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Comment on "Analytic Scattering Theory for Many-Body Systems Below the Smallest Three-Body Threshold"

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Abstract. The proof of [1, Lemmas 2.1–2.3] is completed, showing that the operators of multiplication by k^2 in $H^{t,\ell}$, $|t| \leqq 1$, $\ell = 0, \pm 2$, have spectrum $\bar{R}^+$ and generate the holomorphic semigroups $e^{\zeta k^2}$, $\mathrm{Re}\,\zeta < 0$.

It is pointed out, that [1, (5.54)] does not hold. Accordingly, a new version of [1, Theorem 5.15] is proved, saying that (5.44) defines an isomorphism of $\tilde{\mathcal{N}}(G_+(z,\kappa))/\tilde{\mathcal{N}}_0(G_+(z,\kappa))$ onto $\mathcal{N}(\mathcal{S}_\lambda^*(\bar{z}))$.

1. On the Proof of Lemmas 2.1–2.3

Lemma 2.0. *The operators $\tilde{H}_0$ defined by multiplication by k^2 in the spaces $H^{t,\ell}$ for $t = \pm s$, $0 \leqq s \leqq 1$, $\ell = 0, \pm 2$ with domains $H^{t,\ell+2}$ have the spectrum $\bar{R}^+$ and generate holomorphic semigroups $e^{\zeta \tilde{H}_0}$ defined for* $\mathrm{Re}\,\zeta < 0$.

Proof. Clearly, $(k^2 - z)^{-1} \in \mathcal{B}(H^{1,\ell}) \cap \mathcal{B}(H^{0,\ell})$, $\ell = 0, \pm 2$, $z \notin \bar{R}^+$, and hence by interpolation $(k^2 - z)^{-1} \in \mathcal{B}(H^{s,\ell})$, $0 < s < 1$, $\ell = 0, \pm 2$.

By duality, $(k^2 - z)^{-1} \in \mathcal{B}(H^{-s,-\ell})$. Obviously, $(k^2 - z)^{-1}$ is unbounded in any of these spaces for $z \in \bar{R}^+$, hence $\sigma(\tilde{H}_0) = \bar{R}^+$.

$\tilde{H}_0$ generates the semigroup $\tilde{\mathcal{U}}(\zeta)$ given by

$$\tilde{\mathcal{U}}(\zeta) = e^{\zeta k^2} \quad \text{in} \quad \mathcal{B}(H^{s,\ell}), \qquad \mathrm{Re}\,\zeta < 0.$$

Clearly, $\tilde{\mathcal{U}}(\zeta)$ is a uniformly bounded semigroup. For $s = \ell = 0$, $\zeta < 0$, $\tilde{\mathcal{U}}(\zeta)$ is the bounded semigroup $e^{\zeta H_0}$ generated by the self-adjoint operator H_0 in $\mathcal{L}^2$. Thus, for $f \in H^{0,2}$

$$t^{-1}(e^{-tk^2} - 1)f \xrightarrow[t \to 0]{} -H_0 f \quad \text{in } \mathcal{L}^2, \text{ hence in } H^{-s}. \tag{$*$}$$

From this it follows that $(*)$ holds for all $f \in \mathcal{D}(\tilde{H}_0)$, and it is easy to see that the operator $\tilde{H}_0$ in $H^{-s,0}$ is the infinitesimal generator of $\tilde{\mathcal{U}}(\zeta)$.

A similar argument proves the same for $\tilde{H}_0$ in $H^{-s,-2}$, and using the duality of H^s with H^{-s} and $H^{s,2}$ with $H^{-s,-2}$ the same is proved for $\tilde{H}_0$ in H^s and $H^{s,2}$.

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To prove Lemmas 2.1–2.3 we make the additional induction assumption, that for all systems C of less than n particles $\sigma_d(H^C) \subset \mathbb{R}$, where H^C is the maximal operator in $H^{s,-2}$, with domain $H^{s,0}$.

Ichinose's lemma and Lemma 2.0 yield $\sigma(\tilde{H}_0)=\bar{R}^+$, where $\tilde{H}_0$ is considered as an operator in $H_{p_D}^{-s} \otimes H_{k_D}^{s}$ or $\mathscr{L}_{p_D}^2 \otimes H_{k_D}^s$. Hence $\sigma(\tilde{H}_0(z))=e^{2i\varphi}\bar{R}^+$, and Lemma 2.1 is proved using Ichinose's lemma.

