Commun. Math. Phys. 79, 141–151 (1981)

Communications in
Mathematical
Physics

On the Existence of Antiparticles

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Abstract. Without assuming the existence of interpolating fields, it is shown that any particle in a massive quantum field theory possesses a unique antiparticle and carries parastatistics of finite order. This closes a gap in the hitherto existing theoretical argument leading to particle statistics and to the existence of antiparticles.

1. Introduction

The existence of antiparticles is a well established experimental fact in elementary particle physics. In the conventional framework of quantum field theory, the explanation of this fact is given via the TCP-theorem or, more generally, using the Jost-Lehmann-Dyson representation of the two-point-function (see for example [1, II]). In both cases the existence of local fields, interpolating between vacuum and particle states, is crucial.

However, the assumption about the existence of those fields is not very natural for particles which are separated from the vacuum by some superselection rule. Then an interpolating field is not observable by principle, and the locality assumption for these fields has no obvious physical interpretation.

In fact, it is well known that charged particle states in gauge theories cannot have local interpolating fields. Wheras in abelian gauge theories charged particles only exist in the presence of massless particles [2, 3], for nonabelian gauge theories such a result is not known; there might be charged particles also in cases, where there are no physical massless particles.

Now for particles in theories with a gap in the mass spectrum, localization properties have recently been found [4, I] which admit a nearly complete discussion of the structure of multiparticle states [4, II]. This discussion follows closely the investigations of Doplicher et al. [1]. These authors consider only those states as "being of interest for elementary particle physics" which become vacuumlike on the spacelike complement of a sufficiently large bounded region. This selection criterion for states is in some sense a physical interpretation of the

0010-3616/81/0079/0141/$02.20

assumption of interpolating local fields; in fact it excludes charged states in gauge theories. In this framework, they proved the existence of antiparticles for those particles which carry parastatistics of finite order. But the occurrence of infinite statistics could not be excluded; in this case antiparticles in the usual sense would not exist.

The arguments in [1] can be generalized to the situation discussed in [4]; however, there remains a gap in the theoretical explanation of antiparticles and particle statistics.

In this paper, we shall construct the antiparticle-sector of a given particle directly, using only the spectral properties of particles in a massive theory and local commutativity of observables. Then we shall prove finiteness of statistics and the existence of particles in the antiparticle-sector with the same mass as the original particle.

The argument starts from the proved localization properties of particle states which can be expressed in the following way:

To any spacelike cone S in Minkowski space, there exists a morphism ϱ from the algebra of local observables $\mathfrak{A}$, represented as an operator algebra in the vacuum Hilbert space $\mathscr{H}_0$, into the algebra $\mathscr{B}(\mathscr{H}_0)$ of all bounded operators in $\mathscr{H}_0$, such that ϱ is equivalent to the representation of $\mathfrak{A}$ in the one-particle sector and

$$\varrho(A)=A \tag{1.1}$$

holds for any observable A which is localized in the spacelike complement S' of S.

To recover the vacuum state ω_0 from states in the one-particle representation ϱ, one may translate the local observables to spacelike infinity:

$$(\Psi, U_\varrho(x)\varrho(A)U_\varrho(-x)\Psi)\rightarrow \|\Psi\|^2\omega_0(A) \tag{1.2}$$

as x tends to spacelike infinity, for any $A\in\mathfrak{A}$ and $\Psi\in\mathscr{H}_0$. Here $x\rightarrow U_\varrho(x)$ denotes the representation of the translation group which belongs to the representation ϱ of $\mathfrak{A}$.

Relation (1.2) can be interpreted as "shifting the charge to spacelike infinity". It is therefore tempting, to apply the same procedure to states in the vacuum sector. If

$$(\Psi, U_\varrho(x_n)AU_\varrho(-x_n)\Psi)\rightarrow \|\Psi\|^2\varphi(A) \tag{1.3}$$

for some sequence $\{x_n\}$ which tends to spacelike infinity, we may take the limit state φ as a state in the conjugate sector.

