Commun. Math. Phys. 80, 343–367 (1981)

Communications in
Mathematical Physics

The Existence of Multi-Monopole Solutions to the Non-Abelian, Yang–Mills–Higgs Equations for Arbitrary Simple Gauge Groups

Clifford Henry Taubes* †

Department of Physics, Harvard University, Cambridge, MA 02138, USA

Abstract. We prove that for arbitrary simple gauge groups, the non-Abelian Yang–Mills–Higgs Equations on $\mathbb{R}^3$ in the Prasad–Sommerfield limit have at least a countably infinite set of distinct solutions. These solutions may be interpreted physically as configurations of widely spaced, non-interacting fundamental monopoles. The solutions are generically not spherically symmetric.

1. Introduction

There is an interest in smooth, classical solutions to non-Abelian, Yang–Mills–Higgs equations on Euclidean $\mathbb{R}^3$. The finite action solutions are solitions in a 4-dimensional, Minkowski space theory with spontaneous symmetry breaking; they are known as magnetic monopoles [1]. Until recently, the only solutions resulted from imposing a spherically symmetric ansatz of one form or another [2, 3, 4]. In this paper, we show that if the gauge group is compact, simply connected and simple (with the Higgs scalars in the adjoint representation) and the Prasad–Sommerfield [5] limit of vanishing scalar field self-interaction is taken, there is at least a countably infinite set of distinct solutions. These solutions are generically, not spherically symmetric. They may be interpreted as configurations consisting of arbitrary numbers of the spherically symmetric monopoles previously mentioned.

In a recent paper (see Chap. IV, [6], the author established sufficient conditions for existence of finite action solutions to the Yang–Mills–Higgs equations in the Prasad–Sommerfield limit. These criteria were applied (in [6.IV]) in the case where the gauge group is $SU(2)$ to produce an existence proof for multimonopoles in this $SU(2)$ case. In this paper we apply the criteria of [6.IV] to the general case

* Harvard University, John Parker Fellow

† This work is supported in part by the National Science Foundation under Grant No. PHY 79–16812

0010-3616/81/0080/0343/$05.00

where the gauge group is an arbitrary simple, simply connected, compact Lie group. This paper may be considered as a sequel to [6.IV].

The paper is arranged as follows: Sect. 2 begins by establishing notation and reviewing the properties of the Prasad–Sommerfield limit. The section concludes with the statement of Theorems I and II, which give the existence of multimonopole solutions. The proofs of Theorems I and II are contained in Sect. 3, 4 and 5. In particular Sect. 3 contains a restatement of the necessary results from [6.IV] and a theorem which details sufficient conditions under which two exact solutions can be used to find a third distinct solution. Sections 6–8 are concerned with asymptotic decay estimates, cf. Theorem V.

2. The First-Order Equations

Let $\mathscr{G}$ be a simple, simply connected, compact Lie group and let g denote its Lie algebra. Chose a positive definite, ad-invariant inner product on g and denote the associated by-linear form by $\langle \eta\sigma \rangle$ for $\eta, \sigma \in g$. That is, $\langle\rangle$ is a normalized trace. Let $E = \mathbb{R}^3 \times g$ be the vector bundle associated with the principal $\mathscr{G}$-bundle $\mathbb{R}^3 \times \mathscr{G}$ via the adjoint representation. Because E is flat, the space of smooth connections in E is identical to the space of smooth sections of $T^* \otimes g$; i.e., the space of smooth Lie algebra valued 1-forms. We are concerned with critical points of the action functional

$$\mathscr{a}(A, \Phi) = \frac{1}{2} \int_{\mathbb{R}^3} \langle \Omega_A \wedge *\Omega_A + D_A\Phi \wedge *D_A\Phi \rangle, \tag{2.1}$$

where

$$\Omega_A = dA + A \wedge A,$$
$$D_A\Phi = d\Phi + [A, \Phi],$$

for $(A, \Phi) \in C^\infty(\mathbb{R}^3; T^* \otimes g) \oplus C^\infty(\mathbb{R}^3; g)$. In particular, critical points of $\mathscr{a}(\cdot)$ which satisfy the auxiliary asymptotic condition

$$\lim_{|x| \to \infty} \langle \Phi\Phi(x) \rangle \to 1 \text{ (uniformly)} \tag{2.2}$$

In Eq. (2.1), Ω_A is the curvature of the connection A and $D_A\Phi$ is the covariant derivative of the section Φ.

The variational equations of $\mathscr{a}(\cdot)$ are

$$*D_A*\Omega_A + [\Phi, D_A\Phi] = 0,$$
$$D_A*D_A\Phi = 0. \tag{2.3}$$

As in the case where $g = su(2)$, the set of $(A, \Phi) \in C^\infty(\mathbb{R}^3; T^* \otimes g) \oplus C^\infty(\mathbb{R}^3; g)$ which satisfy added asymptotic conditions decomposes into disjoint subsets labelled by homotopy classes of maps of S^2 into quotient manifolds formed from $\mathscr{G}$ [7]. Let $\hat{x}$ denote a point on the unit sphere S^2. Let

$$\tilde{\Phi}(\hat{x}) = \lim_{t \to \infty} \Phi(t\hat{x})$$

denote a map from the sphere S^2 at infinity in $\mathbb{R}^3$ into g. Furthermore, let $\Phi(p_0)$

denote the asymptotic value of Φ on the positive x_1 axis, $\Phi(p) = \hat{\Phi}(1, 0, 0)$. Then define the Lie subgroup $\mathcal{J}$ as

$$\mathcal{J} = \{g \in \mathcal{G} : g^{-1}\Phi(p)g = \Phi(p)\}. \tag{2.4}$$

Proposition 2.1 (*Theorem II.3.1 of* [6]). *Let A be a continuous connection on $P = \mathbb{R}^3 \times \mathcal{G}$ and $\Phi a C^1$ section of $\mathbb{R}^3 \times \mathfrak{g}$. Assume that*

$$\lim_{R \to \infty} \sup_{|x| = R} (1 - |\Phi|) = 0 \tag{2.5a}$$

and that for some $\delta > 0$,

$$|x|^{1+\delta} |D_A \Phi| \leqq \text{const.} \tag{2.5b}$$

Then

(a) *There exists a gauge such that $\tilde{\Phi}(\hat{x})$ is a continuous map from S^2 into $\mathfrak{g}$.*

(b) *The configuration (A, Φ) defines a homotopy class $[(A, \Phi)] \in \Pi_2(\mathcal{G}/\mathcal{J})$.*

(c) *The class $[(A, \Phi)]$ is invariant under C^1 gauge transformations.*

(d) *Suppose (a, ϕ) are respectively a C^0 $\mathfrak{g}$-valued 1-form and a C^1 section of $\mathbb{R}^3 \times \mathfrak{g}$ which satisfy*

$$\lim_{R \to \infty} \sup_{|x| = R} |\Phi| = \lim_{R \to \infty} \sup_{|x| = R} |x| \, |a| = 0. \tag{2.5c}$$

Then

$$[(A + a, \Phi + \phi)] = [(A, \Phi)]. \qquad \square$$

The class $[(A, \Phi)]$ can be completely specified by $\ell - r \leqq \ell = \text{rank } \mathcal{G}$ integers $\{n_a\}_{a=1}^{\ell - r}$ [8].

These integers in turn are completely determined by surface integrals. [8]

For the cases encountered in this paper, it is convenient to compute the integers $\{n_a\}_{a=1}^{\ell - r}$ differently. Goddard, Nuyts and Olive show that if in addition to (2.5 a, b) one has for some $\delta > 0$,

$$\lim_{|x| \to \infty} 4\pi |x|^2 * \Omega_A(x) = d|x| \beta(x/|x|),$$
$$\lim_{|x| \to \infty} |x|^{1+\delta} |D_A \beta| = 0, \tag{2.6}$$

and

$$\lim_{|x| \to \infty} [\beta(x/|x|), \Phi(x)] = 0,$$

then the integers $\{n_a\}$ may be computed from the value of $\beta(x/|x|)$ at $(1, 0, 0)$. Let

$$\beta(P) = \beta(1, 0, 0). \tag{2.7}$$

The element $\beta(P) \in \mathfrak{g}$ is constrained by

$$\exp(\beta(P)) = 1. \tag{2.8}$$

The constraint (2.8) allows the class $[(A, \Phi)]$ to be characterized in the following way: Choose a maximal torus [9] $T \subset \mathcal{G}$ with generators $(T_1 = \Phi(P), T_2, \ldots, T_\ell) = \vec{T}$, where $\ell = \text{rank } \mathcal{G}$. Let Λ denote the root system of $\mathcal{G}$. Two situations can arise. The first occurs when $\langle \vec{\alpha}\cdot\vec{T}\Phi(P)\rangle \neq 0$ for all $\vec{\alpha}\in\Lambda$. In this case, $\mathscr{J} \approx T$.

We define the positive roots $\Lambda_+ \subset \Lambda$ by the condition $\vec{\alpha}\in\Lambda_+$ iff $\langle \vec{\alpha}\cdot\vec{T}\Phi(P)\rangle > 0$. The positive roots Λ_+ uniquely define the simple roots $\{\vec{\beta}^1, \dots, \vec{\beta}^\ell\} \subset \Lambda_+$. Denote by Λ^* the dual lattice to Λ. There is a basis $\{{}^*\beta^a\}_{a=1}^{\ell}$ of Λ^* such that $({}^*\vec{\beta}^a, \vec{\beta}^b) = \delta^{ab}$; $a, b = \{1, \dots, \ell\}$ where $(\,,)$ is the natural pairing between Λ and Λ^*.

Because $[\beta(P), \Phi(P)] = 0$, and $\mathscr{J}$ is a maximal torus,

$$B(P) = \sum_i b_i T^i = b\cdot\vec{T}. \tag{2.9}$$

Condition (2.8) can be restated as the requirement that $\vec{b}\in\Lambda^*$ and

$$\vec{b} = 4\pi \sum_{a=1}^{\ell} n_a {}^*\vec{\beta}^a; \quad n_a\in\mathbb{Z}. \tag{2.10}$$

The set of integers $\{n_a\}_{a=1}^{\ell}$ is gauge invariant and completely specifies the class $[(A, \Phi)]\in\Pi_2(\mathscr{G}/\mathscr{J})$.

