Better information is available for $\lambda < 0$: $\|H\|_2 = O(\lambda^{-1})$. Now H is inverted by $K - 2\lambda D$ with $K = qD + Dq - (1/2)D^3$, as a routine computation shows. *This will lead to the bound* $\|DH\|_2 = O(N^3)$ *on odd circles.* To begin with, $(K-2\lambda D)H = 1$ implies $DHf = (L-2\lambda)^{-1}(1-q'H)f + g$ with $L = -(1/2)D^2 + 2q$ and $Lg = 2\lambda g$. Next, it is necessary to observe that the distant spectrum of L approximates that of $-(1/2)D^2 + 2(x^2-1)$, namely $4n(n \geqq 0)$. It follows that $g = 0$: indeed, $q'Hf \in \mathbf{L}^\infty$ by the preliminary bound and $Hf \in \mathbf{L}^2$, while any non-vanishing solution of $Lg = 2\lambda g$ is exponentially large if 2λ is not in the spectrum of L and could only unbalance the identity for DHf. The upshot is that $DH = (L-2\lambda)^{-1}(1-q'H)$ *plain.* Now $(L-2\lambda)^{-1}$ and G have a similar behaviour for $|\lambda| = 2N+1$, so the idea of the proof will be adequately conveyed by confirming $\|G(x^2+1)^{1/2}H\|_2 = O(N^3)$ on odd circles. The preliminary bound implies $\|(x^2+1)^{1/2}Hf\|_\infty = O(N^2)\|f\|_2$, so it suffices to overestimate $\|Gh\|_2$ by $O(N)\|h\|_\infty$ for $h \in \mathbf{L}^2 \cap \mathbf{L}^\infty$. That is easy:

$$\begin{aligned}\|Gh\|_2^2 &= \Sigma|\lambda_n - \lambda|^{-2}(e_n, h)^2 \leqq c_2(2N+1)\Sigma|\lambda_n+1|^{-2}(e_n,h)^2\\ &= O(N^2)\|G_{-1}h\|_2^2 = O(N^2)\|G_{-1}^0 1\|_2^2\|h\|_\infty^2,\end{aligned}$$

G_{-1}^0 being the free Green's function for $q^0(x) = x^2 - 1$ and $\lambda = -1$. But $f = G_{-1}^0 1$

23 $G(x,x) \leqq \int_0^\infty e^{\lambda t}e^{ct}[2\pi(1-\varepsilon^2)]^{-1/2}\exp[-x^2(1-\varepsilon)(1+\varepsilon)^{-1}]\,dt$ with $\lambda < 0$, $c \geqq \Delta q$, and $\varepsilon = e^{-2t}$.

looks like x^{-2} far out since it solves $(Q^0 + 1)f = 1$ and vanishes at $\pm\infty$. The proof is finished.

Step 2. *e_i^2 and $e_j f_j$ span* $\mathbf{L}^2$. The residue of H at $\lambda = 2n$ is $R_n = e_n^2 \otimes e_n f_n - e_n f_n \otimes e_n^2$, and the fact that $\|H\|_2 = O(N^2)$ on odd circles justifies the expansion $H_3 = \partial^3 H/\partial\lambda^3 = 6\Sigma(\lambda_n - \lambda)^{-4} R_n$ off spectrum, by a self-evident application of the Cauchy integral for H_3. Now let $f \in \mathbf{L}^2$ be perpendicular to e_i^2 and $e_j f_j$. Then $H_3 f = 0$ and the vanishing of $\|H\|_2$ at $\lambda = -\infty$ implies that $Hf = 0$, too, so that $f = (K - 2\lambda D)Hf = 0$, as well.

Step 3. *De_i^2 and $De_j f_j$ span* $\mathbf{L}^2$. The proof is similar. The extra technicalities required will be found in Step 4.

