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Borel-Summability of the High Temperature Expansion for Classical Continuous Systems

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Abstract. It is shown that for classical gases with stable, bounded and absolutely integrable pair interactions, the Taylor expansions in β of the correlation functions and the pressure are Borel-summable at $\beta = 0$.

1. Introduction

The question of analyticity in β for classical continuous systems was considered some years ago by Lebowitz and Penrose [1]. Among other results they showed that for hard core potentials pressure and correlation functions are analytic at $\beta = 0$. In this paper we treat the case of bounded potentials, where analyticity is not to be expected, as the expansion is around the ideal gas and the negative of a stable potential is unstable, which causes divergence of the pressure for negative β in the finite volume.

2. Infinite Volume Correlation Functions

We assume the interaction potential ϕ to satisfy stability,

$$\sum_{\substack{i,j=1 \\ i<j}}^{m} \Phi(x_i - x_j) \geqq -mB \qquad \text{for some constant } B \tag{1}$$

and

$$\|\Phi\|_\infty < \infty, \tag{2}$$

$$\|\Phi\|_1 < \infty. \tag{3}$$

Eqs. (2) and (3) imply regularity ([2], ch. 4.1):

$$\int |e^{-\beta\Phi(x)} - 1| dx = C(\beta) < \infty \quad \text{for } \beta \in \mathbb{C}. \tag{4}$$

We shall use the representation of the correlation functions given by Ruelle ([2], ch. 4.2.):

On the Banach-spaces E_ξ, $\xi > 0$ of sequences of complex functions $\varphi =$

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$(\varphi(x)_n)_{n\geqq 1}$ with the norm

$$\|\varphi\|_\xi = \sup_{n\geqq 1}(\xi^{-n}\operatorname{ess\,sup}_{(x)_n\in\mathbb{R}^{\nu n}}|\varphi(x)_n|), \tag{5}$$

we define the operator $\mathbf{K}_\beta$ by

$$(\mathbf{K}_\beta\varphi)(x_1) = \sum_{n=1}^{\infty}\frac{1}{n!}\int d(y)_n K_\beta(x_1,(y)_n)\varphi(y)_n \tag{6}$$

$$(\mathbf{K}_\beta\varphi)(x)_m = \exp[-\beta W^i(x)_m] \cdot[\varphi(x)'_{m-1} + \sum_{n=1}^{\infty}\frac{1}{n!}\int d(y)_n K_\beta(x_i,(y)_n)\varphi((x)'_{m-1},(y)_n)], \tag{7}$$

where $(x)'_{m-1} = (x_1,\dots,\hat{x}_i,\dots,x_m)$, the kernel is given by

$$K_\beta(x_i,(y)_n) = \prod_{j=1}^{n}(\exp[-\beta\Phi(x_i-y_j)]-1) \tag{8}$$

and

$$W^i(x)_m = \sum_{\substack{j=1\\ j\neq i}}^{m}\Phi(x_i-x_j). \tag{9}$$

The index i in (7) depends on $(x_1,\dots,x_m)$ and is chosen so as to ensure

$$W^i(x)_m \geqq -2B, \tag{10}$$

which is always possible by (1).

For a linear mapping $\mathbf{A}: E_\eta \to E_\xi$ we define

$$\|\mathbf{A}\|_\eta^\xi = \sup_{\|\varphi\|_\eta=1}\|\mathbf{A}\varphi\|_\xi. \tag{11}$$

If $\operatorname{Re}\beta \geqq 0$, $\mathbf{K}_\beta$ is a bounded operator on E_ξ and

$$\|\mathbf{K}_\beta\|_\xi^\xi \leqq e^{2B\operatorname{Re}\beta}\xi^{-1}\exp[\xi C(\beta)]. \tag{12}$$

For

$$|z| < e^{-2B\operatorname{Re}\beta}\xi\exp[-\xi C(\beta)] \tag{13}$$

the sequence $\rho = (\rho(x)_n)_{n\geqq 1}$ of the infinite volume correlation functions belongs to E_ξ and can be written as

$$\rho(\beta,z) = (\mathbb{1} - z\mathbf{K}_\beta)^{-1}z\alpha, \tag{14}$$

with $\alpha(x_1) = 1$, $\alpha(x)_n = 0$ for $n > 1$.