The second possibility is that the generator $T_1 = \Phi(P)$ is orthogonal to some of the roots $\vec{\alpha}$. In this case the group $\mathscr{J}$ is homotopic to the direct product of a torus T' and a semi-simple Lie subgroup $\mathscr{G}' \subset \mathscr{G}$. Denote by j the Lie algebra of $\mathscr{J}$ and $j_{\mathbb{C}}$ the complexification of j. The algebra $j_{\mathbb{C}}$ is generated by $T_1 = \Phi(P)$, $T_2, \dots, T_\ell, E_{\vec{\alpha}_1}, \dots, E_{\vec{\alpha}_p}$, where $T_1, \dots, T_\ell$ are as in the previous case and $E_{\vec{\alpha}_1}, \dots, E_{\vec{\alpha}_p}$ are eigenvectors in $g_{\mathbb{C}}$ corresponding to the p roots $\{\vec{\alpha}_1, \dots, \vec{\alpha}_p\}\in\Lambda$ orthogonal to $\Phi(P)$. The generators $\{T_1, \dots, T_{\ell-r}\}$ generate the torus T'. A positive root system, $\Lambda_+ \subset \Lambda$ is chosen by requiring $\langle \vec{\alpha}\cdot\vec{T}\Phi(P)\rangle \geq 0$ for $\vec{\alpha}\in\Lambda_+$. The simple roots of Λ_+ are denoted $\{\vec{\beta}^1, \dots, \vec{\beta}^{\ell-r}, \vec{\beta}^{\ell-r+1}, \dots, \vec{\beta}^\ell\}$ where $\langle \vec{\beta}^a\cdot\vec{T}(P)\rangle > 0$ for $a = 1, \dots, \ell - r$ and $\langle \vec{\beta}^a\cdot\vec{T}(P)\rangle = 0$ for $a = \ell - r + 1, \dots, \ell$. There exists an element $g\in\mathscr{G}$ such that $g^{-1}\beta(P)g$ is of the form (2.9). The element g is not unique, the ambiguity in G is expressed in the fact that the vector $\vec{b}$ defined by (2.9) must still satisfy (2.10) but only the set of integers $\{n_a\}_{a=1}^{\ell-r}$ are gauge invariant. This set of integers completely specifies the class $[(A, \Phi)]\in\Pi_2(\mathscr{G}/\mathscr{J})$ (see [8].)

A gross classification of the solutions to (2.3) which satisfy (2.2) and (2.4) is given by specifying the set of integers $\{n_a\}_{a=1}^{\ell-r}$ in the decomposition (2.10). The principal result of this paper is that there are infinitely many distinct solutions for any choice of $\Phi(P)$ and non-negative set of integers $\{n_a\}_{a=1}^{\ell-r}$ satisfying

$$\sum_{a=1}^{\ell-r} n_a > 0.$$

We define a set $\mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r})$ indexed by a unit vector $h\in g$ and a set of integers $\{n_a\}_{a=1}^{\ell-r}$ in the following way:

Definition. $\mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r}) = \{(A, \phi)\in C^\infty(\mathbb{R}^3; T^*\otimes g)\oplus C^\infty(\mathbb{R}^3; g)$
which satisfy:

(a) $\mathscr{A}(A, \Phi) < \infty$.

(b) Equations (2.5a, b) with $\Phi(p) = h$.

(c) With $\mathscr{J} = \{g\in\mathscr{G} | g^{-1}\Phi(p)g = \Phi(p)\}$, the class $[(A, \Phi)]\in\Pi_2(\mathscr{G}/\mathscr{J})$ is completely specified by the integers $\{n_a\}_{a=1}^{\ell-r}$; this as discussed above.$\}$ □

If the two vectors h and h' are conjugate, that is $h = g^{-1}h'g$ for some $g \in \mathscr{G}$, then

$$\mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r}) = g^{-1}\mathscr{H}(h', \{n_a\}_{a=1}^{\ell-r})g.$$

Instead of finding solutions to Eq. (2.3) directly, we shall prove the existence of solutions to the first-order equations

$$*\Omega_A = D_A \Phi. \tag{2.11a}$$

Any solution to (2.11a) is a solution to (2.3). However, if (A, ϕ) is a solution to (2.11a) and in some $\mathscr{H}(h, \{n_a\}_{a=1}^{\ell-1})$, there is a constraint on the possible values of the integers $\{n_a\}_{a=1}^{\ell-r}$:

$$\sum_{a=1}^{\ell-r} n_a \langle * \vec{\beta}^a \cdot \vec{T} h \rangle \geqq 0. \tag{2.11b}$$

Because the action is invariant under the transformation $\Phi \to -\Phi$, every solution to (2.11a) generates two distinct solutions to (2.3). If $(A, \Phi) \in \mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r})$. then $(A, -\Phi)$ is gauge equivalent to an alement in $\mathscr{H}(h, \{-n_a\}_{a=1}^{\ell-r})$. This may be seen most easily by noting that the simple roots defined by $-h$ are -1 times the simple roots defined by h.

Theorem 1. *For every choice of $\mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r})$ such that the set $\{n_a\}_{a=1}^{\ell-r}$ is non-trivial and non-negative there exists at least a countably infinite set of distinct, gauge inequivalent solutions to (2.11a) in $\mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r})$.* □

In Theorem II, we will be more specific about the structure of the solutions given by Theorem I. Before stating Theorem II, we review certain facts about the the $O(3)$ symmetric solutions to Eq. (2.11), [3,10,11].

The group of rotations, $O(3)$, acts on $\mathbb{R}^3$ in the usual way. An action of $O(3)$ on the $\mathscr{G}$-vector bundle $(\mathbb{R}^3 \times T^* \otimes \mathscr{g}) \oplus (\mathbb{R}^3 \otimes \mathscr{g})$ can be defined by giving a lifting of the action of $O(3)$ to the principal bundle $\mathbb{R}^3 \times \mathscr{G}$ which commutes with the natural projection $p: \mathbb{R}^3 \times \mathscr{G} \to \mathbb{R}^3$. Such a lifting is uniquely defined by a homomorphism $L: \mathfrak{su}(2) \to \mathscr{g}$ [4].

The lie algebra $\mathfrak{su}(2)$ has a realization by the differential operators

$$\mathscr{L}_i = -\varepsilon^{ijk} x_j \frac{\partial}{\partial x_k} \tag{2.12}$$

on $C^\infty(\mathbb{R}^3)$. Let $L_i = L(\mathscr{L}_i)$. We assume that the homomorphism $L: \mathfrak{su}(2) \to \mathscr{g}$ is not trivial. A configuration $(A, \Phi) \in C^\infty(\mathbb{R}^3; T^* \otimes \mathscr{g}) \oplus C^\infty(\mathbb{R}^3; \mathscr{g})$ is said to be $O(3)$ symmetric iff

$$(L_i + \mathscr{L}_i)(\Phi) = 0 \tag{2.13a}$$

$$(L_i + \mathscr{L}_i)(A_j dx^j) = \varepsilon_{ijk} A_k dx^j. \tag{2.13b}$$

Given the homomorphism L, $\mathscr{g}$ decomposes into a direct sum of irreducible representations of $\mathfrak{su}(2)$. We denote the integer representations $(\Theta_1, \ldots, \Theta_t)$; each has dimension $(2\ell_j + 1)_{j=1}^t$. Each Θ_j is composed of matrices

$$\mathbb{Y}_m^{\ell_j}, m = -\ell_j, -\ell_j + 1, \ldots, \ell_j.$$

With

$$[L_3, \mathbb{Y}^{\ell_j}_m] = -im\mathbb{Y}^{\ell_j}_m,$$

$$\sum_{a=1}^{3} [L_a[L_a, \mathbb{Y}^{\ell_j}_m]] = -\ell_j(\ell_j+1)\mathbb{Y}^{\ell_j}_m. \tag{2.14}$$

Denote by $(\{Y^{\ell}_m(\hat{x})\}^{\ell}_{m=-\ell})^{\infty}_{\ell=0}$ the standard spherical harmonics; where $\hat{x}^j = x^j/|x|$.

The most general solution [3] to (2.13a) is

$$\Phi(r,\theta,\chi) = \sum_{\alpha=1}^{t} \phi_\alpha(r) \sum_{m=-\ell_\alpha}^{\ell_\alpha} Y^{\ell_\alpha}_{-m}(\phi,\chi)\mathbb{Y}^{\ell_\alpha}_m. \tag{2.15}$$

Using the residual gauge freedom in the ansatz (2.13), the most general solution [3] to (2.13b) may be written down in the following way: Define for $\alpha = 1, \dots, t$,

$$\mathbb{P}_\alpha(\theta,\chi) = \sum_{m=-\ell_\alpha}^{\ell_\alpha} Y^{\ell_\alpha}_{-m}(\theta,\chi)\mathbb{Y}^{\ell_\alpha}_m. \tag{2.16}$$

The generic form of the $O(3)$ symmetric connection as defined by L is:

$$A = \frac{1}{r}\left(\sum_{\alpha=1}^{t} (a_{1\alpha}\varepsilon^{inm}[L_m, \mathbb{P}_\alpha]\hat{x}^n + a_{2\alpha}[L_i, \mathbb{P}_\alpha]) - L_i\right)\varepsilon^{ijk}\hat{x}^j dx^k \tag{2.17}$$

where $(a_{1\alpha}, a_{2\alpha})^t_{\alpha=1}$ are functions of r only.

Given a homomorphism $L: su(2) \to g$ and a set $\mathscr{H}(h, \{n_a\}^{\ell-r}_{a=1})$, the complete set $(A,\Phi) \in \mathscr{H}(h; \{n_a\}^{\ell-r}_{a=1})$ of the form (2.15–2.17) which satisfies equation (2.11) has not been catalogued. Such a catalogue is not the purpose of this paper.

Given a homomorphism L and set $\mathscr{H}(h, \{n_a\}^{\ell-r}_{a=1})$, define a new set

$$\mathscr{C}(L, h, \{n_a\}^{\ell-r}_{a=1}) = \Big\{ (A,\Phi) \in \mathscr{H}(h, \{n_a\}^{\ell-r}_{a=1}) \text{ such that}$$

1) (A,Φ) is defined by (2.15–2.17) and the homomorphism L.
2) (A,Φ) satisfies (2.11) with $n_a \geqq 0$

$$(a = 1, \dots, \ell - r) \text{ and } \sum_{a=1}^{\ell-r} n_a > 0. \tag{2.18}$$

3) There exist constants $0 \leqq C_0(A,\Phi) < \infty$ and $m(A,\Phi) > 0$ such that

$$\sum_{\alpha=1}^{t} (|a_{1\alpha}|^2 + |a_{2\alpha}|^2)^{1/2} \leqq C_0(A,\Phi)\exp(-m(A,\Phi)|x|).\Big\}$$

We remark that if $L: su(2) \to g$ and h define a set $\mathbb{C}(L, h, \{n_a\}^{\ell-r}_{a=1})$ then necessarily $[L_3, h] = 0$.

In the Appendix we express Eq. (2.11) in terms of the variables $(\phi_\alpha, a_{1\alpha}, a_{2\alpha})^t_{\alpha=1}$ defined by (2.15) and (2.17). We prove in the Appendix the following:

Proposition 2.2. *If* $(A,\Phi) \in \mathscr{C}(L, h, \{n_a\}^{\ell-r}_{a=1})$ *then there exist constants* $0 \leqq C_n(A,\Phi) < \infty$, $(n = 0, 1, \dots, \infty)$ *such that*

a) $$\left(\sum_{\alpha=1}^{t}\left(\left|\frac{d^n a_{1\alpha}}{dr^n}\right|^2 + \left|\frac{d^n a_{2\alpha}}{dr^n}\right|^2\right)\right)^{1/2} \leqq C_n(A,\Phi)\exp(-m(A,\Phi)|x|).$$

b) *Along the positive z axis, with* $|z| > 1$,

$$\Phi(z,0,0) = \Phi(P) + \frac{L_3}{z} + \sum_{\alpha=1}^{t} X_\alpha(z) \mathbb{Y}_0^{\ell_\alpha}$$

and

$$\left(\sum_{\alpha=1}^{t} \left| \frac{d^n}{dr^n} X_\alpha(r) \right|^2 \right)^{1/2} \leqq C_n(A,\Phi) \exp(-2m(A,\Phi)|x|). \tag{2.19}$$

We note that since $(A,\Phi) \in \mathscr{C}(L, h, \{n_a\}_{a=1}^{\ell-r})$ must be smooth, $\Phi(0) = 0$.

