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Supergravity and Field Space Democracy

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Abstract. If the action functional is determined uniquely by its symmetry properties, we say that this functional is perfect. We study the perfect functionals in the framework in which the space and field variables are on equal footing. This study leads to the natural multidimensional generalizations of supergravity.

1. Introduction

The formulation of quantum field theory in which field and space variables are on an equal footing is suggested in [1]. In this formulation the action functional is considered as a functional on the space of (m,n)-dimensional submanifolds of (M,N)-dimensional superspace; it is assumed that this functional can be represented in the form

$$S=\int A\left(X(Y), \frac{\partial X^B}{\partial Y^Q}, \frac{\partial^2 X^B}{\partial Y^Q \partial Y^R}\right) dY. \tag{1.1}$$

[We denote by X^B the coordinates in (M,N)-dimensional superspace $\mathscr{E}^{M,N}$ and by Y^R the coordinates in (m,n)-dimensional superspace. The function A in (1.1) must satisfy the conditions ensuring independence of (1.1) on the choice of the parameter equation $X=X(Y)$ of the submanifold Γ. We suppose that the function A depends on the first and second derivatives of $X(Y)$ only; in this case we say that the function A is a (m,n)-density of rank 2.] It is shown in [1] that the Lagrangian of supergravity arises naturally in the framework of field-space democracy. Namely, the action functional of supergravity can be characterized as the functional of the form (1.1), defined on the space of (4,4)-dimensional submanifolds of complex (4,2)-dimensional superspace, which is invariant with respect to supervolume preserving analytic transformations.

In the present paper we show how the Lagrangian of the supergravity in the Ogievetsky-Sokatchev form [2] can be obtained by means of this characterization and describe natural multidimensional generalizations of this Lagrangian. These

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generalizations are based on the notion of a perfect action functional; we believe that this notion deserves a closer study.

The action functional will be called perfect if the symmetry properties determine this functional uniquely. In other words, we consider the symmetry group G_A for every Lagrangian A and say that the Lagrangian A is perfect if all other Lagrangians invariant with respect to the group G_A are equivalent to A.

The definition above is not rigorous. One must restrict the class of Lagrangians under consideration (for instance one can suppose that the derivatives entering into the Lagrangian have order $\leqq k$). One must say which symmetry transformations are permitted (for example, we can require the invariance of the Lagrangian or impose a weaker condition of invariance of equations of motion). Finally one must define the equivalence of Lagrangians (one can say that the Lagrangians are equivalent if they lead to equivalent equations of motion; however, it is convenient to use this term only in the case when the Lagrangian can be obtained from another one by means of multiplication onto a constant multiplier). The rigorous definition of the perfect action functional, determined by means of (m,n)-density in (M,N)-dimensional superspace, will be given in Sect. 2.

One can believe that the action functional in elementary particle theory must be perfect in a certain sense. It follows from [1] that the action functional in supergravity is perfect in the sense of Sect. 2; the Lagrangian describing the interaction of massless fermions with gauge fields is perfect too (in another sense, however). It is reasonable to search the Lagrangian of the unified theory of all interactions among perfect Lagrangians. To find new perfect Lagrangians we can fix a group (or a supergroup) G acting in (M,N)-dimensional superspace and search (m,n)-densities which are invariant with respect to G. If the invariant density is unique (up to a constant multiplier), then it is perfect; we will say that this density is perfect with respect to the group G.

In the present paper we study $(m,2n)$-densities in the superspace $\mathscr{E}^{M,N}$ assuming that $\mathscr{E}^{M,N}$ is provided with complex structure, i.e. one can consider $\mathscr{E}^{M,N}$ as complex superspace $\mathscr{C}^{M/2,N/2}$ of complex dimension $(M/2,N/2)$, and G as a supergroup of analytic supervolume preserving transformations. We will assume that $m=M/2$, $n=N/2$. If $n>3$ we prove under these conditions that $(m,2n)$-density, which is perfect with respect to the supergroup under consideration, exists only in the cases $m=n^2$, $m=n^2-1$, $m=1$. (If $n=3$ there exists such a density in the cases $m=9$, 8, 7, 2, 1 only, if $n=2$ then $m=2,4$.) The proof of this assertion is given in Sect. 2. The perfect densities described in Sect. 2 will be analyzed in the case $m=n^2$ in Sect. 3. In this case the corresponding action functionals can be considered as multidimensional generalizations of the action functional of supergravity. After dimensional reduction these multidimensional action functionals give four-dimensional theories which are invariant with respect to extended Golfand-Likhtman supergroup. The readers interested in supergravity can omit the second half of Sect. 2.

The complete description of densities which are perfect with respect to the supergroup of analytic supervolume preserving transformations will be published later. Forthcoming papers by Gayduk, Khudaverdian, Schwarz, and Tyupkin will describe the densities which are perfect with respect to the superanalogs of Cartan primitive infinite-dimensional groups (the groups of canonical, contact and supervolume preserving transformations).

The m-dimensional linear space will be denoted by $\mathscr{E}^m$, the complex linear space having complex dimension m will be denoted $\mathscr{C}^m$. For (m,n)-dimensional superspace we use the notation $\mathscr{E}^{m,n}$; for complex superspace we use the notation $\mathscr{C}^{m,n}$. The (m,n)-dimensional superspace $\mathscr{E}^{m,n}$ can be considered as a direct sum of m-dimensional linear space $\mathscr{E}^m_{\text{even}}$ and n-dimensional linear space $\mathscr{E}^n_{\text{odd}}$. Let us denote the basis of $\mathscr{E}^m_{\text{even}}$ by $e_1, \ldots, e_m$ and the basis of $\mathscr{E}^n_{\text{odd}}$ by $f_1, \ldots, f_n$. The formal expression

$$x = \sum_{i=1}^{m} a_i e_i + \sum_{j=1}^{n} b_j f_j,$$

where a_i are even elements and b_j are odd elements of an arbitrary Grassmann algebra, will be called a point of superspace $\mathscr{E}^{m,n}$. The set of points of $\mathscr{E}^{m,n}$ can be considered as infinite dimensional linear space[1]. If the superspace $\mathscr{E}^{m,n}$ is provided with the structure of Lie superalgebra, then the set of points of $\mathscr{E}^{m,n}$ can be considered as infinite-dimensional Lie algebra in the usual sense. The elements of the corresponding infinite-dimensional Lie group will be considered as the points of the Lie supergroup corresponding to the Lie superalgebra. (The language of points in the theory of superspaces and supergroups is described in more detail in [3, 4].) Sometimes for the sake of brevity we will use the terms space and group instead of the terms superspace, supergroup, etc. The Berezinian (superdeterminant) of matrix K will be denoted by $\operatorname{Ber} K$.

Let us formulate some assertions which are useful for the study of perfect action functionals. All these assertions can be derived easily from well-known theorems.

Let $\mathscr{G}$ be a group acting in the space $\mathscr{E}$. The transformation corresponding to $g \in \mathscr{G}$ will be denoted by T_g and the isotropy group at $e \in \mathscr{E}$ will be denoted by $\mathscr{H}_e$ (remember that $g \in \mathscr{H}_e$ if $T_g e = e$). We will study the functions φ on $\mathscr{E}$ satisfying

$$\varphi(T_g e) = \varphi(e)\alpha(g), \tag{1.2}$$

where $\alpha(g)$ is a fixed function on $\mathscr{G}$. [Of course one must impose the condition $\alpha(g_1 g_2) = \alpha(g_1)\alpha(g_2)$ on the function α.]

Lemma 1.1. *If $\varphi(e) \neq 0$, $g \in \mathscr{H}_e$ then $\alpha(g) = 1$.*

Lemma 1.2. *Let us suppose that every point of $\mathscr{E}$ can be obtained from the fixed point $e \in \mathscr{E}$ by means of transformation belonging to the group $\mathscr{G}$ (i.e. the orbit of e coincides with $\mathscr{E}$). If $\alpha(g) = 1$ for arbitrary $g \in \mathscr{H}_e$ then there exists one and only one (up to a constant multiplier) function φ satisfying* (1.2). *Conversely, if the function satisfying* (1.2) *is unique (up to a constant multiplier) then the orbit of $e \in \mathscr{E}$ coincides with $\mathscr{E}$.*

Remark. If we impose certain conditions on the function φ Lemma 1.2 must be modified. For example if we suppose that φ is an algebraic function, we must require the existence of such a point $e \in \mathscr{E}$, that the orbit of this point is open in $\mathscr{E}$.

1 To avoid set-theoretical paradoxes one can assume that a_i and b_j belong to a fixed infinite-dimensional Grassmann algebra S. Then the set of points can be considered as a free S-module

(Of course one must assume that $\mathscr{E}$ is an algebraic manifold and $\mathscr{G}$ is an algebraic group in this case.)

If $\mathscr{F}$ is a subset of $\mathscr{E}$ we will denote by $\tilde{\mathscr{F}}$ the set of points which can be obtained from the points of $\mathscr{F}$ by means of transformations of the group $\mathscr{G}$:

$$\tilde{\mathscr{F}}=\{f \mid f=T_g e,\ e\in\mathscr{F},\ g\in\mathscr{G}\}\,.$$

The subset of $\mathscr{G}$ consisting of elements transforming the fixed point $f\in\mathscr{F}$ into the point of $\mathscr{F}$ will be denoted by $\mathscr{H}_f$:

$$\mathscr{H}_f=\{g \mid T_g f\in\mathscr{F}\}\,.$$

We assume that the group $\mathscr{H}$ acting in $\mathscr{F}$ and the function β on $\mathscr{H}$ satisfy the following conditions:

(I) if $f\in\mathscr{F}$, $g\in\mathscr{G}$, $T_g f\in\mathscr{F}$ one can find such an element $h\in\mathscr{H}$ that h transforms f into $T_g f$ and $\beta(h)=\alpha(g)$;

(II) if $h\in\mathscr{H}$, $f\in\mathscr{F}$ one can find such an element $g\in\mathscr{G}$ that h and g transform f into coinciding points of $\mathscr{F}$ and $\beta(h)=\alpha(g)$.

Lemma 1.3. *If every point of $\mathscr{E}$ can be transformed into the points of $\mathscr{F}$ by means of the transformation belonging to $\mathscr{G}$ (i.e. $\tilde{\mathscr{F}}=\mathscr{E}$) and the conditions (I), (II) are fulfilled, then assigning to the function φ on $\mathscr{E}$ the restriction of this function on $\mathscr{F}$ we obtain one-one correspondence between the functions on $\mathscr{E}$ satisfying* (1.2) *for every $g\in\mathscr{G}$ and the functions on $\mathscr{F}$ satisfying a similar equation with $\mathscr{G}$ replaced by $\mathscr{H}$ and α replaced by β.*

Remarks. (I) Lemmas 1.1–1.3 can be applied to supergroups too because one can interpret the action of the supergroup on points of superspace as the action of an infinite-dimensional Lie group in infinite-dimensional space.