3. Estimates

In this section we prove some estimates which we shall use to bound the β-derivatives of ρ.

Proposition 3.1. *For any $\varepsilon > 0$ there is a $R_T > 0$ such that $C(\beta) \leqq \varepsilon$ for $|\beta| \leqq R_T$.*

Proof. By

$$|e^{-\beta\Phi(x)} - 1| \leqq e^{|\beta||\Phi(x)|} - 1 \leqq e^{|\beta|\|\Phi\|_\infty}|\beta||\Phi(x)| \tag{15}$$

we have

$$\int |e^{-\beta\Phi(x)} - 1| dx \leqq |\beta| e^{|\beta| \, \|\Phi\|_\infty} \|\Phi\|_1 . \quad \square \tag{16}$$

Proposition 3.2. *For* $R_F > 0, d > 0, \xi > \eta > R_F + d$, *there is a* $R_T > 0$ *such that for* $|\beta| \leqq R_T, \zeta \in [\eta, \xi]$

$$e^{-2B\mathrm{Re}\,\beta}\zeta \exp[-\zeta C(\beta)] \geqq R_F + d. \tag{17}$$

Proof. By

$$e^{-2B\mathrm{Re}\,\beta}\zeta \exp[-\zeta C(\beta)] \geqq e^{-2B\mathrm{Re}\,\beta}\eta \exp[-\xi C(\beta)] \tag{18}$$

this follows from Proposition 3.1. $\square$

Proposition 3.3. *For* R_F, ξ, η, d, R_T *as in Proposition* 3.2.

$$L = \sup_{\zeta \in [\eta, \xi]} \| (\mathbb{1} - z\mathbf{K}_\beta)^{-1} \|_\zeta^\zeta \leqq 1 + \frac{R_F}{d}, \tag{19}$$

if $|z| \leqq R_F, \beta \in S_{R_T} = \{\beta \,|\, |\beta| \leqq R_T, \mathrm{Re}\,\beta \geqq 0\}, \zeta \in [\eta, \xi]$.

Proof. By Proposition 3.2. and (12)

$$\| \mathbf{K}_\beta \|_\zeta^\zeta \leqq \frac{1}{R_F + d}. \tag{20}$$

Thus the power series expansion of $(\mathbb{1} - z\mathbf{K}_\beta)^{-1}$ converges in the $\| \quad \|_\zeta^\zeta$– norm and

$$\| (\mathbb{1} - z\mathbf{K}_\beta)^{-1} \|_\zeta^\zeta \leqq \sum_{n=0}^{\infty} \left(\frac{R_F}{R_F + d} \right)^n = 1 + \frac{R_F}{d}. \ \square \tag{21}$$

Equation (14) yields

$$\| D_\beta^r \rho(\beta, z) \|_\xi \leqq \| D_\beta^r (\mathbb{1} - z\mathbf{K}_\beta)^{-1} \|_\eta^\xi |z| \, \|\alpha\|_\eta, \tag{22}$$

consequently

$$\| D_\beta^r \rho(\beta, z) \|_\xi \leqq r! \sum_{\substack{r_1, \dots, r_p \geqq 1 \\ \Sigma r_i = r}} |z|^p \| (\mathbb{1} - z\mathbf{K}_\beta)^{-1} \frac{D_\beta^{r_1}\mathbf{K}_\beta}{r_1!} (\mathbb{1} - z\mathbf{K}_\beta)^{-1} \cdot \dots \tag{23}$$

$$\cdot (\mathbb{1} - z\mathbf{K}_\beta)^{-1} \frac{D_\beta^{r_p}\mathbf{K}_\beta}{r_p!} (\mathbb{1} - z\mathbf{K}_\beta)^{-1} \|_\eta^\xi |z| \eta^{-1}$$

$$\leqq |z| \eta^{-1} L r! \sum_{\substack{r_1, \dots, r_p \geqq 1 \\ \Sigma r_i = r}} (|z| L)^p \prod_{i=1}^{p} \frac{\| D_\beta^{r_i}\mathbf{K}_\beta \|_{\zeta_{i-1}}^{\zeta_i}}{r_i!},$$

where we take

$$\zeta_i = \eta \cdot \left(\frac{\xi}{\eta} \right)^{(1/r) \cdot \sum_{j=1}^{i} r_j} \tag{24}$$