The content of Theorem II is that given certain sets $\{(A_i, \Phi_i) \in \mathscr{C}(L^{(i)}, h, \{n_a^{(i)}\}_{a=1}^{\ell-r})\}_{i=1}^K$ there exists a solution (A, Φ) to (2.9) which may be interpreted as a configuration of widely spaced, non-interacting, spherically symmetric monopoles given by $(A_i, \Phi_i)_{i=1}^K$.

Theorem II. *Given a finite set* $\Lambda = \{(A_i, \Phi_i) \in \mathscr{C}(L^{(i)}, h, \{n_a^{(i)}\}_{a=1}^{\ell-r}\}$ *such that* $[L_3^{(i)}, L_3^{(j)}] = 0$, $i,j = 1, \dots, K$, *there exists a constant* $\infty > d_0 \geqq 0$ *such that the following is true: For any set of K points* $\{x_1, \dots, x_K\}$ *satisfying*

$$|x_i - x_j| \geqq d > d_0, \quad i \neq j = 1, \dots, K$$

there is a solution (A, Φ) *to* (2.11) *with*

$$(A,\Phi) \in \mathscr{H}\left(h; \left\{ \sum_{i=1}^{K} n_a^{(i)} \right\}_{a=1}^{\ell-n} \right). \tag{2.20}$$

*) *Further, there exist constants* $\alpha, \beta > 0$ *which are independent of d such that with* $R(d) = \alpha \ell n d$ *and* $\varepsilon(d) = \beta d^{-1/2}$ *the following is true:*

In each open ball

$$B_{R(d)}(x_i) = \{x \in \mathbb{R}^3 \,|\, |x - x_i| < R(d)\},$$

there exists a smooth gauge transformation

$$g_i : B_{R(d)}(x_i) \to \mathscr{G}$$

such that

$$|A(x) - g_i^{-1} A_i(x - x_i) g_i - g_i^{-1} dg_i|^2 + |\Phi(x) - g_i^{-1} \Phi_i(x - x_i) g_i|^2 < \varepsilon^2(d) \qquad \square \tag{2.21}$$

In a recent paper, [12], Weinberg introduced the notion of "fundamental" solutions to (2.11a, b). These are solutions to the first order equations obtained from embeddings of the $SU(2)$ Prasad–Sommerfield solution defined by the simple roots. Specifically let h be an ℓ dimensional unit vector in $\mathscr{g}$ satisfying $\langle \vec{\beta}^a \cdot \vec{T} h \rangle \neq 0$, $a = 1, \dots, \ell$ for all simple roots $\vec{\beta}^a$. Define

$$\begin{aligned} L_3^{(a)} &= \vec{\beta}^a \cdot \vec{T} (\vec{\beta}^a \cdot \vec{\beta}^a)^{-1} \\ L_1^{(a)} &= \tfrac{1}{2}(E_{\vec{\beta}^a} + E_{-\vec{\beta}^a})(\vec{\beta}^a \cdot \vec{\beta}^a)^{-1/2} \\ L_2^{(a)} &= \frac{i}{2}(E_{\vec{\beta}^a} - E_{-\vec{\beta}^a})(\vec{\beta}^a \cdot \vec{\beta}^a)^{-1/2}. \end{aligned} \tag{2.22}$$

The elements $\{L_j^{(a)}\}_{j=1}^3 \in \mathscr{g}$ generate an $su(2)$ subalgebra and define a homomorphism $L^{(a)}: su(2) \to \mathscr{g}$. Given h, there are ℓ fundamental solutions, one for each simple root. They are given by

$$\begin{aligned} \Phi^{(a)}(x) &= -\left(\lambda_{(a)} \coth \lambda_{(a)}|x| - \frac{1}{|x|}\right)\frac{x^k}{|x|}L_k^{(a)} \\ &\quad - (h - \langle \vec{\beta}^a \cdot \vec{T} h \rangle L_3^{(a)}) \\ A^{(a)}(x) &= -\left(\frac{\lambda_{(a)}}{\sinh \lambda_{(a)}|x|} - \frac{1}{|x|}\right)\varepsilon^{ijk}\frac{x^k}{|x|}dx^j L_i^{(a)} \end{aligned} \tag{2.23}$$

where

$$\lambda_{(a)} = \langle \vec{\beta}^a \cdot \vec{T} h \rangle.$$

If the little group $\mathscr{J}$ is the maximal torus, the dimension of the space of moduli to a solution $(A, \Phi) \in \mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r})$ of (2.11a,b) was calculated by Weinberg. He found that if ℓ' of the integers $n_a = 0$ and $\ell - \ell'$ of the $n_a > 0$, the dimensions of the space of moduli is $4\sum_a n_a - \ell + \ell'$. This is consistent with the physical interpretation of a solution $(A, \Phi) \in \mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r})$ as a configuration of n_a monopoles of the form (2.23). This is partially verified by the following corollary to Theorem II:

Corollary 2.3. *Let h be a unit vector in $\mathscr{g}$ orthogonal to $0 \leqq r < \ell$ simple roots and positive inner product with the $\ell - r$ remaining simple roots,* e.g., $\{\langle \vec{\beta}^a \cdot \vec{T}^h \rangle > 0\}_{a=1}^{\ell-r}$ *and* $\{\langle \vec{\beta}^a \cdot \vec{T} h \rangle = 0\}_{a=\ell-r+1}^{\ell}$. *For every set of $\ell - r$ non-negative integers $\{n_a\}_{a=1}^{\ell-r}$ there exists at least a $3\sum_{a=1}^{\ell-r} n_a$ dimensional sublattice of distinct solutions to* (2.11 *a, b*) *in $\mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r})$. In fact, there exists a number $d_0(h) < \infty$ such that for every set of $N = \sum_{a=1}^{\ell-r} n_a$ points $\{x_1^{(a)}, \ldots, x_{n_a}^{(a)}\}_{a=1}^{\ell-r} \in \mathbb{R}^3$ satisfying*

$$\begin{aligned} |x_i^{(a)} - x_j^{(b)}| \geqq d > d_0(h) \text{ with } i \neq j \text{ if } a = b \\ \text{and } i \in \{1, \ldots, n_a\} \\ j \in \{1, \ldots, n_b\}, \end{aligned} \tag{2.24}$$

there is a smooth solution $(A, \Phi) \in \mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r})$ to (2.11).

*Furthermore, the configuration (A, Φ) satisfies statement *) of Theorem II if one substitutes $x_i^{(a)}$ for x_i and $(A^{(a)}, \Phi^{(a)})$ for (A_i, Φ_i) in* (2.21). □

Proof of the Corollary. The Corollary follows directly from Theorem II as, for $a, b \in \{1, \ldots, \ell - r\}$, each

$$(A^{(a)}, \Phi^{(a)}) \in \mathscr{C}(L^{(a)}, h, \{n_b = \delta_{ab}\}_{b=1}^{\ell-r}) \text{ and } [L^{(a)}, L^{(b)}] = 0.$$

We note that the Corollary implies Theorem I.

3. Proof of Theorem II: Analytic Theorems

Let $(A_0, \Phi_0) \in \mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r})$ be a fixed configuration. We write an arbitrary $(A, \Phi) \in \mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r})$ as

$$(A, \Phi) = (A_0 + a, \Phi_0 + \phi). \tag{3.1}$$

If (A, Φ) is a solution to (2.11a, b) then (a, ϕ) satisfies

$$\varepsilon^{ijk} D_j a_k - D_i \phi + [\Phi_0, a_i] = -(*\Omega_{A_0} - D_{A_0}\Phi_0)_i - \varepsilon^{ijk} a_j a_k + [a_i, \phi],$$

where $a = a_i dx^i$ and

$$D_j = \nabla_j + [A_{0_j}, \circ]. \tag{3.2}$$

In order for (3.2) to be an elliptic first-order system, we use the gauge freedom to also require that (a, ϕ) satisfy the background gauge condition

$$D_i a_i + [\Phi_0, \phi] = 0. \tag{3.3}$$

To treat Eqs. (3.2, 3.3) as a single elliptic system for (a, ϕ), we follow [6] in defining a $\mathscr{g}$-valued quaternion

$$\psi = \phi + \tau_j a_j, \tag{3.4}$$

where $\tau_j \tau_k = -\delta_{jk} - \varepsilon_{ijk}\tau_k$, $\tau_j^* = -\tau_j$ and $[\tau_j, \mathscr{g}] = 0$. For example, $\{\tau_j = i\sigma_j\}_{j=1}^3$, with $\{\sigma_j\}_{j=1}^3$ the Pauli matrices.

We define a first-order elliptic operator

$$\mathscr{D} = -\tau_j D_j + [\Phi_0, \cdot]. \tag{3.5}$$

The contents of equations (3.2) and (3.3) are

$$\mathscr{D}\psi = G_0 - \psi \wedge \psi, \tag{3.6}$$

where we define

$$G_0 = \tau_j G_j \equiv -\tau_j (*\Omega_{A_0} - D_{A_0}\Phi_0)_j, \tag{3.7a}$$

and for $u, v \in \mathscr{g} \otimes Q$,

$$u \wedge v \equiv \tfrac{1}{2}\tau_i\{-[u_i, v_0] - [v_i, u_0] + \varepsilon^{ijk}(u_j v_k + v_j u_k)\},$$

hence,

$$\psi \wedge \psi = -\tau_j [a_j, \phi] + \tau_i \varepsilon^{ijk} a_j a_k. \tag{3.7b}$$

Theorem III (Theorem IV.2.1 of [6]) describes sufficient conditions on (A_0, Φ_0) for (3.6) to have solutions.