If (1.1) would be the only information, we have about ϱ, we would have hardly much knowledge about such limit points. But the localization properties of the one-particle representation ϱ, as derived in [4, I], contain more information than used till now. There convergence of the type (1.3) has been shown, using only commutation properties of the operator A with observables in the representation ϱ. By (1.1), any local observable A in the vacuum sector commutes with those observables in the representation ϱ which are localized in S' spacelike to the localization region of A. It will be shown that this property suffices, to prove convergence in (1.3) for all local observables $A\in\mathfrak{A}$ for suitable sequences $\{x_n\}$.

From the state φ we get via the GNS-construction some representation π of $\mathfrak{A}$. This representation is irreducible, translation covariant, the energy-momentum spectrum fulfils the positivity condition and the representation is localizable in the same sense as ϱ. The composite sector, constructed from π and ϱ, contains a subrepresentation equivalent to the vacuum representation. This may be taken as the defining property of the conjugate sector.

We then analyze the analog of the two-point function. This gives finiteness of statistics and equivalence of mass spectra in the representations π and ϱ. So there are particles in the conjugate sector with the same mass as the particle in the representation ϱ.

The used assumptions are quite general:

To any bounded region $\mathcal{O}$ in Minkowski space, we associate a von Neumann-algebra $\mathfrak{A}(\mathcal{O})$, generated by those observables, which can be measured in $\mathcal{O}$. If $\mathcal{O}_1$ is contained in $\mathcal{O}_2$, $\mathfrak{A}(\mathcal{O}_1)$ is contained in $\mathfrak{A}(\mathcal{O}_2)$, and

$$\mathfrak{A} = \bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O}) \tag{1.4}$$

is the algebra of all local observables. For unbounded regions G we set

$$\mathfrak{A}(G) = \bigcup_{\mathcal{O} \subset G} \mathfrak{A}(\mathcal{O}). \tag{1.5}$$

The net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ fulfills the requirements of local commutativity and translation covariance:

(i) If $\mathcal{O}_1$ is contained in the spacelike complement $\mathcal{O}_2'$ of $\mathcal{O}_2$, then $\mathfrak{A}(\mathcal{O}_1)$ commutes with $\mathfrak{A}(\mathcal{O}_2)$.

(ii) There is a representation $x \to \alpha_x$ of the translation group by automorphisms of $\mathfrak{A}$, such that

$$\alpha_x(\mathfrak{A}(\mathcal{O})) \subset \mathfrak{A}(\mathcal{O} + x). \tag{1.6}$$

As discussed in [4, I], a charged particle can be described by some irreducible, translation covariant representation of $\mathfrak{A}$, where the energy-momentum spectrum below a certain mass shell consists of an isolated mass hyperboloid.

Then there exists a unique vacuum representation of $\mathfrak{A}$ with the mentioned localization properties (1.1). The only additional assumption used in the argument is the following duality property in the vacuum sector:

$$\mathfrak{A}(S)' = \mathfrak{A}(S')^{-} \tag{1.7}$$

for any spacelike cone S. Here $'$ on the l.h.s. denotes the commutant and $^{-}$ the weak closure. This assumption can be derived under more special conditions, as discussed in [5].

2. The Conjugate Sector

As mentioned in the introduction, the construction of the conjugate sector is based on localization properties of the one-particle representation ϱ. To formulate these properties in a convenient way, we introduce the following notations:

Let $\mathcal{O}_R$ denote the double cone $\{|x_0| + |\underline{x}| \leqq R\}$ in Minkowski-space. We call an operator $A \in \mathscr{B}(\mathscr{H}_0)$ almost local, if there exists an operator A_R in each algebra

$\mathfrak{A}(\mathcal{O}_R)$ such that

$$\|A-A_R\|$$

decreases faster than any power of R, if R goes to infinity. A spacelike cone S is described by some vertex $a\in\mathbb{R}^4$ and some spacelike closed double cone $\mathcal{O}$:

$$S=a+\bigcup_{\lambda>0}\lambda\mathcal{O}.$$

The automorphisms of $\mathcal{B}(\mathcal{H}_0)$, generated by the translation operators $U_\varrho(x)$, are denoted by β_x:

$$\beta_x(X)=U_\varrho(x)\,XU_\varrho(-x), \qquad X\in\mathcal{B}(\mathcal{H}_0). \tag{2.1}$$

2.1. Lemma. *Let E be the projection, belonging to some sufficiently small nonvoid open set on the one-particle mass shell in the spectrum of U. Then there exists an almost local operator A with $\varrho(A)E\neq 0$ such that*

$$f\mu(R)=\sup_{\substack{X\in\varrho(\mathfrak{A}(\mathcal{O}_R))'\\ \|X\|\leq 1}}\left\|\left\|\frac{\partial}{\partial \mathbf{x}\mu}E\varrho(A)^*\beta_x(X)_\varrho(A)E\right\|\right|_{x=0}$$

decreases faster then any power of R, if R goes to infinity, $i=1, 2, 3$.