Step 4. *e_i^2 and De_j^2 span* $\mathbf{L}^2$. The fact that $\|DH\|_2 = O(N^3)$ on odd circles justifies the expansion $DH_4 = D\partial^4 H/\partial\lambda^4 = 24\Sigma(\lambda_n - \lambda)^{-5} DR_n$. Let $f \in \mathbf{L}^2$ be perpendicular to $e_n^2 (n \geqq 0)$. Then $R_n f$ is a multiple of e_n^2 and $DH_4 f$ belongs to the span of $De_n^2 (n \geqq 0)$. The same follows for DHf, and the fact that $\|2\lambda DHf + f\| = o(1)$ as $\lambda \downarrow -\infty$, finishes the proof. This final point requires justification; it makes use of the identity $DH = (L - 2\lambda)^{-1}(1 - q'H)$ of Step 1. $\|2\lambda(L - 2\lambda)^{-1} f + f\|_2 = o(1)$ is plain, so it suffices to prove $\|(L - 2\lambda)^{-1} q'Hf\|_2 = o(\lambda^{-1})$; as in Step 1, the idea will be adequately conveyed by proving $\|G(x^2 + 1)^{1/2} Hf\|_2 = o(\lambda^{-1})$. The bound $G(x, x) \leqq c_1 (x^2 - \lambda)^{-1}$ for $\lambda < 0$ implies $|Hf| \leqq c_1^2 (x^2 - \lambda)^{-1/2} \int (y^2 - \lambda)^{-1/2} |f| dy = o[\lambda^{-1/4}(x^2 - \lambda)^{-1/2}]$, so

$$\begin{aligned} \|G(x^2 + 1)^{1/2} Hf\|_2^2 &= o(\lambda^{-1/2}) \|G(x^2 + 1)^{1/2} (x^2 - \lambda)^{-1/2}\|_2^2 \\ &= o(\lambda^{-1/2}) \|G(x^2 + 1) G(x^2 - \lambda)^{-1}\|_1 . \end{aligned}$$

But $\|G(x^2 + 1)\|_\infty = O(1)$, as may be confirmed directly for G^0 and carries over to G itself, while

$$\|G(x^2 - \lambda)^{-1}\|_1 = \int dx \int G(x, y)(y^2 - \lambda)^{-1} dy = O(\lambda^{-1}) \int (y^2 - \lambda)^{-1} d\lambda = O(\lambda^{-3/2})$$

by exchange of integrals. The upshot is the necessary bound $\|G(x^2 + 1)^{1/2} Hf\|_2 = o(\lambda^{-1})$ with nothing much to spare.

Step 5. *$e_i f_i$ and $De_j f_j$ span* $\mathbf{L}^2$. The proof is similar. The discussion is finished.

5. Transversal Flows

The present section contains a brief discussion of the transversal flows $\partial q/\partial t = \mathbf{Y}_n q$ produced by the vector fields $\mathbf{Y}_n : q \to D\nabla t_n$: it is desired to move the n^{th} eigenvalue λ_n at speed 1 keeping the rest of the spectrum and all of the norming constants fixed, and there is a simple recipe for that[24].

Individual Flows

Let $n = 0$ and fix q^0. The ground state e_0^0 is positive, so $q^- = q^0 - 2D^2 \ell g e_0^0$ makes sense, and it turns out that the corresponding operator Q^- has a) the same spectrum

24 Deift–Trubowitz [1979] may be consulted for details omitted here.

as Q^0 *but with* λ_0^0 *excised*, b) eigenfunctions $e_n^- = (\lambda_n^0 - \lambda_0^0)^{-1/2}(e_0^0)^{-1}[e_0^0, e_n^0]$ ($n \geqq 1$), and c) the same norming constants, i.e., $e_n^-(x)/e_n^-(-x) \sim e_n^0(x)/e_n^0(-x)$ as $x \uparrow \infty$ for $n \geqq 1$. The eigenvalue λ_0^0 is now *restored to a different place* $\lambda_0^+ < \lambda_1^0$ by a similar recipe: Q^- has an eigenfunction f_0^- with eigenvalue λ_0^+ which is of one signature (+) and satisfies $f_0^-(x)/f_0^-(-x) \sim e^{t_0}$ as $x \uparrow \infty$, and it turns out that the new operator Q^+ with potential $q^+ = q^- - 2D^2 \ell g f_0^-$ has a) spectrum $\lambda_0^+ < \lambda_1^0 < \lambda_2^0 <$ etc., b) eigenfunctions