Before stating Theorem III, we introduce the Hilbert space $H(A_0, \Phi_0)$. If $u = u_0 + \tau_j u_j$ is a $\mathscr{g}$-valued quaternion, we define $u^* = u_0 - \tau_j u_j$. An inner product on the vector space of $\mathscr{g}$-valued quaternions is defined by

$$\langle\langle a, b \rangle\rangle = \langle a^* b \rangle. \tag{3.8}$$

The trace in (3.8) is now over the $\mathscr{g}$ and quaternionic indices. Let $C_0^\infty(\mathbb{R}^3; \mathscr{g} \otimes Q)$

be the space of smooth, compactly supported sections of the vector bundle $\mathbb{R}^3 \times (\mathscr{g} \otimes Q)$. Define the Hilbert space $H(A_0, \Phi_0)$ to be the completion of $C_0^\infty(\mathbb{R}^3; \mathscr{g} \otimes Q)$ in the norm

$$\| u \|_H^2 = \int_{\mathbb{R}^3} \{ \langle\langle D_j u, D_j u \rangle\rangle + \langle\langle [\Phi_0, u], [\Phi_0, u] \rangle\rangle \}. \tag{3.9}$$

It is important to keep in mind that the Hilbert space is different for different choices of (A_0, Φ_0). As in [6. IV] we will denote the standard L_p^m norms by $\| \cdot \|_{L_p^m}$. [14]

Theorem III. *(Theorem IV.2.1 of [6]). Let $(A_0, \Phi_0) \in \mathscr{H}(h_j, \{n_a\}_{a=1}^{\ell-r})$. There exists $\varepsilon_0 > 0$ which is independent of (A_0, Φ_0) such that if*

$$(1 + \| \Phi_0 \|_{L_\infty})^2 (\| G_0 \|_{L_2} + \| G_0 \|_{L_{6/5}}) \equiv \varepsilon < \varepsilon_0, \tag{3.10}$$

then there exists a solution $\psi \in H(A_0, \Phi_0)$ to (3.6). Furthermore, ψ is C^∞ and there exists $c < \infty$ and independent of (A_0, Φ_0) such that

$$\| \psi \|_{L_1} \leqq c\varepsilon (1 + \| \Phi_0 \|_{L_\infty})^{-2}$$
$$\| \psi \|_H \leqq c\varepsilon (1 + \| \Phi_0 \|_{L_\infty})^{-1}. \qquad \square \tag{3.11}$$

Theorem III above states that $(A, \Phi) = (A_0 + a, \Phi_0 + \phi)$ is a smooth finite action configuration satisfying (see Corollary IV.2.2 of [6])

$$* \Omega_A = D_A \Phi. \tag{3.12}$$

Also proved in [6.IV] is the pointwise estimate

Theorem IV. *(Theorem IV.2.3 of [6].). Let (A_0, Φ_0) satisfy the conditions of Theorem III. Suppose that $\mathscr{D}_0^+ G_0 \in L_2$. Then there is a constant c_2, independent of (A_0, Φ_0) such that*

$$\| \psi \|_{L_\infty} \leqq c_2 [\| \mathscr{D}_0^+ G_0 \|_{L_2} + \| \psi \|_{L_2} + (\alpha(A_0, \Phi_0) + \| \psi \|_H^2 (\| \psi \|_H^2 + \| \psi \|_{L_1}^2)]. \qquad \square \tag{3.13}$$

In order to prove Theorem II, we will need an estimate on the pointwise decay of $|\psi(x)|$ for large $|x|$. Such an estimate is given by

Theorem V. *Let (A_0, Φ_0) and ψ be as in Theorem IV. There exists a constant ε_1 independent of (A_0, Φ_0), such that if $(\| \psi \|_H^2 + \| \psi \|_{L_2}^2)^{1/2} < \varepsilon_1$* and in addition if

1) $|x| |G_0|$ *and* $|x| |D_{A_0} G_0| \in L_2(\mathbb{R}^3)$,
2) $|x|^2 (|\Omega_{A_0}|^2 + |D_{A_0} \Phi_0|^2)^{1/2} \in L_\infty(\mathbb{R}^3)$,

then

a) $$\lim_{|x| \to \infty} |x| |\psi(x)| \to 0 \text{ (uniformly in } |x|). \tag{3.14a}$$

if one assumes further that

3) $$|x|^2 |[\Phi_0, G_0]| \text{ and } |x|^2 |[\Phi_0, D_{A_0} G_0]| \in L_2(\mathbb{R}^3)$$
4) $$|x|^2 |(D_{A_0})_k D_{A_0}| \in L_\infty(\mathbb{R}^3),$$

then $|D_A\Phi| \leqq c|x|^{-2}$ *and*

b) $$\lim_{|x|\to\infty} (|x|^2 |D_{A_0}\phi + [a, \Phi_0] + [a, \Phi]|) \to 0 \quad \textit{(uniformly in } |x|). \tag{3.14b}$$

Here, $(A, \Phi) = (A_0 + a, \Phi_0 + \phi)$ *and c is a constant.* □

Theorem V is sufficient to guarantee the following:

Corollary 3.1. *Let* $(A_0, \Phi_0), \psi$ *satisfy all the conditions of Theorem V and* $(A_0, \Phi_0) \in \mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r})$. *Then with* (a, ϕ) *given by (3.4), the configuration* $(A_0 + a, \Phi_0 + \phi) \in \mathscr{H}(h, \{n_a\}_{a=1}^{\ell-r})$ *also.* □

Proof of Corollary 3.1 From (3.14b), $(A_0 + a, \Phi_0 + \phi)$ satisfies (2.5a, b). From (3.14a), with $\Phi = \Phi_0 + \phi$,

$$\Phi(P) = \Phi_0(P). \tag{3.15}$$

Therefore, $(A_0 + a, \Phi_0 + \phi) \in \mathscr{H}(h, \{n'_a\}_{a=1}^{\ell-r})$. The integers n'_a may differ from the integers n_a. However equation (3.14a) and Statement (d) of Proposition 2.1 insure that the integers $n'_a = n_a$, $a = 1, \ldots, \ell = r$.

The technical part of the proof of Theorem II is contained in Theorems III-V of this section. Theorems III and IV were proved in [6]. We postpone the proof of Theorem V to Sect. 6–8. In the next two sections we use Theorems III–V to prove Theorem II.

4. Proof of Theorem II: Patching Together Monopoles

Given Theorems III-V of the last section, the proof of Theorem II becomes straightforward. Theorem II implies that to prove the existence of a solution to (2.11), one need only demonstrate the existence of a sufficiently accurate approximation to a solution. Theorem IV gives an upper bound to the difference between the approximation and the true solution. Theorem IV will be used to prove statement (b) of Theorem II. Theorem V insures that statement (a) of Theorem II will be satisfied. Statement (c) of Theorem II will follow from the construction of the approximation.

To construct approximations to a solution to (2.11) we will patch together spatial translations of $O(3)$ symmetric solutions; members of $\mathscr{C}(L, h, \{n_a\}_{a=1}^{\ell-r})$.

The patching process is similar to that used in [6]. The patching will be done in a singular gauge in which it is convenient to estimate $(*\Omega_{A_0} - D_{A_0}\Phi_0)$. The fact that the patching is done in a singular gauge is not important because of the following well-known result:

Proposition 4.1. (*Proposition IV.6.1 of* [6]). *Let* $\{U_\alpha\}_{\alpha=1}^{\infty}$ *be a uniform open cover of* $\mathbb{R}^3$. *Let* A_0 *be a connection on* $\mathbb{R}^3 \times \mathscr{g}$ *which is not necessarily smooth, and* Φ_0 *a section of* $\mathbb{R}^3 \times \mathscr{g}$, *also not necessarily smooth. Suppose there exist gauge transformations* $\{g_\alpha : U_\alpha \to G\}_{\alpha=1}^{\infty}$ *such that in each* U_α, *both* $g_\alpha^{-1} A_0 g_\alpha + g_\alpha^{-1} dg_\alpha$ *and* $g_\alpha^{-1}\Phi_0 g_\alpha$ *are smooth. Then there exists a gauge transformation* $g : \mathbb{R}^3 \to \mathscr{G}$ *such that* $g^{-1}A_0 g + g^{-1} dg$ *and* $g^{-1}\Phi_0 g$ *are smooth.* □

Proposition 4.2. *Let $(A_{(0)}, \Phi_{(0)})$ be given by (2.15) and (2.17) for some homomorphism $L : \mathfrak{su}(2) \to \mathfrak{g}$. Let u be a singular gauge transformation given in polar coordinates (r, θ, χ) by:*

$$u = \exp(\theta(-\sin\chi L_1 + \cos\chi L_2)). \tag{4.1}$$

Then

$$\begin{aligned}\hat{A}_{(0)} &= u^{-1}A_{(0)}u + u^{-1}du \\ &= -(1-\cos\theta)d\chi L_3 + d\theta \sum_{\alpha=1}^{t}\Big\{a_{1\alpha}[Q_1, \mathbb{Y}_0^{\ell_\alpha}] \\ &\quad + a_{2\alpha}[Q_2, \mathbb{Y}_0^{\ell_\alpha}] + \sin\theta d\chi \sum_{\alpha=1}^{t}\{a_{1\alpha}[Q_2, \mathbb{Y}_0^{\ell_\alpha}] - a_{2\alpha}[Q_1, \mathbb{Y}_0^{\ell_\alpha}]\Big\}.\end{aligned} \tag{4.2}$$

With

$$Q_1 = \cos\chi L_1 + \sin\chi L_2,$$
$$Q_2 = \cos\chi L_2 - \sin\chi L_1.$$

And,

$$\hat{\Phi}_{(0)} = u^{-1}\Phi_0 u = \sum_{\alpha=1}^{t}\phi_\alpha \mathbb{Y}_0^{\ell_\alpha}. \quad \square \tag{4.3}$$

Proof of Proposition 4.2. This straightforward calculation is done explicitly in [3].

We remark that (A_0, Φ_0) fails to be smooth at the origin, $r = 0$ and along the half line

$$s_{(0)} = \{(r, \theta, \chi) \,|\, \cos\theta = -1\}. \tag{4.4}$$

The half line $s_{(0)}$ is called the "Dirac string".

$$\text{Fix } \Lambda = \{(A_i, \Phi_i) \in \mathscr{C}(L^{(i)}, h, \{n_a^{(i)}\}_{a=1}^{\ell-r}\}_{i=1}^{K}$$

as given in the conditions of Theorem II. Recall that one of the requirements is that

$$[L_3^{(i)}, L_3^{(j)}] = 0, \quad i, j = 1, \dots, K. \tag{4.5}$$

The approximate solutions (A_0, Φ_0) to be constructed out of the set $\{A_i, \Phi_i\}_{i=1}^{K}$ will depend on $3K$ parameters, K distinct points in $\mathbb{R}^3$; and a parameter $R > 0$. By adjusting R, the estimates required by Theorems III–V will be achieved.

Each configuration $(A_i, \Phi_i)_{i=1}^{K}$ can be put in the form (4.2) and (4.3) by a gauge transformation given by (4.1). We denote this singular gauge by $(\hat{A}_i, \hat{\Phi}_i)_{i=1}^{K}$. From (2.19b) we have

$$\hat{\Phi}_i(r) = h + \frac{L_3^{(i)}}{r} + X_i(r). \tag{4.6}$$

Equation (4.5) defines $X_i(r)$. (cf. Proposition 2.2 and Appendix 1.)

Define a function $b_R(x) \in C_0^\infty(\mathbb{R}^3)$,

$$b_R(x) = b(x/R) \text{ where}$$

$$1) \qquad 0 \leqq b(x) \leqq 1, \tag{4.7}$$

$$2) \qquad b(x) = \begin{cases} 1 \text{ if } |x| \leqq 1.1, \\ 0 \text{ if } |x| \geqq 1.9. \end{cases}$$

Next define functions

$$b_R(j)(x) = b_R(x - x_J), \quad j = 1, \dots, K.$$
$$w_R(j) = \prod_{\substack{i=1 \\ i \neq j}}^{K} (1 - b_R(i)), \quad j = i, \dots, K \tag{4.8}$$

Denote

$$\Phi_i(x - x_i) = \Phi(i), \quad \hat{\Phi}_i(x - x_i) = \hat{\Phi}(i), \text{ etc.} \tag{4.9}$$

for $i = 1, \dots, K$.