$D_\beta^r \mathbf{K}_\beta$ can be calculated explicitly:

$$(\mathrm{D}_\beta^r \mathbf{K}_\beta \varphi)(x_1) = \sum_{n=1}^{\infty} \frac{1}{n!} \int d(y)_n D_\beta^r K_\beta(x_1, (y)_n) \varphi(y)_n \tag{25}$$

$$(D_\beta^r \mathbf{K}_\beta \varphi)(x)_m = [-W^i(x)_m]^r \exp[-\beta W^i(x)_m]\varphi(x)'_{m-1} \tag{26}$$
$$+\sum_{s=0}^{r}\binom{r}{s}[-W^i(x)_m]^{r-s}\exp[-\beta W^i(x)_m]$$
$$\cdot\left[\sum_{n=1}^{\infty}\frac{1}{n!}\int d(y)_n D_\beta^s K_\beta(x_i,(y)_n)\varphi((x)'_{m-1},(y)_n)\right].$$

Proposition 3.4. *For any $R_T > 0$ there are constants $K_1, K_2(R_T)$ such that for $|\beta| \leqq R_T$*

$$\int d(y)_n |D_\beta^s K_\beta(x,(y)_n)| \leqq K_1^s s! K_2^n. \tag{27}$$

Proof

$$\int d(y)_n |D_\beta^s K_\beta(x,(y)_n)|$$
$$\leqq \sum_{\substack{s_1,\dots,s_n \geqq 0\\ \sum s_i = s}} \frac{s!}{s_1!\cdot\ldots\cdot s_n!}\prod_{i=1}^{n}\int dy_i |D_\beta^{s_i}(e^{-\beta\Phi(x-y_i)}-1)|$$
$$\leqq \sum_{l=1}^{\min(s,n)}\binom{n}{l}C(\beta)^{n-l}.$$
$$\cdot\sum_{\substack{s_1,\dots,s_l \geqq 1\\ \sum s_i = s}} \frac{s!}{s_1!\cdot\ldots\cdot s_l!}\prod_{i=1}^{l}\int dy|\Phi(x-y)|^{s_i}e^{-\mathrm{Re}\beta\Phi(x-y)}. \tag{28}$$

As

$$\int dy|\Phi(x-y)|^{s_i}e^{-\mathrm{Re}\,\beta\Phi(x-y)} \leqq \|\Phi\|_\infty^{s_i-1}e^{|\beta|\,\|\Phi\|_\infty}\|\Phi\|_1, \tag{29}$$

and

$$\sum_{\substack{s_1,\dots,s_l \geqq 1\\ \sum s_i = s}} \frac{s!}{s_1!\cdot\ldots\cdot s_n!} \leqq s! \sum_{\substack{s_1,\dots,s_l \geqq 1\\ \sum s_i = s}} 1 = s!\binom{s-1}{l-1} \leqq 2^s s!, \tag{30}$$

this leads to

$$\int d(y)_n |D_\beta^s K_\beta(x,(y)_n)|$$
$$\leqq (2\|\Phi\|_\infty)^s s! \sum_{l=1}^{n}\binom{n}{l}C(\beta)^{n-l}\left(\frac{e^{|\beta|\,\|\Phi\|_\infty}\|\Phi\|_1}{\|\Phi\|_\infty}\right)^l$$
$$\leqq (2\|\Phi\|_\infty)^s s!\left(C(\beta)+\frac{e^{|\beta|\,\|\Phi\|_\infty}\|\Phi\|_1}{\|\Phi\|_\infty}\right)^n$$
$$\leqq (2\|\Phi\|_\infty)^s s!\left[e^{R_T\|\Phi\|_\infty}\|\Phi\|_1\left(R_T+\frac{1}{\|\Phi\|_\infty}\right)\right]^n. \tag{31}$$