$$e_0^+ = (f_0^-)^{-1}[\textstyle\int (f_0^-)^{-2}]^{-1/2}, e_n^+ = (\lambda_n^0 - \lambda_0^+)^{-1/2}(f_0^-)^{-1}[f_0^-, e_n^-](n \geqq 1),$$

and c) the same norming constants $t_n^+ = t_n^0$ as Q^0; moreover, it is easy to check that the map $q^0 \to q^+$ effects the flow $\partial q/\partial t = D\nabla t_0$ with parameter $t = \lambda_0^+ - \lambda_0^0$. The auxiliary function f_0^- is easily computed from the eigenfunctions of Q^0: it is proportional to $(e_0^0)^{-1}[e_0^0, f_0^0]$ with

$$f_0^0 = (\lambda_0^+ - \lambda_0^0)^{-1} \times [e^{t_0/2}w_+^0 - e^{-t_0/2}w_-^0 \text{ evaluated at } \lambda = \lambda_0^+].$$

The flow now takes the simple form $\Delta q = -2D^2 \ell g[e_0^0, f_0^0]$.

Multiple Flows

The recipe is easily extended to multiple flows: several eigenvalues $\lambda_i^0 (i \leqq n)$ are excised in their *natural* order by successive applications of the map $q^0 \to q^-$ and put back in the *opposite* order at new places $\lambda_0^+ < \lambda_1^+ \ldots < \lambda_n^+$ to the left of λ_{n+1}^0 by use of the map $q^- \to q^+$. The result may be put in the simple form:

$$\Delta q = -2D^2 \ell g[e_0^0, \ldots e_n^0, f_0^0, \ldots, f_n^0],$$

with

$$f_i^0 = (\lambda_i^+ - \lambda_0^0)^{-1} \times [e^{t_i/2}w_+^0 - e^{-t_i/2}w_-^0 \text{ evaluated at } \lambda = \lambda_i^+],$$

$[e_0^0, \ldots, f_n^0]$ being the Wronskian determinant. It is a pleasant feature of the recipe that if $\lambda_i^+ = \lambda_i^0$ for $i \neq j$, then also $[e_i^0, f_i^0] = 1$ for $i \neq j$, so that the big Wronskian simplifies to $[e_j^0, f_j^0]$, and the reduced formula $\Delta q = -2D^2 \ell g[e_j^0, f_j^0]$ expresses the j^{th} individual flow in the same manner as for $j = 0$. The discussion of the case $n = \infty$ is not so simple and will not be entered upon.

Charge

To be candid, the matter is not quite so simple, even for $n < \infty$: starting from the origin, the flow produces in $q(x)$ novel terms $c_2(\pm\infty)x^{-2} + c_3(\pm\infty)x^{-3} +$ etc. near $\pm\infty$ and so leads out of the familiar ambient space $x^2 - 1 + \mathbf{S}$: for example, $c_2(\pm\infty) = \pm(1/2)\Sigma[\lambda_i^+ - \lambda_i^0]$ for any $q^0 \in x^2 - 1 + \mathbf{S}$. This only means that the ambient space must be enlarged to accommodate such behaviour; it seems reasonable to admit terms $c_0(\pm\infty) + c_1(\pm\infty)x^{-1}$ as well. The number $c_0(+\infty) - c_0(-\infty)$ is called the *charge* of q. The spectral interpretation of c_2 should have a counterpart for c_0 since it cannot be moved by any isospectral flow in view of the rapid vanishing of De_n^2 at $\pm\infty$. The subject merits further investigation.