Definition 4.3 The field configuration $(\hat{A}_0, \hat{\Phi}_0)(\Lambda; R, d; x_1, \dots, x_K)$ is defined for all sets

$$\Lambda = \{(A_i, \Phi_i) \in \mathscr{C}(L^{(i)}, h, \{n_a^{(i)}\}_{a=1}^{\ell-r})_{i=1}^K\}$$

which satisfy the conditions in Theorem II; for all $R > 1$; and $\{x_1, \dots, x_K\} \in (\mathbb{R}^3)^K$ subject to the constraint

$$\min_{i \neq j} |x_i - x_j| \geqq d \geqq 8KR. \tag{4.10}$$

This field configuration is:

$$\hat{\Phi}_0 = h + \sum_{i=1}^{K} \left\{ \frac{1}{r_i} L_3^{(i)} w_{2R}(i) + X(i) b_R(i) \right\}$$

$$\hat{A}_0 = \sum_{i=1}^{K} \{ -(1 - \cos\theta_i) d\chi_i L_3^{(i)} w_{2R}(i)$$
$$+ b_R(i) M(i) \}, \tag{4.11}$$

where

$$M(i) = d\theta_i \sum_{\alpha=1}^{t} \{a_{1\alpha}(i)[Q_1(i), \mathbb{Y}_0^{\ell_\alpha}] + a_{2\alpha}(i)[Q_2(i), \mathbb{Y}_0^{\ell_\alpha}]\}$$
$$+ \sin\theta_i d\chi_i \sum_{\alpha=1}^{t} \{a_{1\alpha}(i)[Q_2(i), \mathbb{Y}_0^{\ell_\alpha}]$$
$$- a_{2\alpha}(i)[Q_1(i), \mathbb{Y}_0^{\ell_\alpha}]\}. \tag{4.12}$$

Here (r_k, θ_k, χ_k) is a polar coordinate system centered at x_k with positive orientation, such that the half line

$$s_k = \{(r_k, \theta_k, \chi_k) : \cos\theta_k = -1\} \tag{4.13}$$

satisfies

$$s_k \cap \bigcup_{\substack{i \neq k \\ i=1}}^{K} \bar{B}_{4R}(x_i) = \varnothing \text{ for } k = 1, \dots, K. \qquad \square \tag{4.14}$$

In general, we will write $(\hat{A}_0, \hat{\Phi}_0)$ for the configuration defined by (4.12).

When working with the above configurations, it is convenient to define open sets

$$V = \mathbb{R}^3 \setminus \bigcup_{i=1}^{K} \bar{B}_{2R}(x_i),$$

$$\tilde{V} = \mathbb{R}^3 \setminus \bigcup_{i=1}^{K} \bar{B}_{3.9R}(x_i). \tag{4.15}$$

Proposition 4.4. *The configuration* $(\hat{A}_0, \hat{\Phi}_0)$ *defined above is gauge equivalent to a smooth configuration* (A_0, Φ_0) *on* $\mathbb{R}^3$.

Proof of Proposition 4.4. The proof is similar to the proof of Proposition IV.7.2 of [6]. This is because of the restriction (4.5). We show that the conditions of Proposition 4.1 hold.

In the open set V, $\hat{\Phi}_0$ is smooth and

$$\hat{A}_0 = -\sum_{i=1}^{K} (1 - \cos\theta_i) d\chi_i L_3^{(i)} w_{2R}(i). \tag{4.16}$$

The connection $\hat{A}_0$ is smooth except on the set of strings $\{s_i\}_{i=1}^{K}$. The gauge transformation

$$g_i = \exp(2\chi_i L_3) \tag{4.17}$$

changes s_i to $\bar{s}_i = \{(r_i, \theta_i, \chi_i) | \theta_i = 0\}$ and leaves $s_j, j \neq i$ intact:

$$g_i^{-1} \hat{A}_0 g_i + g_i^{-} dg_i = -\sum_{\substack{j=1 \\ j \neq i}}^{K} (1 - \cos\theta_j) d\chi_j L_3^{(j)} w_{2R}(j) \tag{4.18}$$

$$+ (1 + \cos\theta_i) d\chi_i L_3^{(i)} w_{2R}(i). \tag{4.18}$$

Note that we have used the fact that $s_j \cap w_{2R}(i) = \varnothing$ for $j \neq i$. Because $g_i g_j = g_j g_i$ with g_i defined by (4.17), we can invert each string in the set $\{s_i\}_{i=1}^{K}$ individually.

Definition 4.5. Let $\hat{A}_0$ be the connection defined in (4.11). Let $g = g_{m_1} \cdot \ldots \cdot g_{m_r} (r \leqq K)$ with $m_i \in \{1, \dots, K\}$ and g_{m_i} defined by (4.17). With $\hat{A}_g = g^{-1} \hat{A}_0 g + g^{-1} dg$, define $s_i(\hat{A}_g)$ to be the singular half line of $\hat{A}_g$ with endpoint x_i, $i = 1, \dots, K$. $\square$

The following lemma is self-evident.

Lemma 4.6. *Let* $\hat{A}_0$ *be the connection defined by (4.11). If* $B_{R/10} \subset V$ *is an open ball of radius* $\frac{1}{10}R$, *there exists a gauge transformation* $g = g_{m_1} \cdot \ldots \cdot g_{m_r} (r \leqq K)$ *with* $m_i \in \{1, \dots, K\}$ *and* g_{m_i} *given by (4.15) such that*

$$2d(B_{R/10}, s_i(\hat{A}_g)) \geqq d(B_{R/10}, x_i), \quad i = 1, \dots, K \ \square \tag{4.19}$$

Henceforth, when dealing with the configuration $(\hat{A}_0, \hat{\Phi}_0)$ in any ball $B_{R/10} \subset V$, *we shall implicitly assume that gauge transformations of the type used in Lemma 4.4 have been made so that (4.19) holds.*

In the open sets $B_{2.1R}(x_i)$, $i = 1, \ldots, K$,

$$\hat{\Phi}_0 = h + \frac{L_3}{r_i} L_3^{(i)} + X(i) b_R(i),$$
$$\hat{A}_0 = -(1 - \cos\theta_i) d\chi_i L_3^{(i)} + b_R(i) M(i). \tag{4.20}$$

One can check that the gauge transformation

$$\hat{u}_i = \exp(-\theta_i(-\sin\chi_i L_1^{(i)} + \cos\chi_i L_2^{(i)})) \tag{4.21}$$

renders $(\hat{A}_0, \hat{\Phi}_0)$ smooth. This completes the proof of Proposition 4.4.

With $(\hat{A}_0, \hat{\Phi}_0)$ defined by (4.11), let (A_0, Φ_0) be the smooth configuration gauge equivalent (by a singular gauge) to $(\hat{A}_0, \hat{\Phi}_0)$ as guaranteed by Proposition 4.2.

Proposition 4.7. *Let (A_0, Φ_0) be a smooth configuration which is gauge related to $(\hat{A}_0, \hat{\Phi}_0)(\Lambda; R, d; x_1, \ldots, x_k)$. There exists constant $m > 0$ and $z < \infty$ which depend only on the set Λ such that*

$$\| G_0 \|_{L_p} + \| \mathscr{D}_0^+ G_0 \|_{L_p} \leqq z R^3 (1/d + e^{-mR})$$

for $$p = 6/5, 3/2, 2. \qquad \square \tag{4.22}$$

Proof of Proposition 4.7. Because $|G_0|$ and $|\mathscr{D}_0^+ G_0|$ are invariant under smooth gauge transformations, we are allowed to estimate them locally in any smooth gauge.

We note that in $\tilde{V}$ and in each $B_R(x_i)$, $i = 1, \ldots, K$

$$|G_0| = |\mathscr{D}_0^\dagger G_0| \equiv 0. \tag{4.23}$$

It remains to estimate $|G_0|$ and $|\mathscr{D}_0^\dagger G_0|$ in the annuli $B_{4R}(x_i) \backslash B_R(x_i)$, $i = 1, \ldots, K$. In the open set $B_{4R}(x_i) \backslash B_{1.9R}(x_i)$, $i = 1, \ldots, K$, $w_{2R}(i) = 1$ and $w_{2R}(j) = w_{2R}$ for $j \neq i$. Let $B_{R/10} \subset B_{4R}(x_i) \backslash B_{1.9R}(x_i)$ be an open ball of radius $R/10$. In $B_{R/10}$, (A_0, Φ_0) is gauge equivalent (cf. Lemma 4.6) via a smooth gauge transformation to

$$\tilde{\Phi}_0 = h + \frac{1}{r_i} L_3^{(i)} + \sum_{\substack{j=1 \\ j \neq i}}^{K} \frac{1}{r_j} L_3^{(j)} w_{2R}.$$

$$\tilde{A}_0 = -(1 - \cos\theta_i) \mathrm{d}\chi_i L_3^{(i)} - \sum_{\substack{j=1 \\ j \neq i}}^{K} (1 - \cos\theta_j) d\chi_j L_3^{(j)} w_{2R}.$$

We obtain

$$|G_0|(x) \leqq z_1 \sum_{j \neq i} \frac{1}{r_j} |\nabla w_{2R}| \leqq \bar{z}_1 1/d,$$
$$|\mathscr{D}_0^\dagger G_0|(x) \leqq z_2 \sum_{j \neq i} \left(\frac{1}{r_j} |\nabla\nabla w_{2R}| + \frac{1}{r_j^2} |\nabla w_{2R}| \right) \leqq \bar{z}_2 1/d, \tag{4.24}$$

where $\bar{z}_1$ and $\bar{z}_2$ are numerical constants independent of (A_0, Φ_0) and R.

In the balls $B_{2.1R}(x_i)\backslash B_R(x_i)$, $i = 1, \ldots, K$, the configuration (A_0, Φ_0) is gauge equivalent by a smooth gauge transformation to (cf. Proposition 2.2)

$$\tilde{\Phi}_0 = \Phi(i) - (1 - b_R(i)) \sum_{\alpha=1}^{t} X_\alpha^{(i)}(r)\mathbb{P}_\alpha(i),$$
$$\hat{A}_0 = A(i) - (1 - b_R(i)) \sum_{\alpha=1}^{t} \frac{1}{r_i}\mathbb{M}_\alpha(i),$$

where

$$\mathbb{P}_\alpha(i) = \sum_{m=-\ell_\alpha}^{\ell_\alpha} Y_{-m}^{\ell_\alpha}(\theta_i, \chi_i)\mathbb{Y}_m^{\ell_\alpha}, \tag{4.25}$$

and

$$\mathbb{M}_\alpha(i) = \frac{1}{r_i^2}(a_{1\alpha}^{(i)}(r_i)\varepsilon^{pnm}[L_m^{(i)}, \mathbb{P}_\alpha(i)](x - x_i)^n + a_{2\alpha}^{(i)}(r_i)[L_p^{(i)}, \mathbb{P}_\alpha(i)])\varepsilon^{pjk}(x - x_i)^j dx^k.$$

From Proposition 2.2, there is an estimate on the decay of the functions $(X_\alpha^{(i)}, a_{1\alpha}^{(i)}, a_{2\alpha}^{(i)})_{\alpha=1}^t$. Let

$$m = \min_{i\in(1,\ldots,K)} \{m(A_i, \Phi_i), 1\} \tag{4.26}$$

with $m(A_i, \Phi_i)$ as defined in Proposition 2.2. Then in $B_{2.1R}(x_i)\backslash B_{1R}(x_i)$,

$$|G_0| \leqq z_4 e^{-r_i m}(|\nabla b_R| + 1) \leqq \bar{z}_4 e^{-mR},$$
$$|\mathscr{D}_0^\dagger G_0| \leqq z_5 e^{-r_i m}(|\Delta b_R| + |\nabla\nabla b_R| + 1) \leqq \bar{z}_5 e^{-mR}, \tag{4.27}$$

with $\bar{z}_4, \bar{z}_5$ numerical constants depending on m but independent of (A_0, Φ_0) if $R > 1$.