The proof of this lemma can be found in [4, I].

Now let S_0 be some spacelike cone. Since Lemma 2.1 is valid in any Lorentz frame, we choose a frame such that S_0 has the form

$$S_0=\bigcup_{R>0}(\mathcal{O}_R+Rb), \qquad b\in\mathbb{R}^4, \qquad b_0=0.$$

2.2. Lemma. *There exists a unique state φ on the algebra $\bigcup_x\varrho(\mathfrak{A}(S_0+x))'$ with the following properties:*

i) *$\varphi\circ\beta_x=\varphi$ for any translation x.*
ii) *φ is normal on $\varrho(\mathfrak{A}(S_0+x))'$ for any x.*
iii) *Let $x\in S_0$ such that $x+\mathcal{O}_r\subset S_0$ for some $r>0$. Then*

$$\|E\varrho(A)^*\{\beta_{-x}(X)-\varphi(X)\mathbb{1}\}\varrho(A)E\|\leq f(r)\,\|X\| \tag{2.3}$$

for any $X\in\varrho(\mathfrak{A}(S_0))'$, where f is a function of fast decrease independent of X.

Proof. Let $X\in\varrho(\mathfrak{A}(S_0+x))'$ for some x. Then

$$\beta_{(x+Rb)}(X)\in\varrho(\mathfrak{A}(\mathcal{O}_R))',$$

and from Lemma 2.1 we have for $R_2>R_1$:

$$\|E\varrho(A)^*\{\beta_{-(x+R_2b)}(X)-\beta_{-(x+R_1b)}(X)\}\varrho(A)E\|\leq g(R_1)\,\|X\|, \tag{2.4}$$

where

$$g(R_1)=\int_{R_1}^{\infty}dR\sum_{i=1}^{3}|b_i|f_i(R)$$

is a function of fast decrease. Hence

$$\{E\varrho(A)^* \beta_{-(x+Rb)}(X)\varrho(A)E\}_{R>0}$$

converges, as R tends to infinity. But the weak limit points of $\{\beta_{-(x+Rb)}(X)\}$ are contained in

$$\bigcap_{R>0} \varrho(\mathfrak{A}(\mathcal{O}_R))' = \varrho(\mathfrak{A})',$$

so they are multiples of the identity due to the irreducibility of ϱ. Therefore

$$\underset{R\to\infty}{w\text{-lim}}\, \beta_{-Rb}(X) = \varphi(X)\mathbb{1}, \tag{2.5}$$

where φ is some translation invariant state on the algebra $\bigcup_x \varrho(\mathfrak{A}(S_0+x))'$, and

$$\|E\varrho(A)^*\{\beta_{-(x+Rb)}(X) - \varphi(X)\mathbb{1}\}\varrho(A)E\| \leqq g(R)\|X\| \tag{2.6}$$

for $X \in \varrho(\mathfrak{A}(S_0+x))'$. Thus φ is normal on $\varrho(\mathfrak{A}(S_0+x))'$ as uniform limit of normal states. This proves (i) and (ii). To verify (iii), we note

$$\beta_{-(x+Rb)}(X) \in \varrho(\mathfrak{A}(\mathcal{O}_{r+R}))'$$

for $X \in \varrho(\mathfrak{A}(S_0))'$, so using the translation invariance of φ and Lemma 2.1, we get

$$\|E\varrho(A)^*(\beta_{-x}(X) - \varphi(X)\mathbb{1}\}\varrho(A)E\| \leqq g(r)\|X\|. \quad \text{q.e.d.} \tag{2.7}$$