Example

The proposed enlargement of the ambient space produces a wider spectral class $\mathbf{Q}[x^2-1]$. To see this requires an example. Let $q^0(x)=x^2-1$ and, in $q(x)=e^{tX}q^0 = x^2-1-2D^2\ell g[1+(e^t-1)\int_x^\infty(e_0^0)^2]$, make $t\uparrow+\infty$ or $\downarrow-\infty$. The corresponding limits $q^+(x)$ and $q^-(x)=q^+(-x)$ are

$$q^+(x)=x^2-1-2D^2\int_x^\infty(e_0^0)^2=x^2-1+o(1)\,\text{at}-\infty=x^2+4+o(1)\,\text{at}+\infty,$$

$$q^-(x)=x^2-1-2D^2\int_\infty^x(e_0^0)^2=x^2+3+0(1)\,\text{at}-\infty=x^2-1+o(1)\,\text{at}+\infty.$$

The behaviour at $+\infty$ shows that q^+ and q^- have escaped from $x^2-1+\mathbf{S}$; more drastically, the eigenfunction

$$e_0(x)=e^{t/2}e_0^0(x)[1+(e^t-1)\int_x^\infty(e_0^0)^2]^{-1}$$

tends to 0 as $t\uparrow\infty$ or $\downarrow-\infty$. The explanation is that *the former ground state* 0 *is not in the spectrum of* Q^- *or* Q^+, *the remaining eigenfunctions*

$$e_n^+=e_n^0-e_0^0\int_x^\infty e_0^0e_n^0\times\left[\int_x^\infty(e_0^0)^2\right]^{-1}$$

$$e_n^-=e_n^0-e_0^0\int_{-\infty}^x e_0^0e_n^0\times\left[\int_{-\infty}^x(e_0^0)^2\right]^{-1}$$

of index $n\geqq 1$ *already forming a full set.*

Proof. Let $f\in\mathbf{L}^2$ be perpendicular to e_n^+ $(n\geqq 1)$, say. Then

$$\int fe_n^0=\int fe_0^0\frac{\int_x^\infty e_0^0e_n^0}{\int_x^\infty(e_0^0)^2}=\int e_0^0e_n^0\int_{-\infty}^x\left[fe_0^0/\int_x^\infty(e_0^0)^2\right]$$

for $n\geqq 0$, inclusive, so that

$$f/e_0^0=\int_{-\infty}^x\left[fe_0^0/\int^\infty(e_0^0)^2\right].$$

This leads to the contradictory evaluation of f as a multiple of $(e_0^0)^2\left[\int_x^\infty(e_0^0)^2\right]^{-1}$ which is *not* in $\mathbf{L}^2$, *provided* $h=e_0^0\int_{-\infty}^x\left[fe_0^0/\int^\infty(e_0^0)^2\right]$ belongs to $\mathbf{L}^2$. The latter point requires a speck of ingenuity.

The map $f\to h$ is dual to the map

$$f\to\frac{e_0^0(x)\int_x^\infty fe_0^0}{\int_x^\infty(e_0^0)^2},$$

and the latter is bounded: indeed, with e in place of e_0^0 for simplicity,

$$\left\| f - e\frac{\int_x^\infty fe}{\int_x^\infty e^2}\right\|^2 = \int f^2 + \int\left(\int_x^\infty e^2\right)^{-1} d\left(\int_x^\infty fe\right)^2 + \int e^2 \frac{\left(\int_x^\infty fe\right)^2}{\left(\int_x^\infty e\right)^2}$$

$$= \int f^2 - \frac{\left(\int_x^\infty fe\right)^2}{\int_x^\infty e^2}\Bigg|_{-\infty}^{\infty} = \int f^2 - (e,f)^2 \leqq \|f\|^2,$$

if f vanishes near $+\infty$, and so also in general.

The eigenvalue λ_0^0 may now be restored to its original position by the recipe used for the individual flow of index 0: λ_0^0 lies below the spectrum of Q^+, so that the latter has a positive eigenfunction f_0^+, and the new operator Q with potential $q = q^+ - 2D^2\ell g f_0^+$ has spectrum $\lambda_n = \lambda_n^0\, (n \geqq 0)$, i.e., it belongs to the spectral class $\mathbf{Q}[x^2 - 1]$. But this is not the old but some *enlarged* class: $\ell g\, f_0^+ \sim x^2/2$ at $\pm\infty$, so

$$\begin{aligned} q(x) &= x^2 - 1 + 4 - 2 + o(1) = x^2 - 1 + 2 + o(1) \quad \text{at} \quad +\infty \\ &= x^2 - 1 + 0 - 2 + o(1) = x^2 - 1 - 2 + o(1) \quad \text{at} \quad -\infty; \end{aligned}$$

in particular, q has charge 4.