To compute the L_p norms, we integrate (4.24) and (4.27) over $\bigcup_{i=1}^{K} B_{4R}(x_i)\backslash B_R(x_i)$ and obtain (4.22).

Proposition 4.8. *Let (A_0, Φ_0) be as in Proposition 4.7. Then $|x|^p|G_0| \in L_2$ and $|x|^p|D_{A_0}G_0| \in L_2$ for all $p \geqq 0$. In addition $|x|^2(|\Omega_{A_0}|^2 + |D_{A_0}\Phi_0|^2)^{1/2} \in L_\infty$ and $|x|^2|(D_{A_0})_k D_{A_0}\Phi_0| \in L_\infty$.* □

Proof of Proposition 4.8. $|G_0|$ is compactly supported so $|x|^p|G_0|$ and $|x|^p|D_{A_0}G_0|$ are L_2. The function $|x|^2(|\Omega_{A_0}|^2 + |D_{A_0}\Phi_0|^2)^{1/2}$ is C^0. For large $|x|$, $*\Omega_{A_0} = D_{A_0}\Phi_0$ and $|\Omega_{A_0}| < c_1|x|^{-2}$ for constant c_1. Hence $|x|^2(|\Omega_{A_0}|^2 + |D_{A_0}\Phi_0|^2)^{1/2}$ is bounded. Similarly, for large $|x|$, $|(D_{A_0})_k D_{A_0}\Phi_0| \leqq c_2|x|^{-3}$ so that it is also bounded.

Proposition 4.9. *Let (A_0, Φ_0) be as in Proposition 4.7. Then*

$$(A_0, \Phi_0) \in \mathscr{H}\left(h; \left\{\sum_{i=1}^{K} n_a^{(i)}\right\}_{a=1}^{\ell-r}\right).$$

Proof of Proposition 4.9. It follows from Proposition 4.8 that $\mathscr{A}(A_0, \Phi_0) < \infty$. In addition, Eq. (2.5b) is satisfied.

If $B_{R/10} \subset \tilde{V}$ is any ball, we can find a smooth gauge transformation g in $B_{R/10}$ such that $(\tilde{A}_0 = g^{-1}A_0 g + g^{-1}dg,\ \tilde{\Phi}_0 = g^{-1}\Phi_0 g)$ are smooth in $B_{R/10}$ and

$$\begin{aligned} *\Omega_{\tilde{A}_0} &= -\sum_{i=1}^{K} dr_i 1/r_i^2 L_3^{(i)} \\ \tilde{\Phi}_0 &= h + \sum_{i=1}^{K} 1/r_i L_3^i . \end{aligned} \tag{4.26}$$

It is clear from (4.26) that (2.5a) is also satisfied. Therefore $(A_0, \Phi_0) \in \mathscr{H}(h', \{n'_a\}_{a=1}^{\ell-r})$. It remains to determine h' and the integers $\{n'_a\}_{a=1}^{\ell-r}$. From (4.11), $h' = h$. To determine the integers $\{n'_a\}_{a=1}^{\ell-r}$ we utilize the method developed by Goddard, Nuyts and Olives as outlined in Sect. 2. The verification of (2.6) is straightforward. Along the positive z-axis, we can arrange by a smooth gauge transformation for

$$B(P) = -\sum_{i=1}^{K} L_3^{(i)} 4\pi . \tag{4.27}$$

The corresponding $B^{(i)}(P)$ for the $O(3)$ symmetric solution $(A_i, \phi_i)_{i=1}^{K}$ is $-4\pi L_3^{(i)}$. This fact and (4.27) imply that the eigenvalues of $B(P)$ are found by summing the corresponding eigenvalues of the $B^{(i)}(P)$. The condition $[L_3^{(i)}, L_3^{(j)}]_{i,j=1}^{K} = 0$ insures that this makes sense. Thus

$$n_a = \sum_{i=1}^{K} n_a^{(i)}, \qquad a = 1, \dots, \ell - r.$$

Proposition 4.10. *Let (A_0, Φ_0) be as in Proposition 4.7. Then $\|\Phi_0\|_{L_\infty} < \infty$ depends only on the defining set Λ.* □

Proof of Proposition 4.10. If

$$(A_i, \Phi_i) \in \mathscr{C}(L^{(i)}, h, \{n_a^{(i)}\}_{a=1}^{\ell-r}).$$

Then

$$D_{A_i} * D_{A_i} \Phi_i = 0. \tag{4.28}$$

Let $w = \frac{1}{2}(1 - \Phi_i \Phi_i)$. Since $\lim_{|x| \to \infty} w = 0$ uniformly and w satisfies

$$-\Delta w = |D_{A_i} \Phi_i|^2, \tag{4.29}$$

due to (4.28). The maximum principle [15] implies that $w > 0$. Thus each $\|\Phi_i\|_{L_\infty} = 1$ and $|\Phi_i| < 1$. The proposition now follows from estimate

$$|\Phi_0| \leqq 1 + \sum_{i=1}^{K} |\Phi_i - h| + c \cdot e^{-mR}. \tag{4.30}$$

The constant C depends only on the set Λ. The estimate (4.30) uses Proposition 2.2.

5. Existence of Multimonopoles

We put together the results of Sect. 3 and 4 to complete the proof of Theorem II. Let (A_0, Φ_0) be a smooth configuration which is gauge related to $(\hat{A}_0, \hat{\Phi}_0)(\Lambda; R, d; x_1, \ldots, x_K)$ of Definition 4.3. By Proposition 4.10, $\|\Phi_0\|_{L\infty}$ is independent of $R > 1$ and d. The constants ε_0, c are defined in Theorem III, ε_1 is defined in Theorem V and z, m are defined in Proposition 4.7. These constants are independent of $R \geqq 1$ and d.

Definition 5.1. Define d_0 to be

$$d_0 = \max\left(e^{30}, \left(\frac{8K}{m}\right)^2, \left(\frac{4(1 + \|\Phi_0\|_{L_\infty})^2 z(\varepsilon_1 + 2c\varepsilon_0)}{\varepsilon_1 \varepsilon_0 m^3}\right)^2 \right) \quad \square \quad (5.1)$$

Proposition 5.1. *Let (A_0, Φ_0) be a smooth configuration which is gauge related to $(\hat{A}_0, \hat{\Phi}_0)(\Lambda; R(d), d; x_1, \ldots, x_k)$ with $d > d_0$ and $R(d) = 1/m\ \ell nd$. Then there is a smooth solution $\psi \in H(A_0, \Phi_0) \cap L_2$ to Eq.* (3.6) *which satisfies*

(a) $\|\psi\|_{L_2} + \|\psi\|_H < \varepsilon_1$
(b) $\|\psi\|_{L\infty} < c_1 d^{-1/2}$

where c_1 is independent of d. □ (5.2)

Corollary 5.2. *Let (A_0, Φ_0), ψ be as in Proposition 5.1. Define (a, ϕ) by $\phi = \operatorname{Re} \psi$, $a_j = -\operatorname{Re}(\tau_j \psi)$. Then $(A, \Phi) = (A_0 + a, \Phi_0 + \phi)$ is a smooth finite action solution to (3.12) in*

$$\mathscr{H}\left(h; \left\{ \sum_{i=1}^{K} n_a^{(i)} \right\}_{a=1}^{\ell - r} \right). \quad \square$$

Corollary 5.3. *Let (A, Φ) be as in Corollary 5.2. Then statement *) of Theorem II holds with $\alpha = 1/m$ and $\beta = c_1$.* □

Proof of Proposition 5.1: First, note that if $d > d_0$ then

$$d^{1/2} > (\ell nd)^3 > \ell nd. \quad (5.3)$$

Second,

$$d > \frac{8K}{m} d^{1/2} > \frac{8K}{m} \ell nd = \frac{8K}{m} R(d). \quad (5.4)$$

In addition, (4.22) implies that

$$\varepsilon = (1 + \|\Phi_0\|_{L\infty})^2 (\|G_0\|_{L_2} + \|G_0\|_{L_{6/5}}) \quad (5.5)$$

satisfies

$$\varepsilon \leqq 4(1 + \|\Phi_0\|_{L\infty})^2 \frac{z}{m^3} \frac{(\ell nd)^3}{d} < \frac{\varepsilon_1 \varepsilon_0}{\varepsilon_1 + 2c\varepsilon_0} \left(\frac{d_0}{d}\right)^{1/2} < \varepsilon_0. \quad (5.6)$$

Proposition 5.1 follows directly from Theorems III and IV.

Proof of Corollary 5.2. Because statement (a) of Proposition 5.1 is true, the conditions of Theorem V are met, c.f. Proposition 4.8. Corollary 5.2 is a direct consequence of Proposition 4.9 and Corollary 3.1.

Proof of Corollary 5.3. The corollary is a consequence of statement (b) of Proposition 5.1 and the form of (A_0, Φ_0), c.f. Definition 4.3.

6. A Proposition Concerning Asymptotic Decay

The proof of Statements (a) and (b) of Theorem V are similar. In outline, we verify that under the assumptions of the theorem, both $|x||\psi|$ and $|x|^2|D_{A_0}\phi + [a, \Phi_0] + (a, \Phi]|$ are elements of the Sobolev space $L_6^1(\mathbb{R}^3)$. Then Lemma 6.1 states that elements of $L_p^1(\mathbb{R}^3)$ for $p > 3$ have uniform decay to zero as $|x| \to \infty$. We begin with

Lemma 6.1. *(Proposition III.7.5 of [6]) Let $u \in L_p^1(\mathbb{R}^3; \mathbb{R})$ for some $3 < p < \infty$. Then given $\varepsilon > 0$, there exists $R_\varepsilon < \infty$ such that*

$$\sup_{|x| > R_\varepsilon} |u(x)| < \varepsilon. \qquad \square$$

The proof of Theorem V consists of the verification that first $|x||\psi|$ and second $|x|^2|D_{A_0}\phi + [a, \Phi_0] + [a, \phi]|$ are elements of $L_p^1(\mathbb{R}^3; \mathbb{R})$. It is simpler to treat the generic case. The main tool is

Proposition 6.2. *Let (A_0, Φ_0) be C^∞ and suppose that $\omega \in C^\infty(\mathbb{R}^3; \mathfrak{g} \otimes Q) \cap H(A_0, \Phi_0) \cap L_2$ satisfies*

$$\mathscr{D}_0 \omega = f_0 + f_1 \omega \wedge \omega, \tag{6.2}$$

where $f_1 \in \mathfrak{g} \otimes Q$ and $f_0 \in C^\infty(\mathbb{R}^3; \mathfrak{g} \otimes Q)$. Assume further that for some $p \geqq 1$,

(a) $|x|^{p-1}\omega \in L_2$,
(b) $|x|^p f_0 \in L_2$,
(c) $|x|^2(|\Omega_{A_0}| + |D_{A_0}\Phi_0|) \in L_\infty$.