(II) The requirement $\tilde{\mathscr{F}}=\mathscr{E}$ permits us to restore the values of the function φ on $\mathscr{E}$ if we know the values of this function on $\mathscr{F}$. Imposing certain conditions on the function φ we can weaken this requirement. In particular, if we assume that φ is an algebraic function, it is sufficient to suppose that the set $\tilde{\mathscr{F}}$ contains interior points (i.e. $\mathscr{E}\backslash\tilde{\mathscr{F}}$ is not dense in $\mathscr{E}$). Similar modifications of Lemma 1.3 can be made in the case when $\mathscr{E}$ is a superspace and $\mathscr{G}$ is a supergroup. We will use only the simple assertion that in the case when the points of set $\mathscr{E}\backslash\tilde{\mathscr{F}}$ satisfy an algebraic equation, the values of algebraic function on $\mathscr{E}$ obeying (1.2) can be restored if we know the values of this function on $\mathscr{F}$. (If the points $\mathscr{E}\backslash\tilde{\mathscr{F}}$ satisfy an algebraic equation we say that almost arbitrary point of $\mathscr{E}$ can be transformed into $\mathscr{F}$ by means of transformations belonging to $\mathscr{G}$.)

Let us suppose that the r-dimensional algebraic group $\mathscr{G}$ acts on the m-dimensional algebraic manifold $\mathscr{E}$. The dimension of the isotropy subgroup $\mathscr{H}_e$ will be denoted by t_e; the minimal t_e will be denoted by t:

$$t=\min_{e\in\mathscr{E}} t_e\,.$$

Lemma 1.4. *If $t_e=r-m$ then the orbit of the point $e\in\mathscr{E}$ is open in $\mathscr{E}$. Conversely, if there exists an open orbit of the group $\mathscr{G}$ in $\mathscr{E}$ then one can find such a point $e_0\in\mathscr{E}$ that $t_{e_0}=r-m$ and for other points $t_e\geqq r-m$ (in other words $t=r-m$).*

The following statement can be deduced from Lemma 1.4 and assertions above.

Lemma 1.5. *If* $t=t_{e_0}=r-m$ *and* $\alpha(g)=1$ *for every* $g\in\mathcal{H}_{e_0}$, *then there exists an algebraic function* φ *on* $\mathcal{E}$ *satisfying* (1.2) *and this function is defined uniquely (up to a constant multiplier). Conversely, if there exists one and only one (up to a constant multiplier) algebraic function satisfying* (1.2), *then one can find such a point* $e_0\in\mathcal{E}$ *that* $t=t_{e_0}=r-m$.

2. Perfect Action Functionals

The (m,n)-dimensional quadratic surface in the (M,N)-dimensional superspace is defined by a parametric equation

$$X^B=M^B+L_R^B Y^R+\tfrac{1}{2}T_{RS}^B Y^R Y^S, \tag{2.1}$$

where Y runs over (m,n)-dimensional superspace. The linear reparametrization

$$Y^R\to K_S^R Y^S \tag{2.2}$$

transforms the quadratic surface (2.1) into the quadratic surface

$$X^B=M^B+L'^B_R Y^R+\tfrac{1}{2}T'^B_{RS} Y^R Y^S, \tag{2.3}$$

where

$$L'^B_R=L_S^B K_R^S, \qquad T'^B_{RS}=T_{PQ}^B K_R^P K_S^Q .$$

The quadratic reparametrization

$$Y^R\to Y^R+\tfrac{1}{2}U_{QS}^R Y^Q Y^S \tag{2.4}$$

transforms (2.1) into the quadratic surface

$$X^B=M^B+L_R^B Y^R+\tfrac{1}{2}T'^B_{RS} Y^R Y^S, \tag{2.5}$$

where

$$T'^B_{RS}=T_{RS}^B+L_Q^B U_{RS}^Q .$$

(We do not identify quadratic surfaces coinciding geometrically but having different parametric equations.)

In such a way the supergroup $\mathcal{R}$ generated by linear and quadratic reparametrization acts on the superspace of (m,n)-dimensional quadratic surfaces. The supergroup $\mathcal{R}$ will be called the group of reparametrizations. The superspace of (m,n)-dimensional quadratic surfaces will be denoted by $\mathcal{A}^{m,n}$ or simply by $\mathcal{A}$.

The (m,n)-density of rank 2 in (M,N)-dimensional superspace can be considered as a function A defined on the superspace $\mathcal{A}$ of (m,n)-dimensional quadratic surfaces and satisfying the following conditions:

(I). The value of the function A on the surface (1.1) coincides with the value of A on the surface (2.5) (i.e. the function A is invariant by quadratic reparametrization).

(II). The value of A on the surface (2.3) can be obtained from the value on the surface (2.1) by means of the multiplication on $\sigma(K)\,\mathrm{Ber}\,K$. [Here

$\sigma(K) = \operatorname{sgn}\det K_{11}$ where K_{11} is the transformation of commuting variables induced by K.]

Really, a function A on the superspace of quadratic surfaces can be interpreted as a function $A(M^B, L^B_R, T^B_{RS})$ depending on the coefficients in (2.1). If the function A entering into (1.1) is obtained in such a way from the function on the superspace of quadratic surfaces satisfying the conditions above, then the integral (1.1) does not depend on the choice of the parametrization of the submanifold Γ.

We will assume that the density is an algebraic function on the space $\mathscr{A}$.

The supergroup $\mathscr{D}$ of all transformations of the superspace $\mathscr{E}^{M,N}$ can be considered as a supergroup acting in the superspace $\mathscr{A}$: the map $f:\mathscr{E}^{M,N}\to\mathscr{E}^{M,N}$ transforms the surface (2.1) into the surface

$$X^B = M'^B + L'^B_R Y^R + \tfrac{1}{2} T'^B_{RS} Y^R Y^S,$$

where

$$M'^B = f^B(M), \qquad L'^B_R = \left.\frac{\partial f^B}{\partial X^C}\right|_M L^C_R, \qquad T'^B_{RS} = \left.\frac{\partial f^B}{\partial X^C}\right|_M T^C_{RS} + \left.\frac{\partial^2 f^B}{\partial X^C \partial X^D}\right|_M L^C_R L^D_S.$$

The transformations of the space $\mathscr{A}$ generate as usual transformations of the functions on $\mathscr{A}$. In particular, the transformation of $\mathscr{D}$ generates the transformation of (m,n)-densities. The subgroup of $\mathscr{D}$ consisting of transformations leaving invariant the density A will be denoted by $\mathscr{D}_A$. The density will be called perfect if every density A' satisfying $\mathscr{D}_A \subset \mathscr{D}_{A'}$ has the form $A' = \lambda A$ where λ is a constant factor. The action functional corresponding to the perfect density will be called perfect too.

Let $\mathscr{H}$ be a subgroup of the supergroup $\mathscr{D}$. If there exists a unique (up to a constant multiplier) $\mathscr{H}$-invariant density then this density is perfect. We say that this density is perfect with respect to the group $\mathscr{H}$.

Let us consider the complex superspace $\mathscr{C}^{m,n}$ of complex dimension (m,n). The commuting coordinates in $\mathscr{C}$ will be denoted by $x^1, \ldots, x^m$, and anticommuting ones will be denoted by $\theta^1, \ldots, \theta^n$. The supergroup consisting of supervolume preserving analytic transformations of $\mathscr{C}$ will be denoted by $\mathscr{L}$. In other words the transformations belonging to $\mathscr{L}$ have the form

$$\begin{cases} x^k \to \lambda^k(x,\theta) \\ \theta^\alpha \to \mu^\alpha(x,\theta) \end{cases} \tag{2.6}$$

[the right hand sides in (2.6) do not depend on $\bar{x}, \bar{\theta}$]. The transformation (2.6) is supervolume preserving if $\operatorname{Ber} D = 1$ where

$$D = \begin{pmatrix} \dfrac{\partial \lambda^k}{\partial x^\ell} & \dfrac{\partial \lambda^k}{\partial \theta^\beta} \\ \dfrac{\partial \mu^\alpha}{\partial x^\ell} & \dfrac{\partial \mu^\alpha}{\partial \theta^\beta} \end{pmatrix}. \tag{2.7}$$

We will study the $\mathscr{L}$-invariant (r,s)-densities in the case $r = m$, $s = 2n$, $n > 1$. The $\mathscr{L}$-invariant (r,s)-density can be characterized as a function on the space $\mathscr{A}$ of (r,s)-dimensional quadratic surfaces satisfying (1.2), where $\mathscr{G} = \mathscr{L} \times \mathscr{R}$ and for

$(\ell, r) \in \mathscr{G}$ the function $\alpha(\ell, r)$ is defined as $\sigma(r)\,\mathrm{Ber}\,r$ if r is a linear reparametrization; if r is a quadratic reparametrization than $\alpha(\ell, r) = 1$.

At first we will study the action of the group $\mathscr{L} \times \mathscr{R}$ on the space $\mathscr{A}$.

Lemma 2.1. *Almost arbitrary $(m, 2n)$-dimensional quadratic surface in the space $\mathscr{C}^{m,n}$ can be transformed in the quadratic surface having the form*

$$\begin{cases} x^k = \xi^k + i\Gamma^k_{\alpha\beta} v^\beta \bar{v}^\alpha \\ \theta^\alpha = v^\alpha \end{cases}$$
$$\Gamma^k_{\alpha\beta} = \bar{\Gamma}^k_{\beta\alpha}, \tag{2.8}$$

by means of reparametrizations and transformations belonging to the group $\mathscr{L}$.

In (2.8) $\xi^1, \ldots, \xi^m$ are real commuting coordinates and $v^1, \ldots, v^n$ are complex anticommuting coordinates. The surface (2.8) will be called canonical.