The last inequality follows from (16). □

Proposition 3.5. *For $\beta \in S_{R_T}$, $\varphi \in E_\zeta$*

$$\int d(y)_n |D^s_\beta K_\beta(x_1,(y)_n)\varphi((x)'_{m-1},(y)_n)| \\ \leqq \| \varphi \|_\zeta \zeta^{m-1} K^s_1 s![\zeta K_2(R_T)]^n. \tag{32}$$

$$|[-W^i(x)_m]^s \exp[-\beta W^i(x)_m]| \leqq e^{2BR_T}\left(\frac{K_1}{2}\right)^s (m-1)^s. \tag{33}$$

Proof.
(32) is obvious from Proposition 3.4. and (5), (33) from (10). □

Lemma 3.6. *For $e^{-1}\xi \leqq \eta < \xi$ there is a constant $K_3(R_T,\eta,\xi)$ such that for $\beta \in S_{R_T}$, $\xi \geqq \zeta_2 > \zeta_1 \geqq \eta$*

$$\| D^r_\beta \mathbf{K}_\beta \|^{\zeta_2}_{\zeta_1} \leqq K_3 (2K_1)^r \left(\log\frac{\zeta_2}{\zeta_1}\right)^{-r} r! \tag{34}$$

Proof. For $m>1$, by Proposition 3.5. and (26)

$$\begin{aligned} &|(D^r_\beta \mathbf{K}_\beta \varphi)(x)_m| \\ &\leqq e^{2BR_T} \|\varphi\|_{\zeta_1} \zeta_1^{m-1}\left[\left(\frac{K_1}{2}\right)^r (m-1)^r \right. \\ &\left. + \sum_{s=0}^{r} \binom{r}{s}\left(\frac{K_1}{2}\right)^{r-s}(m-1)^{r-s} K^s_1 s! \cdot \sum_{n=1}^{\infty} \frac{1}{n!}(\zeta_1 K_2)^n \right] \\ &\leqq e^{2BR_T+\zeta_1 K_2} \|\varphi\|_{\zeta_1} \zeta_1^{m-1} K^r_1 \left[(m-1)^r + \sum_{s=1}^{r}\binom{r}{s}(m-1)^{r-s} s!\right]. \end{aligned} \tag{35}$$

By

$$\sup_{m \geqq 1} m^s \left(\frac{\zeta_1}{\zeta_2}\right)^m \leqq \left(\log\frac{\zeta_2}{\zeta_1}\right)^{-s} e^{-s}s^s \leqq \left(\log\frac{\zeta_2}{\zeta_1}\right)^{-s} s! \tag{36}$$

we obtain

$$\| D^r_\beta \mathbf{K}_\beta \|^{\zeta_2}_{\zeta_1} \leqq \zeta_2^{-1} e^{2BR_T + \zeta_1 K_2(R_T)} K^r_1 r! \sum_{s=0}^{r}\left(\log\frac{\zeta_2}{\zeta_1}\right)^{-s} \tag{37}$$

$$\leqq \eta^{-1} e^{2BR_T+\xi K_2(R_T)}(2K_1)^r \left(\log\frac{\zeta_2}{\zeta_1}\right)^{-r} r!. \quad \square$$

Theorem 3.7. *For $z \in \mathbb{C}$, $|z| \leqq R_F$, $\xi > R_F$ there are constants $R_T > 0$, $K_4(R_T, R_F, \xi)$ such that for $\beta \in S_{R_T}$*

$$\| D^r_\beta \rho(\beta, z)\|_\xi \leqq |z| K^r_4 r!^2. \tag{38}$$

Proof. As $\xi > R_F$, we can find η, d, R_T as in proposition 3.2., $\eta \geqq e^{-1}\xi$. Thus, by

(23), (24) and lemma 3.6.