We shall see that φ defines a state on $\mathfrak{A}$, if the spacelike cone S_0 is in the spacelike complement of the localization region S of ϱ. In fact, in this case from (1.1)

$$\varrho(\mathfrak{A}(S_0+x)) = \mathfrak{A}(S_0+x)$$

if S_0+x is contained in S', and

$$\bigcup_x \varrho(\mathfrak{A}(S_0+x))' = \bigcup_{\substack{x \\ S_0+x\subset S'}} \varrho(\mathfrak{A}(S_0+x))' = \bigcup_{\substack{x \\ S_0+x\subset S'}} \mathfrak{A}(S_0+x)' = \bigcup_x \mathfrak{A}(S_0+x). \tag{2.8}$$

Thus φ is defined on $\mathfrak{A}$. As from (1.2)

$$\varphi(\varrho(A)) = \omega_0(A), \qquad A \in \mathfrak{A}, \tag{2.9}$$

φ is a state in some sector conjugate to ϱ.

We shall use in the following the notation

$$\mathscr{B}_{S_0} = \bigcup_x \mathfrak{A}(S_0+x)'. \tag{2.10}$$

$\mathscr{B}_{S_0}$ is an algebra, which contains $\mathfrak{A}$ and $\varrho(\mathfrak{A})$.

Let π_φ be the cyclic representation of $\mathscr{B}_{S_0}$, arising from the GNS construction with the state φ, let $\mathscr{H}$ be the representation space and ξ the cyclic vector such that for $X \in \mathscr{B}_{S_0}$.

$$(\xi, \pi_\varphi(X)\xi) = \varphi(X). \tag{2.11}$$

It is then easy to see that the representation $\pi_\varphi \circ \varrho$ of $\mathfrak{A}$ contains a subrepresentation unitarily equivalent to the vacuum representation. In fact, let V be the isometric mapping from $\mathscr{H}_0$ into $\mathscr{H}$, densely defined by

$$VA\Omega = \pi_\varphi \circ \varrho(A)\xi \tag{2.12}$$

for $A \in \mathfrak{A}$, where $\Omega \in \mathscr{H}_0$ is the vector representing the vacuum. Then the equivalence is given by the relation

$$VA = \pi_\varphi \circ \varrho(A)V \tag{2.13}$$

which holds for any $A \in \mathfrak{A}$. (2.13) enables us to call the representation π_φ, restricted to $\mathfrak{A}$, a representation conjugate to ϱ (compare [1, I, Lemma 3.5]).

A left inverse ϕ of ϱ which is needed for the discussion of the statistics of ϱ, can be defined by

$$\phi(X) = V^* \pi_\varphi(X) V \tag{2.14}$$

for $X \in \mathscr{B}_{S_0}$. Here a left inverse of ϱ is a positive mapping from some algebra containing $\mathfrak{A}$ and $\varrho(\mathfrak{A})$ into $\mathscr{B}(\mathscr{H}_0)$ such that for $A, B, C \in \mathfrak{A}$

$$\begin{aligned} &\text{(i)} \quad \phi(\mathbb{1}) = \mathbb{1} \\ &\text{(ii)} \quad \phi(\varrho(A)B\varrho(C)) = A\phi(B)C\,. \end{aligned} \tag{2.15}$$

For later use, we note some properties of the representation π_φ of $\mathscr{B}_{S_0}$.

2.3. Lemma. (i) *There exists a unitary strongly continuous representation U_φ of the translation group in $\mathscr{H}$ such that*

$$U_\varphi(x)\pi_\varphi(X)U_\varphi(-x) = \pi_\varphi(\beta_x(X)) \tag{2.16}$$

for $X \in \mathscr{B}_{S_0}$ and any translation x.

(ii) *The spectrum of the generators of U_φ is contained in the set*

$$\{0\} \cup \{p \in \mathbb{R}^4 | p^2 \geqq \Delta, p_0 \geqq 0\}\,,$$

where Δ is the upper mass gap in the spectrum of U_ϱ.

(iii) *ξ is the unique groundstate of U_φ.*

Proof. (i) is standard. As $\varphi \circ \beta_x = \varphi$, we define $U_\varphi(x)$ as the unitary mapping which is fixed by

$$U_\varphi(x)\pi_\varphi(X)\xi = \pi_\varphi(\beta_x(X))\xi \tag{2.17}$$

for $X \in \mathscr{B}_{S_0}$. Due to the boundedness of $\{U_\varphi(x)\}$, it suffices to prove the continuity on vectors of the form $\pi_\varphi(X)\xi$, where $X \in \varrho(\mathfrak{A}(S_0 + y))'$ for some y.