6. Theta Sums

The so-called *theta sum* $\theta_x(t_0, t_1, t_2,$ etc.) played a central role. The name is justified by recollecting[25] what happens to the classical Riemann theta function of a non-singular-elliptic curve

$$\mathbf{K}: y^2 = \prod_{i=0}^{g} (\lambda - \lambda_i^-)(\lambda - \lambda_i^+)$$

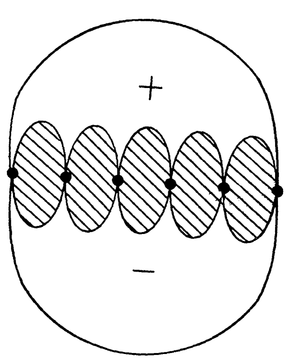

Fig. 3.

25 McKean [1979] gives details.

of genus g as the intervals $[\lambda_i^-, \lambda_i^+]$ are pinched to single points $\lambda_i (i = 0, \dots, g)$. The curve becomes singular as in Fig. 3 $[g = 5]$, falling apart into two sheets labelled *plus* and *minus*; simultaneously, the quadratic form figuring in the theta sum becomes huge, and the principal part of the theta function reduces to one of a number of finite sums θ_k indexed by $k = 0, \dots, g$, corresponding to a breaking up of the Jacobi variety **J** of **K** into $g+1$ connected pieces; this is caused by the breaking up of the space of divisors[26] $\mathfrak{p}_1, \dots, \mathfrak{p}_g$ of degree g into $g+1$ pieces according to the number $0 \leqq k \leqq g$ of points on the upper sheet[27]. The principal parts look like

$$\theta_k(x) = \sum_{\substack{n_i = 0 \text{ or } 1 (1 \leqq i \leqq g) \\ n_1 + \dots + n_g = k,}} e^{(\mathfrak{x} - c) \cdot n} e^{\sum_{i<j} n_i n_j \ell g |\lambda_i - \lambda_j|},$$

in which the constant c depends upon k and $\mathfrak{x} \in \mathbf{J}$ is expressed by means of integrals of differentials of the first kind [**DFK**] summed over the typical divisor having k points on the upper sheet, the sums being construed modulo periods. The forms $\omega_j = [(\lambda - \lambda_j)^{-1} - (\lambda - \lambda_0)^{-1}] d\lambda (j = 1, \dots, g)$ serve as a basis of **DFK**[28] and

$$\mathfrak{x}_j = \sum_{i=1}^{g} \operatorname{sign} \mathfrak{p}_i \times \int_{\infty}^{\mathfrak{p}_i} \omega_j \quad (j = 1, \dots, g),$$

in which sign $\mathfrak{p}$ is $+1[-1]$ on the upper [lower] sheet and, in the i^{th} integral, ∞ is taken on the same sheet as $\mathfrak{p}_i$. *The same type of geometrical interpretation is available for the present theta sums*: There is a singular curve

$$\mathbf{K}: y^2 = [\Delta(\lambda)]^2$$

of infinite genus, some kind of Jacobi variety **J** broken up into uncountably many pieces indexed by the real number x, and a system of theta sums θ_x, one to each piece. But what is the divisor $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{p}_2, \dots$ producing the argument $t_0, t_1, t_2, \dots$ of the sum? and are they related in the classical way via some natural class **DFK**? The answers are not far off. Divide the line into two pieces by a cut at x and let $Q^-[Q^+]$ be the operator Q restricted to functions on the half-line $y \leqq x [y \geqq x]$ vanishing at x. Q^- and Q^+ define *side spectra* $\lambda_n^\pm (n > 0)$, and the divisor is simply the points $\mathfrak{p} = (\lambda_n^+, +1)$ on the upper sheet of **K** and $\mathfrak{p} = (\lambda_n^-, -1)$ on the lower sheet. Now regard $\omega_n = (\lambda - \lambda_n)^{-1} d\lambda (n \geqq 0)$ as a differential of the first kind. Formally,