To prove the lemma we consider the arbitrary $(m, 2n)$-dimensional quadratic surface

$$\begin{cases} x = x_0 + A\xi + Bv + C\bar{v} + \textit{quadratic terms} \\ \theta = \theta_0 + a\xi + bv + c\bar{v} + \textit{quadratic terms}. \end{cases}$$

Without loss of generality we can assume that $x_0 = 0$, $\theta_0 = 0$ (if $x_0 \neq 0$, or $\theta_0 \neq 0$ we must perform the transformation $x \to x + x_0$, $\theta \to \theta + \theta_0$ belonging to the group $\mathscr{L}$). First, making the linear reparametrization

$$\begin{cases} \xi \to \xi \\ a\xi + bv + c\bar{v} \to v, \end{cases} \tag{2.9}$$

we transform this surface into the surface

$$\begin{cases} x = A_1\xi + B_1 v + C_1\bar{v} + \textit{quadratic terms} \\ \theta = v + \textit{quadratic terms}. \end{cases} \tag{2.10}$$

[This is possible only in the case when the reparametrization (2.9) is nondegenerate. However this condition is fulfilled for almost arbitrary quadratic surface.] If the matrix A_1 is invertible we can make a linear transformation belonging to the group

$$\begin{cases} x \to A_1 x \\ \theta \to \lambda\theta \end{cases}$$
$$\lambda^n = \det A_1,$$

and the reparametrization

$$\begin{cases} \xi \to \xi \\ v \to \lambda v. \end{cases}$$

Then the surface (2.10) will be transformed in the surface

$$\begin{cases} x = \xi + B_2 v + C_2\bar{v} + \textit{quadratic terms} \\ \theta = v + \textit{quadratic terms}. \end{cases}$$

Now using the transformation

$$\begin{cases} x \to x + B_2\theta + \bar{C}_2\theta \\ \theta \to \theta, \end{cases}$$

belonging to $\mathscr{L}$ and the reparametrization

$$\begin{cases} \xi \to \xi + \bar{C}_2 v - C_2 \bar{v} \\ v \to v, \end{cases}$$

we obtain that almost every quadratic surface can be transformed in the surface having the form

$$\begin{cases} x = \xi + \textit{quadratic terms} & (2.11a) \\ \theta = v + \textit{quadratic terms}. & (2.11b) \end{cases}$$

Further we use nonlinear transformations belonging to $\mathscr{L}$. Let us consider the transformation

$$\begin{cases} x \to x + \textit{quadratic terms} + \dots \\ \theta \to \theta + \textit{quadratic terms} + \dots, \end{cases} \tag{2.12}$$

where the omitted terms have order $\geqq 3$ with respect to x, θ. Of course the omitted terms are inessential by the transformation of the quadratic surface (2.11) therefore we can replace (2.12) by the quadratic transformation

$$\begin{cases} x \to x + \textit{quadratic terms} & (2.13a) \\ \theta \to \theta + \textit{quadratic terms}. & (2.13b) \end{cases}$$

We say that the quadratic transformation (2.13) belongs to $\mathscr{L}$. In other words we say that the quadratic transformation belongs to $\mathscr{L}$ if one can obtain an element of $\mathscr{L}$ adding higher order terms to this transformation. It is evident that for the quadratic transformation belonging to $\mathscr{L}$

$$\operatorname{Ber} D = \operatorname{Ber}\begin{pmatrix} \dfrac{\partial\lambda}{\partial x} & \dfrac{\partial\lambda}{\partial\theta} \\ \dfrac{\partial\mu}{\partial x} & \dfrac{\partial\mu}{\partial\theta} \end{pmatrix} = 1 + \text{terms of order} \geqq 2. \tag{2.14}$$

Conversely every quadratic transformation (2.13) satisfying (2.14) belongs to $\mathscr{L}$. It is important to note that the coefficients in (2.13a) are arbitrary; in other words for every coefficient in (2.13a), one can choose the coefficients in (2.13b) in such a way that the requirement (2.14) is satisfied. Moreover, we can satisfy the requirement (2.14) changing the coefficients by transformation of two variables only. Using the quadratic transformations belonging to group $\mathscr{L}$, in Eq. (2.11a) we can exclude all terms which are analytic with respect to ξ, v. By means of quadratic reparametrization one can exclude in (2.11a) all real terms. These remarks permit to exclude all terms except $E^k_{\alpha\beta} v^\beta \bar{v}^\alpha$. For example

$$Fvv + G\bar{v}\bar{v} = (F + \bar{G})vv + (G\bar{v}\bar{v} - \bar{G}vv), \tag{2.15}$$

and we can exclude the first sum by means of quadratic transformations belonging to $\mathscr{L}$ and the second sum by means of reparametrization. The expression $E^k_{\alpha\beta} v^\beta \bar{v}^\alpha$ is real if $E^k_{\alpha\beta} = \bar{E}^k_{\beta\alpha}$. Excluding the real part of $Ev\bar{v}$ by means of reparametrization we obtain that the surface (2.11) can be transformed into the surface

$$\begin{cases} x^k = \xi^k + i\Gamma^k_{\alpha\beta} v^\beta \bar{v}^\alpha \\ \theta^\alpha = v^\alpha + quadratic\ terms\,, \end{cases} \tag{2.16}$$

where $\Gamma^k_{\alpha\beta} = \bar{\Gamma}^k_{\beta\alpha}$ [2]. Using quadratic reparametrization

$$v^\alpha \to v^\alpha + quadratic\ terms\,,$$

we see that every quadratic surface having the form (2.16) can be transformed in (2.8). This completes the proof of Lemma 2.1.

Let us consider the case $m = n^2$. It is convenient to represent the variable x by means of $(n \times n)$-matrix $x^{\alpha\beta}$. The variable ξ parametrizing quadratic surface will be considered as a Hermitian $(n \times n)$-matrix $\xi^{\alpha\beta}$. Slight modification of considerations used by the proof of Lemma 2.1 permits us to check that almost arbitrary quadratic surface can be transformed into the surface

$$\begin{cases} x^{\alpha\beta} = \xi^{\alpha\beta} + iE^{\alpha\beta}_{\gamma\delta} v^\gamma \bar{v}^\delta \\ \theta^\alpha = v^\alpha\,, \end{cases} \tag{2.17}$$

where $E^{\alpha\beta}_{\gamma\delta} = \bar{E}^{\beta\alpha}_{\delta\gamma}$ by means of reparametrizations and transformations belonging to $\mathscr{L}$. It is easy to check that the transformation

$$\begin{cases} x^{\alpha\beta} \to |\lambda|^2 E^{\alpha\beta}_{\gamma\delta} x^{\gamma\delta} \\ \theta^\alpha \to \lambda\theta^\alpha\,, \end{cases} \tag{2.18a}$$

$$\begin{cases} \xi^{\alpha\beta} \to |\lambda|^2 E^{\alpha\beta}_{\gamma\delta} \xi^{\gamma\delta} \\ v^\alpha \to \lambda v^\alpha\,, \end{cases} \tag{2.18b}$$

where $\lambda^n = (\det \hat{E}) \cdot |\lambda|^{2n^2}$, transforms the surface (2.17) into the surface

$$\begin{cases} x^{\alpha\beta} = \xi^{\alpha\beta} + iv^\alpha \bar{v}^\beta \\ \theta^\alpha = v^\alpha\,. \end{cases} \tag{2.19}$$

Here $\hat{E}$ is the operator in the space of matrices transforming $f^{\alpha\beta}$ into $E^{\alpha\beta}_{\gamma\delta} f^{\gamma\delta}$. The surface (2.19) will be denoted by Q_0. We see that almost every quadratic surface can be obtained from the surface (2.19) with the aid of elements of the supergroup $\mathscr{L} \times \mathscr{R}$ (i.e. by means of reparametrizations and transformations belonging to $\mathscr{L}$). It follows from this assertion that in the case under consideration the $\mathscr{L}$-invariant density is completely determined if the value of this density on the surface Q_0 is known. In other words there exists at most one (up to a constant multiplier) $\mathscr{L}$-invariant density. To prove that a $\mathscr{L}$-invariant density exists one must study the isotropy subgroup $\mathscr{H}$ of $\mathscr{L} \times \mathscr{R}$ at the point of $\mathscr{A}$ defined by the surface Q_0.

One can check that if the pair $(\ell, r) \in \mathscr{H}$ and ℓ is defined by the formula (2.6) then

$$\lambda^{\alpha\beta}(0,0) = 0\,, \quad \mu^\alpha(0,0) = 0\,, \quad \left.\frac{\partial \lambda^{\alpha\beta}}{\partial \theta^\gamma}\right|_0 = 0$$

$$A^{\alpha\beta}_{\gamma\delta} = \bar{A}^{\beta\alpha}_{\delta\gamma}\,, \tag{2.20}$$

$$b^\alpha_\gamma \bar{b}^\beta_\delta = A^{\alpha\beta}_{\gamma\delta}\,. \tag{2.21}$$

2 One can explain the important role of the form $\Gamma^k_{\alpha\beta} v^\beta \bar{v}^\alpha$ in our considerations noting that this form can be interpreted as the Levi form of the surface (2.16). In a forthcoming paper we will show how the complex geometry can be applied to the study of supergravity

[We use the notations $A^{\alpha\beta}_{\gamma\delta} = \left.\frac{\partial \lambda^{\alpha\beta}}{\partial x^{\gamma\delta}}\right|_0$, $b^{\alpha}_{\beta} = \left.\frac{\partial \mu^{\alpha}}{\partial \theta^{\beta}}\right|_0$. As before, we consider $A^{\alpha\beta}_{\gamma\delta}$ as a linear operator $\hat{A}$ transforming the matrix $f^{\alpha\beta}$ into the matrix $\tilde{f}^{\alpha\beta} = A^{\alpha\beta}_{\gamma\delta} f^{\gamma\delta}$. The condition (2.20) means that this operator transforms the Hermitian matrix $f^{\alpha\beta}$ to the Hermitian: if $f^{\alpha\beta} = \bar{f}^{\beta\alpha}$ then $\tilde{f}^{\alpha\beta} = \bar{\tilde{f}}^{\beta\alpha}$.] If ℓ is defined by (2.6) and $(\ell, r) \in \mathscr{H}$ then the reparametrization r has the form

$$
\begin{aligned}
&\xi^{\alpha\beta} \to \varrho^{\alpha\beta}(\xi, \nu, \bar{\nu}) \\
&\nu^{\alpha} \to \sigma^{\alpha}(\xi, \nu, \bar{\nu}),
\end{aligned}
\tag{2.22}
$$

where

$$
\varrho^{\alpha\beta}(0,0,0) = 0, \qquad \sigma^{\alpha}(0,0,0) = 0; \tag{2.23}
$$

$$
\begin{gathered}
\left.\frac{\partial \varrho^{\alpha\beta}}{\partial \bar{\nu}^{\gamma}}\right|_0 = 0, \quad \left.\frac{\partial \sigma^{\alpha}}{\partial \bar{\nu}^{\gamma}}\right|_0 = 0, \quad \left.\frac{\partial \varrho^{\alpha\beta}}{\partial \xi^{\gamma\delta}}\right|_0 = \left.\frac{\partial \lambda^{\alpha\beta}}{\partial x^{\gamma\delta}}\right|_0, \\
\left.\frac{\partial \varrho^{\alpha\beta}}{\partial \nu^{\gamma}}\right|_0 = \left.\frac{\partial \lambda^{\alpha\beta}}{\partial \theta^{\gamma}}\right|_0, \quad \left.\frac{\partial \sigma^{\alpha}}{\partial \xi^{\gamma\delta}}\right|_0 = \left.\frac{\partial \mu^{\alpha}}{\partial x^{\gamma\delta}}\right|_0, \quad \left.\frac{\partial \sigma^{\alpha}}{\partial \nu^{\beta}}\right|_0 = \left.\frac{\partial \mu^{\alpha}}{\partial \theta^{\beta}}\right|_0.
\end{gathered}
\tag{2.24}
$$

It follows from (2.21) that

$$
\det \hat{A} = |\det b|^{2n}, \tag{2.25}
$$

and from (2.14) that

$$
\det \hat{A} = \det b. \tag{2.26}
$$

Combining (2.26) with (2.25) we see that

$$
\det \hat{A} = |\det b| = 1. \tag{2.27}
$$

From (2.27) we obtain that the Berezinian of the linear part of the reparametrization (2.22) is equal to 1.