$$\|D^r_\beta \rho(\beta,z)\|_\xi \leqq |z|\eta^{-1}L(2K_1)^r r!. \tag{39}$$

$$\cdot \sum_{\substack{r_1,\dots,r_p\geqq 1\\ \Sigma r_i=r}} (R_F L K_3)^p \prod_{i=1}^{p}\left[\log\left(\frac{\xi}{\eta}\right)^{r_i/r}\right]^{-r_i}$$

$$\leqq |z|\eta^{-1}L[2K_1\max(1,R_F L K_3)]^r\left(\log\frac{\xi}{\eta}\right)^{-r} r!$$

$$\cdot \sum_{\substack{r_1,\dots,r_p\geqq 1\\ \Sigma r_i=r}} \prod_{i=1}^{p}\left(\frac{r}{r_i}\right)^{r_i}$$

$$\leqq |z|\eta^{-1}L[4eK_1\max(1,R_F L K_3)]^r\left(\log\frac{\xi}{\eta}\right)^{-r} r!^2,$$

where we have used

$$\sum_{\substack{r_1,\dots,r_p\geqq 1\\ \Sigma r_i=r}} \prod_{i=1}^{p}\left(\frac{r}{r_i}\right)^{r_i} \leqq r^r \sum_{\substack{r_1,\dots,r_p\geqq 1\\ \Sigma r_i=r}} 1 \leqq r^r \sum_{p=1}^{r}\binom{r-1}{p-1} \leqq (2e)^r r!. \quad \Box \tag{40}$$

Theorem 3.8. *For z, β, ξ as in Theorem 3.7.*

$$|D^r_\beta(\beta p(\beta,z))| \leqq |z|\xi K_4^r r!^2, \tag{41}$$

where $p(\beta,z)$ is the thermodynamic limit of the pressure.

Proof. From Theorem 3.7. it follows that $\rho_1(x)$, which is translation-invariant, i.e. a constant, satisfies

$$|D^r_\beta \rho_1(\beta,z)| \leqq |z|\xi K_4^r r!^2. \tag{42}$$

For $|z| < e^{-2B\mathrm{Re}\,\beta}\xi\exp[-\xi C(\beta)]$ $\dfrac{\rho_1(\beta,z)}{z}$ is analytic in z by (14) and

$$\beta p(\beta,z) = \int_0^z \frac{dz'}{z'}\rho_1(\beta,z') \tag{43}$$

(see [2], ch.4.3.). Consequently

$$|D^r_\beta(\beta p(\beta,z))| \leqq \left|\int_0^z dz'\,\xi K_4^r r!^2\right| = |z|\xi K_4^r r!^2. \quad \Box \tag{44}$$

4. Borel-summability

Theorem 4.1. *For $z\in\mathbb{C}$, $|z| \leqq R_F$, $\xi > R_F$, f a continuous linear functional on E_ξ there is a $R_T > 0$ such that for β in the circle $C_{R_T} = \{\beta \mid \mathrm{Re}\,\beta^{-1} \geqq R_T^{-1}\}$ the Borel-sums of the Taylor-series in β at the origin for the functions*

$$f(\rho(\beta,z)): S_{R_T} \to \mathbb{C}$$
$$\beta p(\beta,z): S_{R_T} \to \mathbb{C}$$

converge absolutely and uniformly in β, z.

Proof. This follows from Theorems 3.7., 3.8. and Nevanlinna's theorem ([3], see also [4]). $\Box$

Corollary 4.2. *Let R_F, ξ, R_T be as in Theorem 4.1., A a set of finite Lebesgue-measure in* $\mathbb{R}^{\nu n}$ (e.g. *a product of balls centered at points* $x_1, \ldots, x_n$). *Then the Borel-sum of the Taylor-series in β at the origin for*

$$\int_A \rho((x)_n; \beta, z) d(x)_n$$

converges absolutely and uniformly in β, z for $\beta \in C_{R_T}$, $|z| \leqq R_F$.

Proof. This is a direct consequence of Theorem 4.1., as the mapping

$$\varphi \mapsto \int_A \varphi(x)_n d(x)_n$$

is obviously a continuous linear functional on E_ξ. □

Remark. If Φ is a continuous function, the $\rho((x)_n; \beta, z)$ are continuous in $(x)_n$ ([2], p.79) and we can replace the integral in corollary 4.2. by $\rho((x)_n; \beta, z)$, $(x)_n$ fixed.

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