$$\sum \int_{\infty}^{\mathfrak{p}} \omega_n = \sum_{k=0}^{\infty} \int_{\lambda_k^-}^{\lambda_k^+} \frac{d\lambda}{\lambda - \lambda_n} = \ell g \prod_{k=0}^{\infty} \frac{\lambda_k^+ - \lambda_n}{\lambda_k^- - \lambda_n},$$

provided the side spectra are disjoint from the fixed spectrum $\lambda_n = 2n$. But $\lambda_k^\pm (k \geqq 0)$ are the roots of $w_\pm(x, \lambda) = 0$, while $e^{-t_n} w_+(x, \lambda) = w_-(x, \lambda) \neq 0$ for $\lambda = \lambda_n$, so that

$$\sum \int_{\infty}^{\mathfrak{p}} \omega_n = \ell g \frac{w_+(x, \lambda_n)}{w_-(x, \lambda_n)} = -t_n.$$

26 $\mathfrak{p}$ is the typical point $[\lambda(\mathfrak{p}), y(\mathfrak{p})]$ of **K**.
27 $\mathfrak{p}_i = \lambda_j$ is disallowed.
28 The presence of poles looks odd but their residues are really *vestigial periods*.

Amplification 1. The side spectra move with x as in Fig. 4: For $x = -\infty$, the left-hand spectrum is absent and $\lambda_n^+ = 2n$ ($n \geqq 0$). These points move steadily to the right as x comes in from $-\infty$; simultaneously, the left-hand spectrum enters from $+\infty$, crossing the right-hand spectrum only at the points 2, 4, 6, etc. and occupying, for $x = \infty$, the place $\lambda_n^- = 2n$ ($n \geqq 0$), the right-hand spectrum having disappeared. The rule that the side spectra cross only at 2, 4, 6, etc. follows from the fact that if, e.g., $\lambda_1^+ = 6 = \lambda_3^-$, then $e_3(x) = 0$, so that e_3 is also an eigenfunction of Q^-, i.e., 6 is a left-hand eigenvalue.

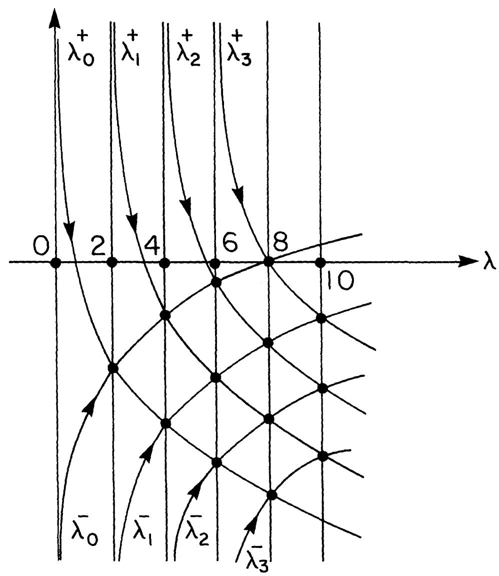

Fig. 4.

Appendix: Numerical Results

The following pictures of $q(x) = x^2 - 1 - 2D^2\ell g\,\theta$ were kindly made for us by O. McBryan by numerical evaluation of θ for five active parameters $t_j (0 \leqq j \leqq 4)$; their values are indicated under each figure. Figures 5–7 display an unexpected progressively deep well. This appears to contradict the fact that as $t = t_0 \uparrow \infty$ in $e^{tX_0}(x^2 - 1) = q(x)$, the final potential $q^+(x)$ has spectrum $\lambda_n^+ = 2n (n \geqq 1)$ *higher* than that of $q^0(x) = x^2 - 1$: actually, the well moves off to $+\infty$ and disappears at $t = \infty$.