As we noted above the $\mathscr{L}$-invariant density can be characterized as a function on $\mathscr{A}$ satisfying (1.2) for a particular choice of a function β. It follows from the description of the group $\mathscr{H}$ that $\beta(\ell, r) = 1$ for $(\ell, r) \in \mathscr{H}$. Using Lemma 1.2 we obtain that a $\mathscr{L}$-invariant density exists in the case under consideration.

The perfect $\mathscr{L}$-invariant density in the case $m = n^2$ has a larger symmetry group than $\mathscr{L}$. Let us consider the supergroup $\mathscr{L}_1$ of analytic transformations of $\mathscr{C}^{m,n}$ satisfying

$$
\operatorname{Ber} D \cdot \overline{\operatorname{Ber} D} = 1, \tag{2.28}
$$

where D is defined by (2.7). Really, we can repeat for the group $\mathscr{L}_1$ all the arguments used for the group $\mathscr{L}$. The only change is the following. We must replace (2.26) by the weaker condition

$$
|\det \hat{A}| = |\det b|,
$$

which can be deduced from (2.28). However this condition is sufficient to check (2.27). We obtain that a $(n^2, 2n)$-density exists which is perfect with respect to $\mathscr{L}_1$. Of course this density coincides with the perfect $\mathscr{L}$-invariant density.

Let us now study $(m, 2n)$-densities in $\mathscr{C}^{m,n}$ in the general case using Lemmas 1.3 and 2.1. We can identify the set of canonical surfaces with the space $\mathscr{B}$ of sequences $\{\Gamma^1, \ldots, \Gamma^m\}$ of Hermitian $(n \times n)$-matrices. In other words $\mathscr{B}$ can be considered as the space of linear operators Γ acting from m-dimensional space $\mathscr{E}^m$ into n^2-dimensional space of Hermitian matrices $\mathscr{M}$. If T is a linear operator in $\mathscr{C}^n$ we define an operator $\tilde{T}$ in $\mathscr{M}$ as an operator transforming the Hermitian matrix M into T^+MT. For every pair (A, T) consisting of nondegenerate operators acting in $\mathscr{E}^m$ and $\mathscr{C}^n$ respectively, we define a transformation $\tau_{(A,T)}$ of $\mathscr{B}$ by the formula

$$\tau_{(A,T)}\Gamma = \tilde{T}\Gamma A^{-1}.$$

We obtain the group acting in $\mathscr{B}$; this group will be denoted by $\mathscr{K}$. The pairs (A, T) where $\det A = \det T$ form a subgroup of $\mathscr{K}$. This subgroup will be denoted by $\mathscr{K}_1$. The subgroup of $\mathscr{K}$ consisting of pairs (A, T) where the operator T has a real determinant will be denoted by $\mathscr{K}_0$.

Lemma 2.2. *There exists one-one correspondence between $\mathscr{L}$-invariant $(m, 2n)$-densities in $\mathscr{C}^{m,n}$ and the functions $K(\Gamma)$ on $\mathscr{B}$ satisfying*

$$K(\tilde{T}\Gamma A^{-1}) = K(\Gamma)|\det A|^{-1}, \tag{2.29}$$

for every pair $(A, T) \in \mathscr{K}_1$.

To prove this assertion we will use Lemma 1.3, taking $\mathscr{E} = \mathscr{A}$, $\mathscr{F} = \mathscr{B}$, $\mathscr{G} = \mathscr{L} \times \mathscr{R}$, $\mathscr{H} = \mathscr{K}_1$. To check the conditions of Lemma 1.3 we must study the set of pairs $(\ell, r) \in \mathscr{L} \times \mathscr{R}$ transforming one canonical surface into another canonical surface. If the pair $(\ell, r) \in \mathscr{L} \times \mathscr{R}$ consisting of transformation (2.6) and the reparametrization

$$\begin{cases} \xi^k \to \varrho^k(\xi, v, \bar{v}) \\ v^\alpha \to \sigma^\alpha(\xi, v, \bar{v}) \end{cases}$$

transforms the quadratic surface (2.8) into the quadratic surface having the same form then one can verify that

$$\lambda^k(0,0) = \varrho^k(0,0,0) = 0, \qquad \mu^\alpha(0,0) = \sigma^\alpha(0,0,0) = 0;$$

$$\left.\frac{\partial \lambda^k}{\partial \theta^\alpha}\right|_0 = \left.\frac{\partial \varrho^k}{\partial v^\alpha}\right|_0 = 0, \qquad \left.\frac{\partial \sigma^\alpha}{\partial \bar{v}^\beta}\right|_0 = 0, \qquad \left.\frac{\partial \varrho^k}{\partial \bar{v}^\beta}\right|_0 = 0,$$

$$\left.\frac{\partial \lambda^k}{\partial x^\ell}\right|_0 = \left.\frac{\partial \varrho^k}{\partial \xi^\ell}\right|_0, \qquad \left.\frac{\partial \mu^\alpha}{\partial x^\ell}\right|_0 = \left.\frac{\partial \sigma^\alpha}{\partial \xi^\ell}\right|_0, \qquad \left.\frac{\partial \mu^\alpha}{\partial \theta^\beta}\right|_0 = \left.\frac{\partial \sigma^\alpha}{\partial v^\beta}\right|_0.$$

In other words

$$\begin{cases} x^k \to \lambda^k(x, \theta) = A_n{}^k x^n + \ldots \\ \theta^\alpha \to \mu^\alpha(x, \theta) = T^\alpha_\beta \theta^\beta + C^\alpha_\ell x^\ell + \ldots, \end{cases} \tag{2.30}$$

$$\begin{cases} \xi^k \to \varrho^k(\xi, v, \bar{v}) = A_\ell{}^k \xi^\ell + \ldots \\ v^\alpha \to \sigma^\alpha(\xi, v, \bar{v}) = T^\alpha_\beta v^\beta + C^\alpha_\ell \xi^\ell + \ldots, \end{cases} \tag{2.31}$$

where the omitted terms have order $\geqq 2$ with respect to x, θ, ξ, v. The matrix

$$A_\ell{}^k = \left.\frac{\partial \lambda^k}{\partial x^\ell}\right|_0$$

is real and the matrix

$$T^\alpha_\beta = \left.\frac{\partial \mu^\alpha}{\partial \theta^\beta}\right|_0$$

satisfies

$$\det A = \det T$$

[this follows from (2.14)].

The change of the coefficients in (2.8) by the transformations (2.30), (2.31) is given by the formula

$$\Gamma^i \to T^+ \Gamma^j T (A^{-1})^i_j,$$

or briefly

$$\Gamma \to \tilde{T} \Gamma A^{-1}.$$

Using those assertions and Lemma 1.3 we obtain the statement of Lemma 2.2.

Let us denote by Q_{Γ_0} the isotropy subgroup of $\mathscr{K}_0$ at the point $\Gamma_0 \in \mathscr{B}$, i.e. the group of pairs $(A, T) \in \mathscr{K}_0$ satisfying

$$\tilde{T} \Gamma_0 A^{-1} = \Gamma_0 . \tag{2.32}$$

The isotropy subgroup of $\mathscr{K}_1$ at Γ_0 will be denoted by $Q^1_{\Gamma_0}$

$$Q^1_{\Gamma_0} = Q_{\Gamma_0} \cap \mathscr{K}_1 .$$

If $(A, T) \in Q_{\Gamma_0}$ then $(|\lambda|^2 A, \lambda T)$ belongs to Q_{Γ_0} too. If $m \neq n/2$ we can take

$$|\lambda|^{2m} \det A = \lambda^n \det T$$

and obtain from $(A, T) \in Q_{\Gamma_0}$ a pair $(A', T' = (|\lambda|^2 A, \lambda T) \in Q^1_{\Gamma_0}$. If $K(\Gamma)$ satisfies (2.29) and $(A', T') \in Q^1_{\Gamma_0}$ then $|\det A'| = 1$. Noting that

$$\det A' = |\lambda|^{2m} \det A = \lambda^n \det T = \det T' ,$$

we obtain that for $(A, T) \in Q_{\Gamma_0}$, $m \neq n/2$ we have

$$|\det A| = |\det T|^{2m/n} . \tag{2.33}$$

Lemma 2.3. *If* Γ_0 *satisfies* (2.32) *for* $(A, T) \in \mathscr{K}_1$, $K(\Gamma_0) \neq 0$, *and* $m \neq n/2$ *then* $\operatorname{Ker} \Gamma_0 = 0$ *(i.e. the equation* $\Gamma_0 f = 0$ *has only zero solution).*

Let us suppose that $\operatorname{Ker} \Gamma_0 \neq 0$. Then there exists such an operator C that $\Gamma_0 C = 0$, $\operatorname{Sp} C \neq 0$. If $(A, T) \in Q^1_{\Gamma_0}$ then $(A + \lambda C, T) \in Q_{\Gamma_0}$ for every λ. It follows from (2.33) that $|\det(A + \lambda C)|$ does not depend on λ. Taking $A = 1$, $T = 1$ we obtain a contradiction.

If $\mathscr{T}$ is a linear subspace of $\mathscr{M}$ we denote by $\mathscr{R}_{\mathscr{T}}$ the group of unimodular linear operators in $\mathscr{C}^n$ leaving the subspace $\mathscr{T}$ invariant. In other words $U \in \mathscr{R}_{\mathscr{T}}$ if $\det U = 1$ and for every matrix $M \in \mathscr{T}$ we have $\tilde{U} M = U^+ M U \in \mathscr{T}$. If $\mathscr{T}$ is an image of Γ, $\Gamma \in \mathscr{B}$ then $U \in \mathscr{R}_{\mathscr{T}}$ if $\det U = 1$ and for every $f \in \mathscr{E}^m$ one can find $\tilde{f} \in \mathscr{E}^m$ in such a way that $\tilde{U} \Gamma f = U^+ (\Gamma f) U = \Gamma \tilde{f}$.