In the proof of Lemma 2.2 it follows in the same way from Ichinose's lemma, that $\sigma(\tilde{H}_0)=\bar{R}^+$, where now $\tilde{H}_0$ is considered as an operator in

$$H_{\alpha_1}^{-s,-2} \otimes H_{M_1,\alpha_2}^{-s,-2} \otimes \ldots \otimes H_{M_{k-1},\alpha_k}^{-s,-2} \otimes H_{M_k,\alpha_{k+1}}^{s,-2} \otimes \ldots \otimes H_{M_{n-2},\alpha_{n-1}}^{s,-2},$$

and hence

$$V_{\alpha_1} R_0(\zeta) \in \mathscr{B}(H_{\alpha_1}^{-s,-2}, H_{\alpha_1}^{s,-2}) \otimes \mathscr{B}(H_{M_1,\alpha_2}^{-s,-2}) \otimes \ldots \otimes \mathscr{B}(H_{M_{k-1},\alpha_k}^{-s,-2}) \otimes \mathscr{B}(H_{M_k,\alpha_{k+1}}^{s,-2}) \otimes \ldots \otimes \mathscr{B}(H_{M_{n-2},\alpha_{n-1}}^{s,-2}).$$

The fact that $V_{\alpha_1} R_0(\zeta)$ belongs to the same space with the first factor replaced by $\mathscr{C}(H_{\alpha_1}^{-s,-2}, H_{\alpha_1}^{s,-2})$ is proved as in the proof of Lemma 2.2. Here the formula (2.8) is valid because by Lemma 2.0, H_0 generates a semigroup of bounded operators

$$e^{-H_0 t} = e^{-H_0^{\alpha_1} t} e^{-H_0^{M_1,\alpha_2} t} \ldots e^{-H_0^{M_{n-2},\alpha_{n-1}} t}$$

defined for $t>0$ in the above space. The validity of Lemma 2.2 for all $\zeta \in C \setminus \bar{R}^+$ then follows from the analyticity and Hahn-Banach's theorem by the argument given in [29].

For the proof of Lemma 2.3 we notice that by the $H_0-\varepsilon$-boundedness of $V_{D_{n-i}}$, the operator $H_{D_{n-i}}$ is a closed operator in

$$H_{k_{D_{n-i}}}^{s,-2} \otimes H_{M_i,\alpha_{i+1}}^{-s,-2} \otimes \ldots \otimes H_{M_{n-2},\alpha_{n-1}}^{-s,-2}$$

with domain

$$H_{k_{D_{n-i}}}^{s} \otimes H_{M_i,\alpha_{i+1}}^{-s} \otimes \ldots \otimes H_{M_{n-2},\alpha_{n-1}}^{-s}.$$

By Ichinose's lemma and the induction assumption,

$$\sigma(H_{D_{n-i}})=[\lambda_{e,D_{n-i}}, \infty),$$

where

$$\lambda_{e,D_{n-i}} = \sum_{j=1}^{n-i} \min\{\lambda \in \sigma(H^{C_k})\}, \qquad D_{n-i}=\{C_1, \ldots, C_{n-i}\}.$$

Hence, by the ε-boundedness of $V_{D_{n-i}}$, for $\zeta \notin [\lambda_{e,D_{n-i}}, \infty)$

$$V_{D_{n-i}} R_{D_{n-i}}(\zeta) \in \mathscr{B}(H_{k_{D_{n-i}}}^{s,-2}) \otimes \mathscr{B}(H_{M_i,\alpha_{i+1}}^{-s,-2}) \otimes \ldots \otimes \mathscr{B}(H_{M_{n-2},\alpha_{n-1}}^{-s,-2}).$$

Finally it remains to verify the additional induction assumption, i.e. that $\sigma_d(H) \subset R$, where H is considered as an operator in $H^{s,-2}$ with domain H^s.

We first note that this holds by Lemma 2.9, when H is considered as an operator in H^{-s} with domain $H^{-s,2}$. Hence the induction proof shows that the spectrum of this operator is contained in $\mathbb{R}$. Denote for the present this operator by $\tilde{H}$, while H is the operator in $H^{s,-2}$ with domain H^s.

Assume $\lambda \notin \mathbb{R}$, $\phi \in H^s$, $(H-\lambda)\phi = 0$. Then

$$\langle (H-\lambda)\phi, \psi\rangle = \langle \phi, (\tilde{H}-\bar{\lambda})\psi\rangle = 0$$

for all $\psi \in H^{-s,2}$, where $\langle \cdot, \cdot \rangle$ denotes the duality of $H^{s,-2}$ and $H^{-s,2}$. Hence $\mathscr{R}(\tilde{H}-\bar{\lambda}) \neq H^{-s}$, so $\bar{\lambda} \in \sigma(\tilde{H})$, a contradiction.