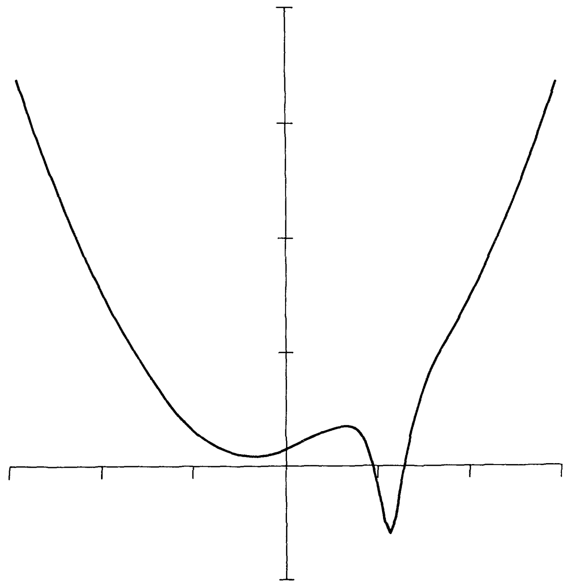

Fig. 5. (7, 0, 0, 0, 0)

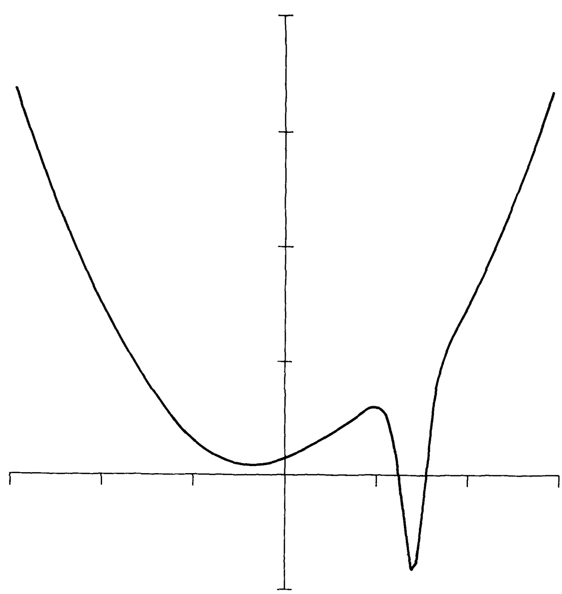

Fig. 6. (10, 0, 0, 0, 0)

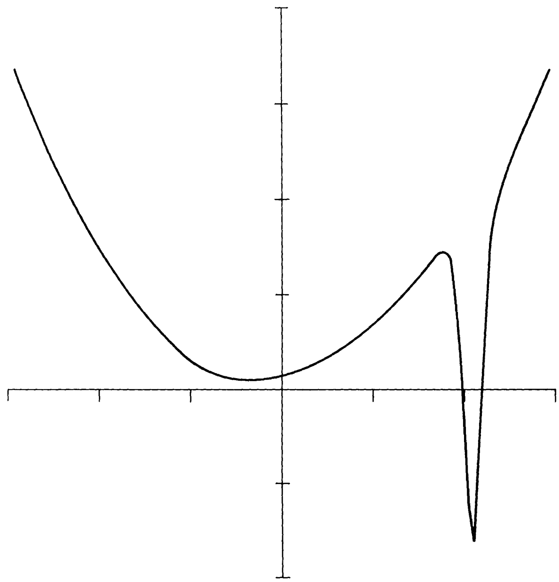

Fig. 7. (20, 0, 0, 0, 0)