If $\operatorname{Ker}\Gamma=0$ then $\tilde{f}$ is determined uniquely and one can consider a linear operator A_U in $\mathscr{E}^m$ transforming f into $\tilde{f}$. We see that $U\in\mathscr{R}_{\operatorname{Im}\Gamma}$ if there exists a linear operator A in $\mathscr{E}^m$ satisfying

$$\tilde{U}\Gamma=\Gamma A_U\,.$$

If $\operatorname{Ker}\Gamma=0$, $m\neq n/2$ we can consider for every $U\in\mathscr{R}_{\operatorname{Im}\Gamma}$ a pair $(A,T)\in Q^1_\Gamma$ where

$$A=|\lambda|^2A_U\,,\qquad T=\lambda U\,,\qquad |\lambda|^{2m}\det A_U=\lambda^n\,.$$

It is easy to check that this correspondence between Q^1_Γ and $\mathscr{R}_{\operatorname{Im}\Gamma}$ is an isomorphism. We obtain

Lemma 2.4. *If* $\operatorname{Ker}\Gamma=0$ *and* $m\neq n/2$ *then* $\mathscr{R}_{\operatorname{Im}\Gamma}$ *is isomorphic to* Q^1_Γ.

We will say that the pair (m,n) is perfect if there exists $(m,2n)$-density in $\mathscr{C}^{m,n}$ which is perfect with respect to the group $\mathscr{L}$. Taking into account Lemma 2.2 we see that the pair (m,n) is perfect if and only if there exists a unique (up to a constant multiplier) function $K(\Gamma)$ on $\mathscr{B}$ satisfying (2.29) for every $(A,T)\in\mathscr{K}_1$. We apply Lemma 1.5 to list the perfect pairs (m,n). Using this lemma for $\mathscr{E}=\mathscr{B}$, $\mathscr{G}=\mathscr{K}_1$ and noting that $\dim\mathscr{B}=mn^2$, $\dim\mathscr{K}_1=m^2+2n^2-2$ we see that the pair (m,n) is perfect if and only if

$$(m^2+2n^2-2)-mn^2=\dim Q^1_{\Gamma_0}=\min_{\Gamma\in\mathscr{B}}\dim Q^1_\Gamma\,,$$

and $|\det A|=1$ for every $(A,T)\in Q^1_{\Gamma_0}$. By means of Lemma 2.4 we obtain.

Lemma 2.5. *If* $m\neq n/2$ *then the pair* (m,n) *is perfect if and only if*

$$m(n^2-m)=2n^2-2-\min\dim\mathscr{R}_{\mathscr{T}}=2n^2-2-\dim\mathscr{R}_{\mathscr{T}_0}\,,\tag{2.34}$$

and $|\det A_U|=1$ *for every* $U\in\mathscr{R}_{\mathscr{T}_0}$. *(in* (2.34) $\mathscr{T}$ *runs over all m-dimensional subspaces of* $\mathscr{M}$.*)*

It follows from (2.34) and Lemma 2.3 that for a perfect pair (m,n)

$$m(n^2-m)\leqq 2n^2-2\,,\tag{2.35}$$

$$n^2\geqq m\,.\tag{2.36}$$

(If $m>n^2$, then $\operatorname{Ker}\Gamma_0\neq 0$ for every $\Gamma_0\in\mathscr{B}$ and we can conclude from Lemmas 2.3 and 2.2 that an $\mathscr{L}$-invariant density cannot exist.) It is easy to check that (2.35), (2.36) can be satisfied for $m=1$, $m=2$, $m=n^2-2$, $m=n^2-1$, $m=n^2$ only. It is proved above that in the case $m=n^2$, the pair (m,n) is perfect. In the case $m=1$ the space $\mathscr{B}$ can be considered as the space of Hermitian $(n\times n)$-matrices. For $n\neq 2$ the function

$$K(\Gamma)=|\det\Gamma|^{\frac{1}{n-2}}\tag{2.37}$$

satisfies (2.29) for $(A,T)\in\mathscr{K}_1$. [This follows from relations

$$\det(\tilde{T}\Gamma)=\det(T^+\Gamma T)=|\det T|^2\det\Gamma$$

and $\det\Gamma A=(A)^n\det\Gamma$ where T is an $(n\times n)$-matrix and A is a one-dimensional matrix.]

The arbitrary positive Hermitian matrix Γ can be represented in the form B^+B, $\det B>0$. Hence in the case $n\neq 2$

$$\Gamma=\tilde{T}\Gamma_0 A^{-1}, \tag{2.38}$$

where $(A,T)\in\mathscr{K}_1$, $\Gamma_0=1$. [To verify (2.38) we must define as a one-dimensional matrix

$$A=(\det B)^{\frac{2}{2-n}}$$

and T as $(\det B)^{\frac{1}{2-n}}B$.] We see that the group $\mathscr{K}_1$ has open orbits in $\mathscr{B}$ (one of these orbits is the set of positive matrices). We can conclude from Lemma 1.5 that all algebraic functions on $\mathscr{B}$ satisfying (2.29) for $(A,T)\in\mathscr{K}_1$ coincide with (2.37) (up to a constant multiplier) and therefore the pair $(1,n)$ is perfect for $n\neq 2$. To study the case $m=n^2-1$ we use the following assertion.

Lemma 2.6. *If the pair (m,n) is perfect and $m\neq n/2$, $m\neq n^2-n/2$ then the pair (n^2-m,n) is perfect too.*

To prove this assertion we note that if $U\in\mathscr{R}_{\mathscr{T}}$ then $U^+\in\mathscr{R}_{\mathscr{T}^\perp}$ (here $\mathscr{T}^\perp$ denotes the orthogonal complement of $\mathscr{T}$ in $\mathscr{M}$. The scalar product in $\mathscr{M}$ is defined as usual by the formula $\langle E,E'\rangle=\mathrm{Sp}(EE')$. We see that one-one correspondence exists between $\mathscr{R}_{\mathscr{T}}$ and $\mathscr{R}_{\mathscr{T}^\perp}$, hence

$$\dim\mathscr{R}_{\mathscr{T}}=\dim\mathscr{R}_{\mathscr{T}^\perp}. \tag{2.39}$$

Let us consider the operator $U\in\mathscr{R}_{\mathscr{T}}$ and the corresponding operator $\tilde{U}$ in $\mathscr{M}$. We choose the orthonormal bases $E^1,\ldots,E^{n^2}$ in such a way that $E^i\in\mathscr{T}$ for $1\leqq i\leqq m$ where $m=\dim\mathscr{T}$ and $E^{m+1},\ldots,E^{n^2}$ is the basis of $\mathscr{T}^\perp$. The subspace $\mathscr{T}\subset\mathscr{M}$ is invariant with respect to U. Therefore the matrix of operator U in the basis under consideration has the form

$$\begin{pmatrix} V_{11} & 0 \\ V_{21} & V_{22}\end{pmatrix},$$

where V_{11} is the matrix of the operator A_U acting in $\mathscr{T}$ in the basis $E^1,\ldots,E^m$. The subspace $\mathscr{T}^\perp$ is invariant with respect to $\widetilde{U^+}$ and therefore the matrix of $\widetilde{U^+}$ has the form

$$\begin{pmatrix} W_{11} & W_{12} \\ 0 & W_{22}\end{pmatrix},$$

where W_{22} is the matrix of A_{U^+} in the basis $E^{m+1},\ldots,E^{n^2}$. It is easy to check that the operators $\tilde{U}$ and $\widetilde{U^+}$ are adjoint and therefore $V_{11}=W_{11}^*$, $V_{22}=W_{22}^*$. We see that

$$\det\tilde{U}=\det V_{11}\det V_{22}=\det V_{11}\det W_{22}^*=\det A_U\det A_{U^+}.$$

From the other side

$$\det\tilde{U}=(\det UU^+)^n=1.$$

It follows from these equations that if $|\det A_U|=1$ then $|\det A_{U^+}|=1$. Using this assertion, (2.39) and Lemmas 2.4, 2.5 we obtain the statement of the lemma.

It follows from Lemma 2.6 that the pair (n^2-1,n) is perfect for $n\neq 2$.

Let us consider the case $m=2$. The element $\Gamma\in\mathscr{B}$ can be identified in this case with the pair Γ_1,Γ_2 of Hermitian matrices. If the matrices Γ_1,Γ_2 are diagonal then the pair (A,B) where $A=1$ and B is a diagonal matrix with elements $b_1,\ldots,b_n$ satisfying $|b_i|^2=1$, $b_1b_2\ldots b_n=1$ belongs to Q^1_Γ. We see that

$$t_\Gamma=\dim Q^1_\Gamma\geqq n-1\,. \tag{2.40}$$

For almost every pair of Hermitian matrices Γ_1,Γ_2 one can find such a matrix B that the matrices $\Gamma'_1=B^+\Gamma_1B$ and $\Gamma'_2=B^+\Gamma_2B$ are diagonal. Using this assertion and (2.40) we obtain

$$t=\min t_\Gamma\geqq n-1\,.$$

If the pair $(2,n)$ is perfect then one can conclude from (2.34) that $t=2$ and therefore such a pair can be perfect only in the case $n\leqq 3$.

In the case $n=2$, we consider the function $E(\lambda,\Gamma)=\det(\lambda_1\Gamma_1+\lambda_2\Gamma_2)$. The function $E(\lambda,\Gamma)$ can be regarded as a quadratic form with respect to λ_1,λ_2; the determinant of this form will be denoted by $\mathscr{E}(\Gamma)$. The change of $\mathscr{E}(\Gamma)$ by the transformation $(A,T)\in\mathscr{K}_1$ is given by the formula

$$\mathscr{E}(\tilde{\Gamma})=\mathscr{E}(\tilde{T}\Gamma A^{-1})=\mathscr{E}(\Gamma)(\det T)^4(\det A)^{-2}=\mathscr{E}(\Gamma)(\det A)^2\,.$$

We see that $\mathscr{D}(\Gamma)=|\mathscr{E}(\Gamma)|^{-1/2}$ satisfies (29) for $(A,T)\in\mathscr{K}_1$.