There exists $\varepsilon_1 > 0$ which is independent of $(A_0, \Phi_0), f_0, f_1$ such that the condition

$$\|\omega\|_H + \|\omega\|_{L_2} < \varepsilon_1 |f_1|^{-1} \tag{6.3}$$

ensures that $|x|^p|\omega| \in L_6$ and $||x|^p D_{A_0}\omega|, ||x|^p[\Phi_0, \omega]| \in L_2$. $\square$

Proof of Proposition 6.2. Define $\sigma_R(x) \in C_0^\infty(B_{2R}(0))$ by

$$\sigma_R(x) = b_R(x)(1 + |x|^2)^{1/2}. \tag{6.4}$$

We denote $\sigma(x) \equiv \sigma_\infty(x) = (1 + |x|^2)^{1/2}$.

Equation (6.2) implies that

$$\mathscr{D}_0(\sigma_R^p \omega) = \sigma_R^p f_0 + \sigma_R^p f_1 \omega \wedge \omega - \tau_j(\nabla_j \sigma_R^p)\omega. \tag{6.5}$$

Squaring both sides of (6.5) and integrating yields

$$\|\mathscr{D}_0(\sigma_R^p \omega)\|_{L_2} \leqq \|\sigma^p f_0\|_{L_2} + |f_1| \, \|\sigma_R^p |\omega|^2\|_{L_2} + K\|\sigma^{p-1}\omega\|_{L_2} \tag{6.6}$$

where $K = p\|\nabla_R \sigma_R\|_{L_\infty} < \infty$. The constant K is independent of R if $R \geqq 1$.

The following identity holds for any $v \in C_0^\infty(\mathbb{R}^3; \mathscr{g} \otimes Q)$:

$$\|\mathscr{D}_0 v\|_{L_2}^2 = 2\|v\|_H^2 - \int_{\mathbb{R}^3} \langle\langle v, \tau_j[\bar{G}_{0j}, v]\rangle\rangle$$

where

$$\bar{G}_{0j} = (*\Omega_{A_0} + D_{A_0}\Phi_0)_j. \tag{6.7}$$

The identity follows from the definition of $\mathscr{D}_0$ and a series of integration by parts.

Utilizing (6.7) we obtain

$$2\|\sigma_R^p \omega\|_H \leqq (2\|\sigma^2 \bar{G}_0\|L_\infty^{1/2} + K)\|\sigma^{p-1}\omega\|_{L_2} + \|\sigma^p f_0\|_{L_2} + |f_1|\,\|\sigma_R^p|\omega|^2\|_{L_2}. \tag{6.8}$$

It is necessary to estimate $\|\sigma_R^p|\omega|^2\|_{L_2}$ in terms of known quantities. Hölder's inequality yields:

$$\|\sigma_R^p|\omega|^2\|_{L_2} \leqq \|\sigma_R^p \omega\|_{L_6}\|\omega\|_{L_6}^{1/2}\|\omega\|_{L_2}. \tag{6.9}$$

An a priori estimate of $\|\sigma_R^p|\omega|^2\|_{L_2}$ results from the following lemma:

Lemma 6.3. *If* $u \in H(A_0, \Phi_0)$ *then*

a) $\|u\|_H^2 \geqq \frac{1}{2}\int_{\mathbb{R}^3} (\nabla_K|u|)(\nabla_K|u|)$.

b) *There exists a constant* $c < \infty$, *independent of* (A_0, Φ_0) *such that*

$$\|u\|_{L_6}^2 \leqq c\|u\|_H^2. \qquad \square \tag{6.10}$$

For a proof, see [6], Chap. VI.6.

With Lemma 6.3, the right hand side of (6.9) may be replaced by

$$\|\sigma_R^p|\omega|^2\|_{L_2} \leqq (c)^{3/4}\|\sigma_R^p\omega\|_H(\|\omega\|_H + \|\omega\|_{L_2}). \tag{6.11}$$

Together (6.11) and (6.14) imply that

$$\begin{aligned}&\|\sigma_R^p\omega\|_H(2 - |f_1|c^{3/4}(\|\omega\|_H + \|\omega\|_{L_2}))\\ &\quad\leqq (2\|\sigma^2\bar{G}_0\|_{L_\infty} + K)\|\sigma^{p-1}\omega\|_{L_2} + \|\sigma^p f_0\|_{L_2}.\end{aligned} \tag{6.12}$$

Therefore, if $\|\omega\|_H + \|\omega\|_{L_2} < \varepsilon_1|f_1|^{-1} < 2c^{-3/4}|f_1|^{-1}$, then $\|\sigma_R^p\omega\|_H$ is bounded independent of R.

Statement (b) of Lemma 6.3 implies that $\|\sigma_R^p\omega\|_{L_6}$ is bounded independently of R. Hence $\sigma^p\omega \in L_6$. Furthermore, since (c.f. (3.9))

$$\|\sigma_R^p D_{A_0}\omega\|_{L_2} \leqq 2\|\sigma_R^p\omega\|_H + K\|\sigma^{p-1}\omega\|_{L_2}$$

we conclude that $\|\sigma_R^p D_{A_0}\omega\|_{L_2}$ and $\|\sigma_R^p[\Phi_0, \omega]\|_{L_2}$ are bounded independently of R also. Hence $|\sigma^p D_{A_0}\omega|, |\sigma^p[\Phi_0, \omega]| \in L_2$ as claimed.

7. Proof of Theorem V: Asymptotic Decay of ψ

We prove statement (a) of Theorem V in this section and prove statement (b) in Sect. 8. To make use of Proposition 6.2, set $\omega = \psi, f_0 = G_0$ and $f_1 = 1$. Then the assumptions of Theorem V and Proposition 5.2 imply that

$$|x|\psi\in L_6,\ |x|(|D_{A_0}\psi|+[\Phi_0,\psi])\in L_2. \tag{7.1}$$

Upon taking derivatives of both sides of (3.6) we obtain

$$D_i\mathscr{D}_0\psi=D_iG_0-D_i(\psi\wedge\psi) \tag{7.2}$$

where $D_i\equiv(D_{A_0})_i$.
Commuting derivatives gives

$$\mathscr{D}_0D_i\psi=-\tau_k[(\Omega_{A_0})_{ki},\psi]-[(D_{A_0}\Phi_0)_i,\psi]+D_iG_0-D_i(\psi\wedge\psi). \tag{7.3}$$

We remark that (7.3) is similar in form to (6.2). To make the similarity manifest, set

$$\begin{aligned}&\omega=D_i\psi,\\&f_0=\tau_k[(\Omega_{A_0})_{ki},\psi]+[(D_{A_0}\Phi_0)_i,\psi]+D_iG_0-D_i(\psi\wedge\psi),\\&f_1=0.\end{aligned} \tag{7.4}$$

Lemma 7.1. *Let (A_0,Φ_0) and ψ be as stated in Theorem V. Then $|D_i(\sigma\psi)|\in L_6$.*

Proof of Lemma 7.1. By Proposition 6.2 it is enough to prove that $\sigma f_0\in L_2$, with f_0 given by 7.4. In fact, with this established, Proposition 6.2 ensures that $\sigma|D_i\psi|\in L_6$. But $|D_i(\sigma\psi)|\leqq\sigma|D_i\psi|+|\psi|$, and $|\psi|\in L_6$ so that $|D_i\sigma\psi|\in L_6$ as claimed. To establish that $\sigma f_0\in L_2$, note that

$$\begin{aligned}\|\sigma f_0\|_{L_2}\leqq{}&\|\sigma(|\Omega_{A_0}|+|D_{A_0}\Phi_0|)\|_{L_\infty}\|\psi\|_{L_2}+\|\sigma D_iG_0\|_{L_2}\\&+4\|\psi\|_{L_\infty}\|\sigma D_i\psi\|_{L_2}<\infty.\end{aligned} \tag{7.5}$$

The fact that $\sigma D_{A_0}\psi\in L_2$ is used here. To prove statement (a) of Theorem V, note that we have established the facts

(i) $|\sigma\psi|\in L_6$,
(ii) $|D_i\sigma\psi|\in L_6$.

But $|D_i\sigma\psi|\geqq\nabla_i|\sigma\psi|$ (see [6], Chap. VI. 6) so that $|\sigma\psi|\in L_p^1$. Now use Lemma 6.1.

8. Asymptotic Decay of $|D_A\Phi|$

We now turn to the proof of statement (b) of Theorem V. The proof uses Proposition 6.2 also. Set

$$\omega=\tau_i[(D_{A_0}\phi)_i+[a_i,\Phi_0+\phi]]. \tag{8.1}$$

Because $(A,\Phi)=(A_0+a,\Phi_0+\phi)$ satisfies (3.12), $D_A\Phi$ satisfies

$$\begin{aligned}&D_A{*}D_A\Phi=0\\&{*}D_AD_A\Phi+[\Phi,D_A\Phi]=0.\end{aligned} \tag{8.2}$$

Upon writing out (8.2) in longhand we find that ω satisfies

$$\begin{aligned}\mathscr{D}_0\omega={}&[\Phi_0,G_0]-[a_i,(D_{A_0}\Phi_0)_i]-\varepsilon^{ijk}\tau_i[a_j,(D_{A_0}\Phi_0)_k]-\tau_i[\phi,(D_{A_0}\Phi_0)_i]\\&-[\phi,\omega]-[a_i,\omega]-\varepsilon^{ijk}\tau_i[a_j,\omega_k].\end{aligned} \tag{8.3}$$

It is the similarity between equation (8.3) and equation (6.2) which allows us to use the same techniques to prove (b) of Theorem V.

Lemma 8.1. *Let* $(A_0, \Phi_0), \psi$ *satisfy the assumptions of Theorem V with* $\|\psi\|_H + \|\psi\|_{0,2} < \varepsilon_1$. *Then*

$$\sigma^2 \omega \in L_6 \tag{8.4a}$$

$$\sigma^2 D_{A_0} \omega \in L_2 \tag{8.4b}$$

$$\sigma^2 [\Phi_0, \omega] \in L_2 \quad \square \tag{8.4c}$$

Proof of Lemma 8.1. Let F denote the right hand side of (8.3). We establish that $\sigma^2|F| \in L_2(\mathbb{R}^3)$. In fact, this follows from the assumption that $\sigma^2|[\Phi_0, G_0]| \in L_2$, $\sigma^2|D_{A_0}\Phi_0| \in L_\infty$ and the facts established previously concerning ψ and ω; namely that $\psi \in L_2$, $\sigma\psi \in L_\infty$ and $\sigma\omega \in L_2$. Thus

$$\begin{aligned} \|\sigma^2 F\|_{L_2} \leqq \|\sigma^2[\Phi_0, G_0]\|_{L_2} + 8\beta\|\psi\|_{L_2} \\ + 8\|\sigma\psi\|_{L_\infty}\|\sigma\omega\|_{L_2}. \end{aligned} \tag{8.5}$$

Here $\beta \equiv \|\sigma^2(|\Omega_{A_0}| + |D_{A_0}\Phi_0|)\|_{L_\infty}$. Now use Proposition 6.2 with $f_0 = F$ and $f_1 \equiv 0$.