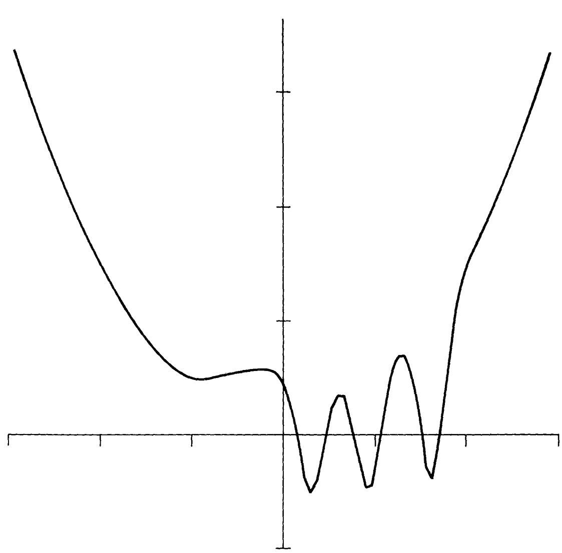

Fig. 8. (7, 7, 7, 0, 0)

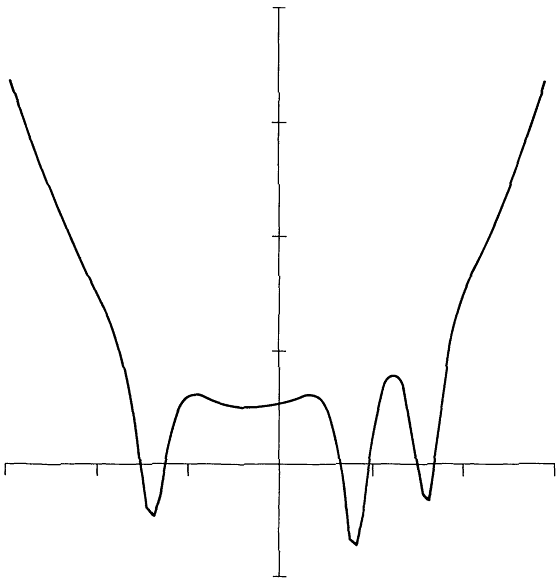

Fig. 9. (7, − 7, 7, 0, 0)

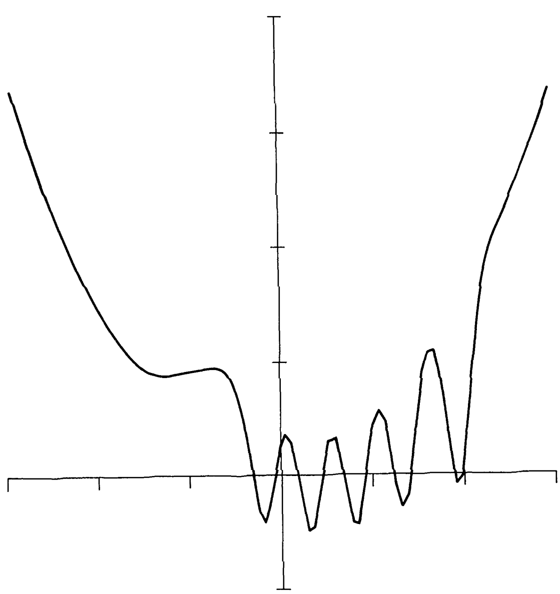

Fig. 10. (7, 7, 7, 7, 7)

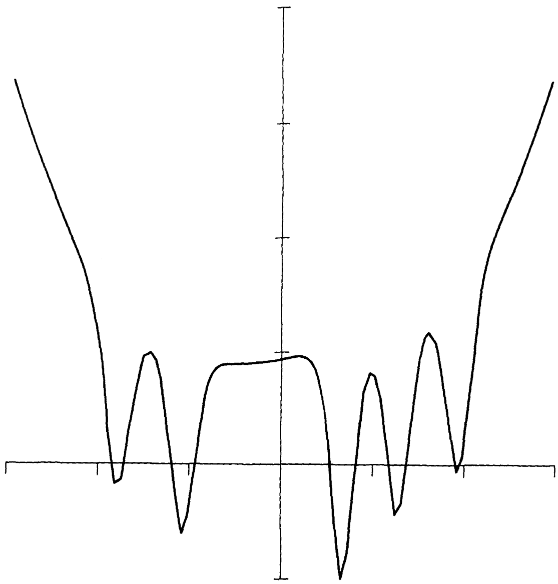

Fig. 11. (7, − 7, 7, − 7, 7)

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