Let us study the case $n=3$. At first we construct the function satisfying (2.29) for $(A,T)\in\mathscr{K}_1$. As follows from Lemma 2.2, such a function determines an $\mathscr{L}$-invariant density. The function

$$E(\lambda,\Gamma)=\det(\lambda_1\Gamma_1+\lambda_2\Gamma_2)$$

can be considered as a cubic form with respect to λ_1,λ_2. The Hessian $H(\lambda,\Gamma)$ of $E(\lambda,\Gamma)$ is a quadratic form

$$H(\lambda,\Gamma)=\begin{vmatrix}\dfrac{\partial^2E}{\partial\lambda_1^2} & \dfrac{\partial^2E}{\partial\lambda_1\partial\lambda_2}\\[2ex] \dfrac{\partial^2E}{\partial\lambda_2\partial\lambda_1} & \dfrac{\partial^2E}{\partial\lambda_2^2}\end{vmatrix}\,.$$

The change of the determinant $D(\Gamma)$ of the quadratic form $H(\lambda,\Gamma)$ by the transformation $(A,T)\in\mathscr{K}_1$ is given by the formula

$$D(\tilde{\Gamma})=D(\tilde{T}\Gamma A^{-1})=D(\Gamma)(\det T)^8(\det A)^{-6}=D(\Gamma)(\det A)^2\,.$$

We see that $K(\Gamma)=|D(\Gamma)|^{-1/2}$ satisfies (2.29) for $(A,T)\in\mathscr{K}_1$. To prove that every function satisfying (2.29) is proportional to $|D(\Gamma)|^{-1/2}$ we must check that $t=\min\dim\mathscr{R}_{\mathscr{T}}$, where $\mathscr{T}$ runs over two-dimensional subspaces of $\mathscr{M}$, is equal to 2. If Γ_1 is a positive matrix and Γ_2 is an arbitrary Hermitian matrix then one can find such a matrix B that $\Gamma'_1=B^+\Gamma_1B=1$ and $\Gamma'_2=B^+\Gamma_2B$ is a diagonal matrix. It follows from this assertion that it is sufficient to study the case when $\mathscr{T}$ is spanned on the matrices $\Gamma_1=1$ and $\Gamma_2=\operatorname{diag}(\lambda_1,\lambda_2,\lambda_3)$ where $\lambda_1>\lambda_2>\lambda_3$. [We use the notation $\operatorname{diag}(\lambda_1,\ldots,\lambda_n)$ for a diagonal matrix with elements $\lambda_1,\ldots,\lambda_n$.] We will prove that in this case $\dim\mathscr{R}_{\mathscr{T}}=2$. Let us consider the operator $T\in\mathscr{R}_{\mathscr{T}}$. This operator transforms the matrix $\Gamma_1=1\in\mathscr{T}$ into the matrix $\Delta=T^+T$, $\Delta\in\mathscr{T}$. If the matrix Δ is not scalar then the space $\mathscr{T}$ is spanned on $L_1=1$ and Δ. The operator T transforms $\Delta\in\mathscr{T}$ into the matrix $T^+\Delta T=(T^+)^2T^2\in\mathscr{T}$, hence

$$(T^+)^2T^2=\alpha T^+T+\beta\cdot 1. \tag{2.41}$$

Conversely, if T satisfies (2.41) and $\det T=1$, then $T\in\mathscr{R}_{\mathscr{T}}$ where $\mathscr{T}$ is spanned on 1 and T^+T. The operator T is unimodular, therefore one can represent T in the form $T=US$ where S is a Hermitian matrix, U is a unitary matrix, $\det S=\det U=1$. Using this representation we obtain

$$U^+S^2U=\alpha+\beta S^{-2}, \tag{2.42}$$

where $S^2=T^+T=\Delta$. Noting that $S^2\in\mathscr{T}$ we obtain that S^2 is a diagonal matrix, and the diagonal elements $\sigma_1,\sigma_2,\sigma_3$ of S^2 can be represented in the form

$$\sigma_i=k+\ell\lambda_i.$$

Therefore $\sigma_1>\sigma_2>\sigma_3$ or $\sigma_1<\sigma_2<\sigma_3$. It follows from (2.42) that S^2 is unitarily equivalent to $\beta S^{-2}+\alpha$. Hence for $\beta<0$

$$\begin{cases}(k+\ell\lambda_1)=\alpha+\beta(k+\ell\lambda_1)^{-1}\\(k+\ell\lambda_2)=\alpha+\beta(k+\ell\lambda_2)^{-1}\\(k+\ell\lambda_3)=\alpha+\beta(k+\ell\lambda_3)^{-1},\end{cases} \tag{2.43}$$

and for $\beta>0$

$$\begin{cases}(k+\ell\lambda_1)=\alpha+\beta(k+\ell\lambda_3)^{-1}\\(k+\ell\lambda_2)=\alpha+\beta(k+\ell\lambda_2)^{-1}\\(k+\ell\lambda_3)=\alpha+\beta(k+\ell\lambda_1)^{-1}.\end{cases} \tag{2.44}$$

Adding to (2.43) and (2.44) the equation

$$(k+\ell\lambda_1)(k+\ell\lambda_2)(k+\ell\lambda_3)=1 \tag{2.45}$$

which follows from $\det S^2=1$, we obtain two systems of equations for k,ℓ,α,β. These systems can have only a finite number of solutions; therefore there exists only a finite number of matrices S^2. If the matrix S^2 is known, then there exists at most a two parameter family of matrices U satisfying (2.42). [The matrix U transforms the vectors $e_1=(1,0,0)$, $e_2=(0,1,0)$, $e_3=(0,0,1)$ into the vectors u_1e_1, u_2e_2, u_3e_3 if $\beta<0$ and into the vectors v_1e_3, v_2e_2, v_3e_1 if $\beta>0$. Noting that $|u_i|=1$, $|v_i|=1$, $u_1u_2u_3=1$, $v_1v_2v_3=1$ we obtain that the set of matrices U is at most two dimensional.]

We have considered such matrices $T\in\mathscr{R}_{\mathscr{T}}$, that the matrix $\Delta=T^+T$ is not scalar, and proved that the set of these matrices is two-dimensional. To complete the calculation of $\dim\mathscr{R}_{\mathscr{T}}$ we must study matrices $T\in\mathscr{R}_{\mathscr{T}}$ such that T^+T is a scalar matrix. Using that $\det T=1$ we obtain $\det T^+T=1$ and therefore $T^+T=1$, i.e. $T\in\mathrm{SU}(3)$. The matrix T transforms the diagonal matrix

$$\Gamma_2=\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3)\in\mathscr{T}$$

into the diagonal matrix $T^+\Gamma_2T=T^{-1}\Gamma_2T\in\mathscr{T}$, hence T transforms eigenvectors e_1,e_2,e_3 of Γ_2 into $u_1e_{\sigma(1)}$, $u_2e_{\sigma(2)}$, $u_3e_{\sigma(3)}$, and σ is an arbitrary permutation. We see that the set of matrices T under consideration is at most two-dimensional too. We obtain $\dim \mathscr{R}_{\mathscr{T}} = 2$ and therefore the pair $(2,3)$ is perfect; by analogy one can obtain that the pair $(2,2)$ is perfect. The case $m = n^2 - 2$ can be reduced to the case $m = 2$ by means of Lemma 2.6.

We have proved the main statement of present section.

Theorem 2.1. *The pair (m,n) is perfect only in the cases $m = n^2$, $n > 1$; $m = 1$ or $m = n^2 - 1$, $n \geqq 3$; $m = 2$ or $m = 7$, $n = 3$; $m = 2$, $n = 2$.*

The supergroup $\mathscr{L}$ is imbedded in the supergroup $\mathscr{L}_\alpha$ of transformations satisfying

$$\mathrm{Ber}\, D (\overline{\mathrm{Ber}\, D})^\alpha = 1$$

[here D is defined by (2.7)]. We have considered this supergroup in the case $\alpha = 1$ above. It is easy to check that $\mathscr{L}_\alpha \subset \mathscr{L}_1$ for $\alpha \neq -1$.

We have proved above that the $(n^2, 2n)$-density which is perfect with respect to $\mathscr{L}$ is $\mathscr{L}_1$-invariant. This assertion can be generalized in the following way.

Theorem 2.2. *Every $\mathscr{L}$-invariant $(m, 2n)$-density in $\mathscr{C}^{m,n}$ is $\mathscr{L}_1$-invariant.*

The proof of this theorem is based on the modification of Lemma 2.2.

Lemma 2.2′. *There exists one-one correspondence between $\mathscr{L}_1$-invariant $(m, 2n)$-densities in $\mathscr{C}^{m,n}$ and the functions $K(\Gamma)$ of $\mathscr{B}$ satisfying*

$$K(\tilde{T}\Gamma A^{-1}) = K(\Gamma)|\det A| \cdot |\det T|^{-2} = K(\Gamma)|\det A|^{-1},$$

for every pair $(A, T) \in \mathscr{K}_2$. (We use here the notation $\mathscr{K}_2$ for subgroup of $\mathscr{K}$ consisting of elements satisfying $|\det A| = |\det T|$, $A = A^+$.)

The proof of Lemma 2.2′ is similar to the proof of Lemma 2.2.

If $(A, T) \in \mathscr{K}_2$ then $\det A = \exp(i\varphi)\det T = \det T'$ where $T' = \exp(i\varphi/n)T$. Noting that $\tilde{T}'\Gamma A^{-1} = \tilde{T}\Gamma A^{-1}$ for every $\Gamma \in \mathscr{B}$ and $(A, T') \in \mathscr{K}_1$, we obtain that every function satisfying (2.29) for $(A, T) \in \mathscr{K}_1$ satisfies (2.29) for $(A, T) \in \mathscr{K}_2$. Using Lemmas 2.2 and 2.2′ we obtain that every $\mathscr{L}$-invariant $(m, 2n)$-density is $\mathscr{L}_1$-invariant.

It follows from Theorem 2.2 that $(m, 2n)$-densities in $\mathscr{C}^{m,n}$ which are perfect with respect to the supergroup $\mathscr{L}_\alpha$ for $\alpha \neq -1$ coincide with perfect $\mathscr{L}$-invariant densities. One can check that the $(m, 2n)$-density which is perfect with respect to $\mathscr{L}_{-1}$ cannot exist if $n > 1$.

3. Generalized Supergravity

In the present section we study the $\mathscr{L}$-invariant $(m, 2n)$-density $A(Q)$ in $\mathscr{C}^{m,n}$ in the case $m = n^2$ (as was shown earlier, the case $m = 4$, $n = 2$ corresponds to the supergravity). First of all we will give an explicit expression for the density $A(Q)$.

Let us normalize the density under consideration by the condition $A(Q_0) = 1$. Here Q_0 as in Sect. 2 denotes the quadratic surface defined by the equation

$$\begin{cases} x^{\alpha\beta} = \xi^{\alpha\beta} + i v^\alpha \bar{v}^\beta \\ \theta^\alpha = v^\alpha \end{cases} \tag{3.1}$$

($x^{\alpha\beta}$ and $\xi^{\alpha\beta}$ are even variables, θ^α and v^α are odd variables, $\bar{\xi}^{\alpha\beta}=\xi^{\beta\alpha}$). It is shown in Sect. 2 that the density $A(Q)$ is perfect, i.e. there exists a unique $\mathscr{L}$-invariant $(n^2,2n)$-density $A(Q)$ satisfying the normalization condition $A(Q_0)=1$.