It follows that $\sigma_d(H) \subset \mathbb{R}$.

2. On Lemma 5.14 and Theorem 5.15

The identity (5.54) does not hold. The correct identity is

$$G_-(z,\kappa)\tilde{Y}_-(z,\kappa) + G'_-(z,\kappa)A_- = I\,.$$

Hence Theorem 5.15 cannot be proved as Theorem 5.13, replacing $Y_-(z,\kappa)$ by $\tilde{Y}_-(z,\kappa)$. The Theorem holds with T_κ being given by (5.44), but with no explicit expression for the inverse Z_κ. This can be proved as follows.

It is seen as in the proof of Lemma 5.14, that $\mathscr{S}_\lambda^*(\bar{\zeta})$ is regular at $\zeta = z$ and is given by (5.53) or, in view of (5.56)

$$\mathscr{S}_\lambda^*(\bar{z}) = 1 + \lim_{\zeta\to z} 2\pi i m_D \zeta^{-2}\gamma_D(1)E_\lambda Y_-\left(\zeta, \lambda + \frac{\zeta^2}{2_{m_D}}\right) W_D^i(\zeta)\gamma_D^*(1)\,.$$

Assume $\Omega \in \tilde{\mathcal{N}}(G_+(z,\kappa))/\tilde{\mathcal{N}}_0(G_+(z,\kappa))$ and let $\sigma = T_\kappa\Omega$. Then

$$\mathscr{S}_\lambda^*(\bar{z})\sigma = \lim_{\zeta\to z} 2\pi i\zeta^{-2}\gamma_D(1)E_\lambda\left\{\Omega + Y_-\left(\zeta,\lambda+\frac{\zeta^2}{2m_D}\right) W_D^i(\zeta) 2\pi i m_D \zeta^{-2}\gamma_D^*(1)\gamma_D(1)E_\lambda\Omega\right\}$$

$$= \lim_{\zeta\to z} 2\pi i\zeta^{-2}\gamma_D(1)E_\lambda$$

$$\cdot\left\{\Omega + Y_-\left(\zeta,\lambda+\frac{\zeta^2}{2m_D}\right)\left(G_+\left(\zeta,\lambda+\frac{\zeta^2}{2m_D}\right) - G_-\left(\zeta,\lambda+\frac{\zeta^2}{2m_D}\right)\right)\Omega\right\} = 0\,,$$

because (5.56) holds, $G_+(z,\kappa)\Omega = 0$ and

$$\gamma_D(1)E_\lambda \lim_{\zeta\to z} Y_-\left(\zeta,\lambda+\frac{\zeta^2}{2m_D}\right) G_-\left(\zeta,\lambda+\frac{\zeta^2}{2m_D}\right) = \gamma_D(1)E_\lambda \quad \text{by Lemma 4.3}\,,$$

since

$$JR_{1-}(z,\kappa)\lim_{\zeta\to z} Y_-\left(\zeta,\lambda+\frac{\zeta^2}{2m_D}\right) G_-\left(\zeta,\lambda+\frac{\zeta^2}{2m_D}\right)$$

$$= \lim_{\zeta\to z} R_-\left(\zeta,\lambda+\frac{\zeta^2}{2m_D}\right) G_-\left(\zeta,\lambda+\frac{\zeta^2}{2m_D}\right) = JR_{1-}(z,\kappa)\,.$$

This shows that T_κ is an isomorphism of $\tilde{\mathcal{N}}(G_+(z,\kappa))/\tilde{\mathcal{N}}_0(G_+(z,\kappa))$ into $\mathcal{N}(\mathcal{S}_\lambda^*(\bar{z}))$. A simple argument, utilizing the expression (5.39) for $\mathcal{S}_\lambda^{-1*}(\bar{z})$, shows that

$$\dim \tilde{\mathcal{N}}(G_+(z,\kappa))/\tilde{\mathcal{N}}_0(G_+(z,\kappa)) \geqq \dim \mathcal{N}(\mathcal{S}_\lambda^*(\bar{z})).$$

Hence the isomorphism T_κ is onto $\mathcal{N}(\mathcal{S}_\lambda^*(\bar{z}))$, and the theorem is proved.

The same proof applies to establish a similar version of [2, Theorem 7.9].

References

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2. Balslev, E.: Ann. Inst. Henri Poincaré **32**, 125–160 (1980)

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