We have

$$\|(U_\varphi(x) - \mathbb{1})\pi_\varphi(X)\xi\|^2 = \varphi((\beta_x(X) - X)^*(\beta_x(X) - X))\,. \tag{2.18}$$

Since $x \to \beta_x(X)$ is strongly continuous and uniformly bounded, the strong continuity of U_φ follows from the fact that φ is normal on

$$\bigcup_{x \in U} \varrho(\mathfrak{A}(S_0 + y + x))'\,,$$

where U is some bounded neighbourhood of the origin.

(ii) Let $q \notin \{0\} \cup \{p^2 \geqq \Delta, p_0 \geqq 0\}$. We shall show that there is a neighbourhood U of q such that for any infinitely often differentiable function f with support contained in U the integral

$$\int d^4x \tilde{f}(x) U_\varphi(x) := f(P_\mu) \tag{2.19}$$

vanishes. Here $\tilde{f}$ denotes the Fourier transform of f.

There is some point p on the one-particle mass shell such that $p+q$ does not belong to the spectrum of U_φ.

Choose neighbourhoods V of p and U of q such that the same is true for $V+U$. Then

$$\beta_{-Rb}(X_f)\Psi = 0 \tag{2.20}$$

for any $X \in \mathscr{B}_{S_0}$, $\operatorname{supp} f \subset U$ and spectral support of Ψ contained in V. ($X_f = \int d^4x \tilde{f}(x) \beta_x(X)$.)

But

$$\beta_{-Rb}(X_f^* X_f) \xrightarrow[R\to\infty]{W} \varphi(X_f^* X_f)\mathbb{1} = \|f(P_\mu)\pi_\varphi(X)\xi\|^2 \mathbb{1}.$$

Thus (2.20) implies (2.19).

(iii) Let f be an infinitely often differentiable function with support contained in some sufficiently small neighbourhood U of the origin. Let $X, Y \in \mathscr{B}_{S_0}$. Then from Lemma 2.1 for some $\Psi \in \mathscr{H}_0$

$$\begin{aligned}&(\pi_\varphi(Y)\xi, f(P_\mu)\pi_\varphi(X)\xi)\\ &\quad= \lim_{R\to\infty} (\varrho(A)E\Psi, \beta_{-Rb}(Y^* X_f)\varrho(A)E\psi).\end{aligned} \tag{2.21}$$

Using the identity

$$\beta_{-Rb}(X_f)\varrho(A) = [\beta_{-Rb}(X_f), \varrho(A)] + \varrho(A)\beta_{-Rb}(X_f)$$

we decompose the right hand side of (2.21) into a sum of two terms, where the first term vanishes, since $\tilde{f}$ is a function of fast decrease and A is almost local. To handle the second term, we choose U and the energy-momentum support V of E such that Lemma 2.1 also holds for E replaced by the spectral projection F of $U+V$ with the same almost local operator A. Then

$$\beta_{-Rb}(X_f)E\Psi \in F\mathscr{H}_0$$

and from Lemma 2.2

$$n\text{-}\lim_{R\to\infty} E\varrho(A)^* \beta_{-Rb}(Y^*)\varrho(A)F = \varphi(Y^*)E\varrho(A^*A)F$$

and

$$w\text{-}\lim_{R\to\infty} \int d^4x \tilde{f}(x)\beta_{x-Rb}(X) = \varphi(X)f(0)\mathbb{1}$$

So we get

$$f(P_\mu)\pi_\varphi(X)\xi = f(0)(\xi, \pi_\varphi(X)\xi)\xi$$

which proves the uniqueness of the groundstate. q.e.d.

From Lemma 2.3, we conclude, according to standard results of axiomatic field theory [6], that ξ is a cyclic vector for $\pi_\varphi(\varrho(\mathfrak{A}(S_0+x))')$ for any x and that the representation π_φ of $\mathscr{B}_{S_0}$ is irreducible. On the other hand, it can be shown that the representation $\pi_\varphi \circ \varrho$ of $\mathfrak{A}$ is irreducible if and only if the representation ϱ fulfills the duality condition (1.7) which means that ϱ has pure Fermi- or Bose-statistics (compare [1, I, Lemma 2.2]).