The quadratic surface Q_E

$$\begin{cases} x^{\alpha\beta}=\xi^{\alpha\beta}+iE^{\alpha\beta}_{\gamma\delta}v^\gamma\bar{v}^\delta \\ \theta^\alpha=v^\alpha, \end{cases}$$

where $E^{\alpha\beta}_{\gamma\delta}=\bar{E}^{\beta\alpha}_{\delta\gamma}$ can be transformed into Q_0 with the aid of transformations (2.18).

Using Eq. (2.18), (2.29), and $\det(\lambda^2\hat{E})=\lambda^{2n^2}\det\hat{E}$ we obtain

$$A(Q_E)=|\det\hat{E}|^{1/(1-2n)}. \tag{3.2}$$

Let us consider now the quadratic surface Q defined by the equations

$$\begin{cases} x^k=\xi^k+i(b^k_\ell\xi^\ell+v^\gamma\varrho^k_\gamma+\sigma^k_\gamma\bar{v}^\gamma+e^k_{\gamma\delta}v^{\gamma\delta}+f^k_{\gamma\delta}v^\gamma v^\delta \\ \qquad +g^k_{\gamma\delta}\bar{v}^\gamma\bar{v}^\delta+v^\gamma\varphi^k_{\ell\gamma}\xi^\ell+\psi^k_{\ell\gamma}\bar{v}^\gamma\xi^\ell+d^k_{\ell m}\xi^\ell\xi^m) \\ \theta^\alpha=v^\alpha. \end{cases} \tag{3.3}$$

[The Greek indices run over the integers $1,2,\ldots,n$. Latin letters denote the pairs of Latin indices; for example $k=(\alpha,\beta)$.] If $b^k_\ell=0$, $\varrho^k_\gamma=0$, $\sigma^k_\gamma=0$ then the surface (3.3) can be transformed into the surface Q_E with

$$E^{\alpha\beta}_{\gamma\delta}=\tfrac{1}{2}(e^{\alpha\beta}_{\gamma\delta}+\bar{e}^{\beta\alpha}_{\delta\gamma})$$

by means of quadratic transformations belonging to $\mathscr{L}$ and quadratic reparametrizations. (This follows from arguments used by the proof of Lemma 2.1.) We obtain that in this case

$$A(Q)=A(Q_E)=|\det(\tfrac{1}{2}(e^{\alpha\beta}_{\gamma\delta}+\bar{e}^{\beta\alpha}_{\delta\gamma}))|^{1/(1-2n)}. \tag{3.4}$$

If $b^{\alpha\beta}_{\gamma\delta}=\bar{b}^{\beta\alpha}_{\delta\gamma}$, $\varrho^{\alpha\beta}_\gamma=\bar{\sigma}^{\beta\alpha}_\gamma$ then the surface (3.3) can be transformed into the quadratic surface with $b=0$, $\varrho=0$, $\sigma=0$, with the aid of the linear transformation

$$\begin{cases} x^k\to(1+ib)^k_\ell x^\ell+2ip\theta^\gamma[(1-ib)^{-1}]^k_\ell\varrho^\ell_\gamma \\ \theta^\alpha\to p\theta^\alpha, \end{cases} \tag{3.5}$$

and the linear reparametrization

$$\begin{cases} \xi^k\to\xi^k+ipv^\beta[(1-ib)^{-1}]^k_\ell\varrho^\ell_\beta-i\bar{p}[(1+ib)^{-1}]^k_\ell\sigma^\ell_\gamma\bar{v}^\gamma \\ v^\alpha\to pv^\alpha \end{cases} \tag{3.6}$$

(here $(1+ib)^k_\ell=\delta^k_\ell+ib^k_\ell$, $p=[\det(1+ib)]^{1/n}$).

Using (3.5) and (3.6) we obtain that the value of the density $A(Q)$ on the surface (3.3) is given by the formula

$$A(Q)=[\det(1+b^2)]^{-n/(1-2n)}|\det\hat{\mathscr{E}}|^{1/(1-2n)}, \tag{3.7}$$

where $(1+b^2)^k_\ell=\delta^k_\ell+b^k_m b^m_\ell$,

$$\mathscr{E}^k_{\alpha\beta}=\tilde{e}^k_{\alpha\beta}+\tilde{\varphi}^k_{\ell\alpha}\lambda^\ell_\beta-\tilde{\psi}^k_{\ell\beta}\mu^\ell_\alpha+2\tilde{d}^k_{\ell m}\lambda^\ell_\alpha\mu^m_\beta;$$

$$\tilde{e}^{\alpha\beta}_{\gamma\delta}=\tfrac{1}{2}(e^{\alpha\beta}_{\gamma\delta}+\bar{e}^{\beta\alpha}_{\delta\gamma}),\qquad \tilde{\varphi}^{\alpha\beta}_{\gamma\delta,\sigma}=\tfrac{1}{2}(\varphi^{\alpha\beta}_{\gamma\delta,\sigma}+\bar{\varphi}^{\beta\alpha}_{\delta\gamma,\sigma}),$$

$$\tilde{\psi}^{\alpha\beta}_{\gamma\delta,\sigma}=\tfrac{1}{2}(\psi^{\alpha\beta}_{\gamma\delta,\sigma}+\bar{\psi}^{\beta\alpha}_{\delta\gamma,\sigma}),\qquad \tilde{d}^{\alpha\beta}_{\gamma\delta,\varrho\sigma}=\tfrac{1}{2}(d^{\alpha\beta}_{\gamma\delta,\varrho\sigma}+\bar{d}^{\beta\alpha}_{\delta\gamma,\sigma\varrho});$$

$$\lambda^k_\alpha=i[(1-ib)^{-1}]^k_\ell\varrho^\ell_\alpha,\qquad \mu^k_\alpha=-i[(1+ib)^{-1}]^k_\ell\sigma^\ell_\alpha.$$

Similar considerations permit us to calculate $A(Q)$ for more general quadratic surfaces.

Using the results above we can write the action functional for every $(n^2, 2n)$-dimensional surface Γ in the space $\mathscr{C}^{n^2,n}$. Without essential loss of generality we suppose that the parametric equation of the surface Γ is represented in the form

$$\begin{cases} x^k = \xi^k + i\mathscr{H}^k(\xi, v, \bar{v}) \\ \theta^\alpha = v^\alpha, \end{cases} \tag{3.8}$$

where $\mathscr{H}^{\alpha\beta} = \bar{\mathscr{H}}^{\beta\alpha}$. At the point of Γ corresponding to parameters ξ_0, v_0 we can construct the tangential quadratic surface to Γ; this surface can be defined by Eq. (3.3) where

$$b^k_\ell = \partial_\ell \mathscr{H}^k, \quad \varrho^k_\gamma = \partial_\gamma \mathscr{H}^k, \quad \sigma^k_\gamma = \bar{\partial}_\gamma \mathscr{H}^k, \quad e^k_{\gamma\delta} = \partial_\gamma \bar{\partial}_\delta \mathscr{H}^k,$$
$$f^k_{\gamma\delta} = \tfrac{1}{2}\partial_\gamma \partial_\delta \mathscr{H}^k, \quad g^k_{\gamma\delta} = \tfrac{1}{2}\bar{\partial}_\gamma \bar{\partial}_\delta \mathscr{H}^k, \quad \varphi^k_{\ell\gamma} = \partial_\ell \partial_\gamma \mathscr{H}^k,$$
$$\psi^k_{\ell\gamma} = \bar{\partial}_\gamma \partial_\ell \mathscr{H}^k, \quad d^k_{\ell m} = \tfrac{1}{2}\partial_\ell \partial_m \mathscr{H}^k,$$
$$\partial_\gamma = \frac{\partial}{\partial v^\gamma}, \quad \bar{\partial}_\gamma = \frac{\partial}{\partial \bar{v}^\gamma}, \quad \partial_\ell = \frac{\partial}{\partial \xi^\ell},$$

all derivatives are calculated at the point (ξ_0, v_0). The value of the density $A(Q)$ on the tangential quadratic surface can be calculated with the aid of (3.7). We obtain that the value of the action functional on the surface (3.8) is given by

$$\begin{aligned} S(\Gamma) = \int d\xi dv d\bar{v} [\det(1 + \mathscr{H}^2)]^{1/2} \\ \cdot [|\det(\Delta_\alpha \bar{\Delta}_\beta \mathscr{H}^k)|]^{1/(2-4n)} [|\det(\bar{\Delta}_\beta \Delta_\alpha \mathscr{H}^k)|]^{1/(2-4n)}, \end{aligned} \tag{3.9}$$

where

$$(1 + \mathscr{H}^2)^k_\ell = \delta^k_\ell + \mathscr{H}^k_m \mathscr{H}^m_\ell, \quad \mathscr{H}^k_\ell = \partial_\ell \mathscr{H}^k,$$
$$\Delta_\alpha = \partial_\alpha + i\partial_\alpha \mathscr{H}^k [(1 - i\mathscr{H})^{-1}]^\ell_k \partial_\ell,$$
$$\bar{\Delta}_\alpha = -\bar{\partial}_\alpha + i\bar{\partial}_\alpha \mathscr{H}^k [(1 + i\mathscr{H})^{-1}]^\ell_k \partial_\ell.$$

For $m=4$, $n=2$ the action functional (3.9) coincides with the action functional in supergravity [2].

In a similar way one can calculate the value of the action functional on the surface defined by the parametric equation

$$\begin{cases} x^k = F^k(\xi, v, \bar{v}) = G^k(\xi, v, \bar{v}) + iH^k(\xi, v, \bar{v}), \\ \theta^\alpha = v^\alpha, \end{cases} \tag{3.10}$$

where $G^+ = G$, $H^+ = H$. We obtain

$$\begin{aligned} S(\Gamma) = \int d\xi dv d\bar{v} [\det(G^2 + H^2)]^{(n-1)/(2n-1)} \\ \cdot |\det(G^k_\ell A_{\alpha\beta} H^\ell - H^k_\ell A_{\alpha\beta} G^\ell)|^{1/(1-2n)}, \end{aligned} \tag{3.11}$$

where

$$(G^2 + H^2)^k_\ell = G^k_m G^m_\ell + H^k_m H^m_\ell,$$
$$G^k_\ell = \partial_\ell G^k, \quad H^k_\ell = \partial_\ell H^k, \quad F^k_\ell = \partial_\ell F^k,$$
$$\begin{aligned} A_{\alpha\beta} = \partial_\alpha \bar{\partial}_\beta + \bar{\partial}_\alpha F^\ell [(F)^{-1}]^m_\ell \partial_\beta \partial_m - \partial_\alpha (F^+)^\ell [(F^+)^{-1}]^m_\ell \bar{\partial}_\beta \partial_m \\ + \partial_\alpha (F^+)^\ell \bar{\partial}_\beta F^m [(F^+)^{-1}]^s_\ell (F^{-1})^t_m \partial_s \partial_t. \end{aligned}$$

It is proved in Sect. 2 that the density under consideration is invariant with respect to analytic transformations satisfying (2.28). We have denoted the supergroup consisting of these transformations by $\mathscr{L}_1$. The transformations belonging to $\mathscr{L}_1$ can be considered as local symmetry transformations.