Applying the operator D_{A_0} to both sides of (8.3) we obtain

$$\mathscr{D}_0 D_i \omega = D_i F + \tau_k[(\Omega_{A_0})_{ki}, \omega] + [(D_{A_0}\Phi_0)_i, \omega] \tag{8.6}$$

Lemma 8.2. *Let* (A_0, Φ_0) *and* ψ *be as stated in Theorem V. Then* $|D_i(\sigma^2\omega)| \in L_6$. $\square$

Proof of Lemma 8.2. Let f_0 denote the right hand side of (8.6). Then as in the proof of Lemma 7.1, the result follows from Proposition 6.2 upon verification that $\sigma^2 f_0 \in L_2$. Thus we need to establish an L_2 bound for $\sigma^2 D_i F$. In fact,

$$\begin{aligned} |D_i F| \leqq |[\Phi_0, D_i G_0]| + 2|G_0||D_{A_0}\Phi_0| + 8|D_i\psi||D_{A_0}\Phi_0| \\ + 8|\psi||D_i D_{A_0}\Phi_0| + 8|D\psi||\omega| + 8|\psi||D\omega|. \end{aligned} \tag{8.7}$$

From (5.40) we find that

$$\begin{aligned} \|\sigma^2 D_i F\|_{L_2} \leqq \|\sigma^2[\Phi_0, D_{A_0}G_0]\|_{L_2} + 2\beta\|G_0\|_{L_2} \\ + 8\beta\|D_{A_0}\psi\|_{L_2} + 8\|\sigma^2 D_i D_{A_0}\Phi_0\|_{L\infty}\|\psi\|_{L_2} \\ + 8\|D\psi\|_{L_3}\|\sigma^2\omega\|_{L_6} + 8\|\psi\|_{L\infty}\|\sigma^2 D_{A_0}\omega\|_{L_2}. \end{aligned} \tag{8.8}$$

The right hand side of (8.8) is finite: By assumption; β, $\|\sigma^2[\Phi_0, D_{A_0}G_0]\|_{L_2}$, $\|G_0\|_{L_2}$, and $\|\sigma^2 D_i D_{A_0}\Phi_0\|_{L\infty}$ are finite. From Theorem III; $\|D_{A_0}\psi\|_{L_2}$ and $\|\psi\|_{L_2}$ are finite. From Theorem IV; $\|\psi\|_{L\infty}$ is finite. From Lemma 7.1, $\|D\psi\|_{L_3}$ is finite and from Lemma 8.1, $\|\sigma^2 D_{A_0}\omega\|_{L_2}$ and $\|\sigma^2\omega\|_{L_6}$ are finite.

To finish the proof of statement (b) of Theorem V note that we have established that $|\sigma^2\omega| \in L_6$ and $|D_{A_0}(\sigma^2\omega)| \in L_6$. Since

$$|D_i(\sigma^2\omega)| \geqq \nabla_i|\sigma^2\omega| \tag{8.9}$$

we conclude that $\sigma^2|\omega| \in L_6^1$. Now use Lemma 6.1 to complete the proof.

Appendix

Proof of Proposition 2.2.

Let $(A, \Phi)\in\mathscr{C}(L, h, \{n_a\}_{a=1}^{\ell-r})$. The configuration (A, Φ) is of the form (2.15) and (2.17). We rewrite (2.15) and (2.17) as

$$\Phi = \sum_{\alpha=1}^{t} \phi_\alpha(r)\mathbb{P}_\alpha(\theta, \chi),$$

$$A = \frac{1}{r}(\mathbb{M}^i - L^i)\varepsilon^{ijk}\hat{x}^j dx^k, \tag{A.1}$$

where

$$\mathbb{M}^i = \sum_{\alpha=1}^{t} (a_{1\alpha}(r)\varepsilon^{inm}[L_m, \mathbb{P}_\alpha]\hat{x}^n + a_{2\alpha}(r)[L_i, \mathbb{P}_\alpha]).$$

We remind the reader that if we define differential operators $\left(\mathscr{L}_i = -\varepsilon^{ijk}x_j\frac{\partial}{\partial x^k}\right)_{i=1}^3$, then

$$(L_i + \mathscr{L}_i)\Phi = 0,$$
$$(L_i + \mathscr{L}_i)A^j = \varepsilon^{ijk}A^k,$$
$$(L_i + \mathscr{L}_i)\mathbb{M}^j = \varepsilon^{ijk}\mathbb{M}^k.$$
$$\nabla_i = \hat{x}^i\hat{x}^j\nabla_j + \frac{1}{r}\varepsilon^{ijk}\hat{x}^j\mathscr{L}_k. \tag{A.2}$$

Using (A.2) we obtain

$$D_A\Phi = \frac{\partial\Phi}{\partial r}dr + \frac{1}{r}[\mathbb{M}^i, \Phi]\varepsilon^{ijk}\hat{x}^j dx^k$$

$$*\Omega_A = \frac{1}{r^2}\hat{x}^n\{\varepsilon^{nij}\mathbb{M}^i\mathbb{M}^j - L^n\}dr + \frac{1}{r}\left(\frac{\partial}{\partial r}\mathbb{M}^\ell\right)(\delta^{\ell k} - \hat{x}^\ell\hat{x}^k)dx^k. \tag{A.3}$$

With Eq. (A.3) Eq. (2.11) can be written in terms of the variables $(\phi_\alpha, a_{1\alpha}, a_{2\alpha})_{\alpha=1}^t$. Because of the spherical symmetry, it is sufficient to consider the equations for $(\Phi, \mathbb{M}^1, \mathbb{M}^2)$ on the z-axis. [3] On the z-axis,

$$\Phi = \sum_{\alpha=1}^{t} \phi_\alpha(r)\mathbb{Y}_0^{\ell_\alpha}$$

$$\mathbb{M}^1 = \sum_{\alpha=1}^{t} (-a_{1\alpha}(r)[L_2, \mathbb{Y}_0^{\ell_\alpha}] + a_{2\alpha}(r)[L_1, \mathbb{Y}_0^{\ell_\alpha}]) \tag{A.4}$$

$$\mathbb{M}^2 = \sum_{\alpha=1}^{t} (a_{1\alpha}(r)[L_1, \mathbb{Y}_0^{\ell_\alpha}] + a_{2\alpha}(r)[L_2, \mathbb{Y}_0^{\ell_\alpha}]).$$

The equations for $(\Phi, \mathbb{M}^1, \mathbb{M}^2)$ are

$$\frac{1}{r}\frac{d}{dr}\mathbb{M}^2 = -\frac{1}{r}[\mathbb{M}^1, \Phi] \tag{A.5a}$$

$$\frac{1}{r}\frac{d}{dr}\mathbb{M}^1 = \frac{1}{r}[\mathbb{M}^2, \Phi] \tag{A.5b}$$

$$\frac{d}{dr}\Phi = \frac{1}{r^2}([\mathbb{M}^1, \mathbb{M}^2] - L_3) \tag{A.5c}$$

We prove Proposition 2.2 by induction on the integer n. Write

$$\Phi(r, 0, 0) = \Phi(P) + \frac{L_3}{r} + X(r), \tag{A.6}$$

and substitute this last expression into (A.5c). The result is an equation for $X(r)$:

$$\frac{d}{dr}X(r) = \frac{1}{r^2}[\mathbb{M}^1, \mathbb{M}^2] \tag{A.7}$$

with the boundary condition $X(r) \to 0$ as $r \to \infty$. Integrating (A.7) gives

$$X(r) = -\int_r^\infty \frac{dt}{t^2}[\mathbb{M}^1, \mathbb{M}^2]. \tag{A.8}$$

From Eq. (2.18), a bound on $|X(r)|$ is found to be

$$|X(r)| \leqq \frac{1}{r^2}\frac{c_0^2(A, \Phi)}{m(A, \Phi)} e^{-2m(A,\Phi)r}. \tag{A.9}$$

Therefore Statements (a) and (b) of Proposition 2.2 are true for $n = 0$. Assume that they are true for $n \leqq i$. By differentiating both sides of (A.5a, b) and (A.7) i times we obtain

$$\begin{aligned} \frac{d^{i+1}}{dr^{i+1}}\mathbb{M}^2 &= -\sum_{j=0}^{i}\binom{i}{j}\left[\frac{d^j}{dr^j}\mathbb{M}^1, \frac{d^{i-j}}{dr^{i-j}}\Phi\right] \\ \frac{d^{i+1}}{dr^{i+1}}\mathbb{M}^1 &= \sum_{j=0}^{i}\binom{i}{j}\left[\frac{d^j}{dr^j}\mathbb{M}^2, \frac{d^{i-j}}{dr^{i-j}}\Phi\right] \\ \frac{d^{i+1}}{dr^{i+1}}\Phi &= \sum_{j=0}^{i}\binom{i}{j}(j+1)!(-1)^j\cdot\frac{1}{r^{j+2}}\sum_{k=0}^{i-j}\left[\frac{d^k}{dr^k}\mathbb{M}^1, \frac{d^{i-j-k}}{dr^{i-j-k}}\mathbb{M}^2\right]. \end{aligned} \tag{A.10}$$

Equation (A.10) expressed the $i+1$'st derivative of $(\Phi, \mathbb{M}^1, \mathbb{M}^2)$ in terms of derivatives of order less than or equal to i. Taking the absolute value of both sides of (A.10) and using the induction hypothesis proves Statements (a) and (b) of Proposition (2.2) for $n = i + 1$. This induction argument completes the proof of Proposition (2.2).

References

1. 't Hooft, G.: Nuc. Phys. **B79**, 276 (1976)
   Polyakov, A.M.: JETP Lett. **20**, 194 (1974)
2. Goddard, P., Olive, D.: Rep. Prog. Phys. **41** 1357 (1978)
3. Wilkinson, D., Goldhaber, A. S.: Phys. Rev. D. **16**, 1221 (1977)
4. Manton, N. S., Forgacs, P.: Commun. Math. Phys. **72**, 15 (1980)

5. Prasad, M. K., Sommerfield, C. M. : Phys. Rev. Lett. **35** 760 (1975)
6. Jaffe, A., Taubes, C. H. : Vortices and monopoles. Boston : Birkhauser 1980
7. Goddard, P., Nuyts, J., Olive, D. : Nucl. Phys. **B125**, 1 (1977)
8. Taubes, C. H. : Commun. Math. Phys. (to appear)
9. Pontryagin, L. S.: Topological groups. New York: Gordon and Breach 1966
10. Bais, F. A., Weldon, H. A. : Phys. Rev. Lett. **11**, 601 (1978)
11. Leznov, A., Saveliev, M. : Lett. Math. Phys. **3**, 207 (1979)
12. Weinberg, E. : Nucl. Phys. B **167**, 500 (1980)
13. Bias, F. A. : Phys. Rev. **D18**, 1206 (1978)
14. Palais, R. : Foundations of global non-linear analysis. New York : W. A. Benjamin Inc. 1968
15. Gilbarg, G., Trudinger, S. : Elliptic partial differential equations of second order. Berlin, Heidelberg, New York : Springer 1977

Communicated by A. Jaffe

Received May 15, 1980, in revised form October 20, 1980