The group $\mathscr{L}_1$ acts in natural way in the superspace of all surfaces. The action functional $S(\Gamma)$ is $\mathscr{L}_1$-invariant; in other words if an element of $\mathscr{L}_1$ transforms the surface Γ into the surface Γ' then the values of the action functional on Γ and Γ' coincide.

In the case of supergravity ($m=4$, $n=2$) the surface Q_0 can be considered as flat space or as classical vacuum (see [2, 1]). In a more general case one can interpret Q_0 in a similar way. The transformations belonging to $\mathscr{L}_1$ and leaving invariant the surface Q_0 can be interpreted as global symmetries; the supergroup consisting of these transformations will be denoted by $\mathscr{P}$. (It is important to note that we do not consider Q_0 as a quadratic surface here.) Let us list transformations belonging to $\mathscr{P}$. For every unimodular $(n\times n)$-matrix a^α_β we can construct the transformation

$$\begin{cases} x^{\alpha\beta}\to a^\alpha_\gamma \bar{a}^\beta_\delta x^{\gamma\delta}+b^{\alpha\beta} \\ \theta^\alpha\to a^\alpha_\gamma\theta^\gamma, \end{cases} \tag{3.12}$$

where $b^{\alpha\beta}=\bar{b}^{\beta\alpha}$, leaving invariant Q_0. In the case of supergravity ($n=2$) the transformations (3.12) can be identified with Poincaré transformations. Further, the transformations

$$\begin{cases} x^{\alpha\beta}\to x^{\alpha\beta}+2i\theta^\alpha\bar{\varepsilon}^\beta+i\varepsilon^\alpha\bar{\varepsilon}^\beta \\ \theta^\alpha\to\theta^\alpha+\varepsilon^\alpha \end{cases} \tag{3.13}$$

belong to $\mathscr{P}$ too. The supergroup $\mathscr{P}$ is generated by (3.12) and (3.13). In the case $n=2$ we obtain that $\mathscr{P}$ coincides with the Golfand-Likhtman supergroup.

The action functional defined on (r,s)-dimensional surfaces can be considered as an action functional on the boson and fermion fields depending on the point of r-dimensional space (see [1]). Therefore a $(n^2,2n)$-density in $\mathscr{C}^{n^2,n}$ determines a field theory in n^2-dimensional space. In such a way the densities under consideration determine a field theory in physical four-dimensional space only in the case $n=2$ (the supergravity). However one can think in certain cases that multidimensional theory can be equivalent to a four-dimensional theory for low energies. (To construct a quantum theory we must study a functional integral over the space of (r,s)-dimensional surfaces of the theory as described by a (r,s)-density. Let us consider surfaces lying in a small neighborhood of a $(4,s)$-dimensional surface; these surfaces can be interpreted as tubes having small size in $(r-4)$ directions. It is possible that for low energies only such tubes are essential in the functional integral; then the multi-dimensional theory can be reduced to the four-dimensional one. This reduction is similar to the spontaneous compactification described in [5].) It is not easy to find cases when the multidimensional theory admits dimensional reduction for low energies and to describe the reduced theory. However we can use formal dimensional reduction as in [6]. Namely we fix a subspace of the space of surfaces in such a way that the surfaces belonging to this subspace can be characterized by means of functions depending on four even variables. Then the action functional on this subspace generates a four-dimensional field theory.

Let us describe the dimensional reduction of $\mathscr{L}_1$-invariant $(n^2, 2n)$-density $A(Q)$ in $\mathscr{C}^{n^2,n}$. We will assume that $n=2r$ and consider the $(n^2, 2n)$-dimensional surfaces having a parameter equation of the form

$$\begin{cases} x^{\alpha\beta} = \xi^{\alpha\beta} + i\mathscr{H}^{\alpha\beta}(\tilde{\mathrm{tr}}\,\xi, v, \bar{v}) \\ \theta^\alpha = v^\alpha . \end{cases} \tag{3.14}$$

Here the symbol $\tilde{\mathrm{tr}}\,\xi$ denotes (2×2)-matrix with matrix elements

$$(\tilde{\mathrm{tr}}\,\xi)^{\mu\nu} = \sum_{\varkappa=1}^{r} \xi^{2\varkappa+\mu,\, 2\varkappa+\nu},$$

where $\mu, \nu = 0, 1$.

It is convenient to represent the $(2r\times 2r)$-matrix ξ as a $(r\times r)$-matrix Ξ consisting of (2×2)-matrices $\Xi^{\tau\omega}$ (the matrix $\Xi^{\tau\omega}$ has matrix elements $\xi^{2\tau+\mu,\, 2\omega+\nu}$ where $\tau, \omega = 1, \ldots, r$). Then $\tilde{\mathrm{tr}}\,\xi$ can be considered as the trace of Ξ:

$$\tilde{\mathrm{tr}}\,\xi = \Xi^{11} + \ldots + \Xi^{rr} .$$

The surface (3.14) is invariant by the translation $x^{\alpha\beta} \to x^{\alpha\beta} + b^{\alpha\beta}$ where $b^{\alpha\beta} = \bar{b}^{\beta\alpha}$, $\tilde{\mathrm{tr}}\, b = 0$. One can replace the Eq. (3.14) by this invariance property.

The functions $\mathscr{H}^{\alpha\beta}(\tilde{\mathrm{tr}}\,\xi, v, \bar{v})$ depend on four even variables. Therefore one can consider the $(n^2, 2n)$-density $A(Q)$ on (3.14) as a Lagrangian L_n defined on the fields in four-dimensional space. If the element of $\mathscr{L}_1$ transforms every surface of the form (3.14) into the surface having the same form then this element can be considered as a symmetry transformation of the Lagrangian L_n. The supergroup consisting of these elements will be denoted by $\mathscr{L}'$. It is easy to check that the transformation

$$\begin{cases} x^{\alpha\beta} \to \lambda^{\alpha\beta}(x, \theta) \\ \theta^\alpha \to \mu^\alpha(x, \theta) \end{cases} \tag{3.15}$$

belongs to $\mathscr{L}'$ when the functions $\mu^\alpha(x, \theta)$,

$$\tilde{\mathrm{tr}}\, \mathrm{Re}\,\lambda(x, \theta) = \tilde{\mathrm{tr}}\, \frac{\lambda(x, \theta) + \lambda^+(x, \theta)}{2}, \qquad \mathrm{Im}\,\lambda(x, \theta) = \frac{\lambda(x, \theta) - \lambda^+(x, \theta)}{2i}$$

satisfy

$$\mu^\alpha(x+b, \theta) = \mu^\alpha(x, \theta),$$
$$\tilde{\mathrm{tr}}\, \mathrm{Re}\,\lambda(x+b, \theta) = \tilde{\mathrm{tr}}\, \mathrm{Re}\,\lambda(x, \theta),$$
$$\mathrm{Im}\,\lambda(x+b, \theta) = \mathrm{Im}\,\lambda(x, \theta)$$

if $b^{\alpha\beta} = \bar{b}^{\beta\alpha}$, $\tilde{\mathrm{tr}}\, b = 0$ (in other words these functions must depend on $\tilde{\mathrm{tr}}\, \mathrm{Re}\, x$, $\mathrm{Im}\, x$, $\theta^1, \ldots, \theta^n$). The subgroup of $\mathscr{L}'$ consisting of elements transforming every surface of the form (3.14) into the same surface will be denoted by $\mathscr{L}''$. The group $\mathscr{L}'/\mathscr{L}''$ can be considered as the group of local symmetries of the Lagrangian L_n. The surface Q_0 has the form (3.14); one can consider this surface as the classical vacuum of the Lagrangian L_n. The supergroup $\mathscr{P}\cap\mathscr{L}'/\mathscr{P}\cap\mathscr{L}''$ can be interpreted as the group of global symmetries of the Lagrangian L_n. The transformation (3.13) belongs to

$\mathscr{P} \cap \mathscr{L}'$. The transformation (3.12) belongs to $\mathscr{P} \cap \mathscr{L}'$ in the case when there exists such a unitary $(r \times r)$-matrix A^{τ}_{ω} and unimodular (2×2)-matrix c^{μ}_{ν} that

$$a^{2\tau+\mu}_{2\omega+\nu} = A^{\tau}_{\omega} c^{\mu}_{\nu}$$

for $\mu, \nu = 0, 1$; $\tau, \omega = 1, \ldots, r$ [in other words the matrix a^{α}_{β} can be considered as $(r \times r)$-matrix consisting of (2×2)-matrices $A^{\tau}_{\omega} c$]. The group $\mathscr{P} \cap \mathscr{L}''$ consists of transformations

$$x^{\alpha\beta} \to x^{\alpha\beta} + b^{\alpha\beta},$$

where $b^{\alpha\beta} = \bar{b}^{\beta\alpha}$, $\tilde{\operatorname{tr}}\, b = 0$. It follows from this description of $\mathscr{P} \cap \mathscr{L}'$ and $\mathscr{P} \cap \mathscr{L}''$ that the Lie superalgebra of $\mathscr{P} \cap \mathscr{L}' / \mathscr{P} \cap \mathscr{L}''$ is a directed sum of the extended Golfand-Likhtman algebra and Lie algebra of the group $U(r)$.

We see that the global symmetry group of L_n is similar to the global symmetry group in extended supergravity. However the Lagrangian L_n describes many fields having spin 2 and fields having higher spins.

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When finishing this paper I learned about the most tragic and utterly unexpected death of my close friend F. A. Berezin. His death has interrupted extremely fruitful and very promising scientific activities. His contribution to different branches of mathematics and mathematical physics is outstanding.

He was the first to realize that simultaneously with the usual analysis of functions which depend on commuting variables one can construct also the analysis of functions depending on anticommuting variables. This led him naturally to the new notion of a Lie group with anticommuting parameters. Known now as supergroups, these objects were in fact introduced by Berezin earlier than by physicists. Later he developed nice and coherent theories of supergroups and supermanifolds. There is no doubt that these theories will be very useful for supersymmetric quantum field theories which are now becoming more and more important.

We dedicate this work to the bright memory of F. A. Berezin. A. S. Schwarz

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