We now study the conjugate representation $\pi := \pi_\varphi|_{\mathfrak{A}}$. We have the following result:

2.4. Theorem. (i) *π is irreducible*

(ii) *The equivalence class of π does not depend on S_0 nor on the choice of ϱ in its equivalence class.*

(iii) *To any spacelike cone S_1, there exists a unitary mapping U from $\mathscr{H}$ onto $\mathscr{H}_0$ such that*

$$U\pi(A) = AU \tag{2.22}$$

holds for any $A \in \mathfrak{A}(S_1)$.

(iv) *π is translation covariant, i.e. there exists a strongly continuous representation U_π of the translation group in $\mathscr{H}$ such that for $A \in \mathfrak{A}$*

$$U_\pi(x)\pi(A)U_\pi(-x) = \pi(\alpha_x(A)). \tag{2.23}$$

(v) *The spectrum of the generators of U_π is contained in the forward light cone.*

Proof. (i) From the definition of π it is clear that

$$\pi(\mathfrak{A})^- \supset \bigcup_{S_0+x \subset S'} \pi_\varphi(\mathfrak{A}(S_0'+x))^-,$$

where S is some localization region of ϱ. Since φ is normal on $\varrho(\mathfrak{A}(S_0+x))'$ and $\pi_\varphi(\varrho(\mathfrak{A}(S_0+x))')\xi$ is dense in $\mathscr{H}$, π_φ is a W^*-representation of $\pi(\mathfrak{A}(S_0+x))'$ [7, 1.16] and thus

$$\pi_\varphi(\mathfrak{A}(S_0'+x))^- \supset \pi_\varphi(\mathfrak{A}(S_0'+x)^-)$$

if $S_0+x \subset S'$, further

$$\mathfrak{A}(S_0'+x)^- = \mathfrak{A}(S_0+x)'$$

by duality (1.7). Hence

$$\pi(\mathfrak{A})^- \supset \bigcup_{S_0+x \subset S'} \pi_\varphi(\mathfrak{A}(S_0+x)') = \pi_\varphi(\mathscr{B}_{S_0}),$$

thus the irreducibility of π_φ implies the irreducibility of π.

(ii) Let us repeat the construction of the conjugate representation with some spacelike cone $\hat{S}_0$ such that $S_0 \subset \hat{S}_0 \subset S'$, and let us denote the associated state of $\mathscr{B}_{\hat{S}_0}$ by $\hat{\varphi}$ and the arising conjugate representation by $\hat{\pi}$. Since $\hat{\varphi}$ is the restriction of φ to the subalgebra $\mathscr{B}_{\hat{S}_0}$ of $\mathscr{B}_{S_0}$, $\hat{\pi} = \pi_{\hat{\varphi}|\mathfrak{A}}$ is equivalent to some subrepresentation of π. But π is irreducible, so $\hat{\pi}$ is equivalent to π. Iteration of this argument gives the equivalence for any spacelike cone $\hat{S}_0 \subset S'$.

Now replace ϱ by some equivalent morphism $\hat{\varrho}$ with localization region $\hat{S} \subset S_0'$. There is a unitary intertwiner $\hat{U} \in \mathscr{B}(\mathscr{H}_0)$ such that

$$\hat{U}\varrho(A) = \hat{\varrho}(A)\hat{U}$$

for any $A \in \mathfrak{A}$, and the state $\hat{\varphi}$, corresponding to $\hat{\varrho}$, is given by

$$\hat{\varphi}(X) = \varphi(\hat{U}^{-1} X \hat{U})$$

for $X \in \mathscr{B}_{S_0}$. As $\hat{U} \in \mathscr{B}_{S_0}$, the equivalence of the corresponding conjugate representation to π follows.

(iii) According to (ii), we are free to choose S_0 and S, such that $S_1 \subset S'$ and S_1 contains S_0 together with some cone spacelike to S_0. As $S_0 \subset S'$, φ is normal on $\mathfrak{A}(S_0)' = \varrho(\mathfrak{A}(S_0))'$. Now the commutant of $\mathfrak{A}(S_0)'$ possesses some cyclic vector in $\mathscr{H}_0$, so there is some vector $\phi \in \mathscr{H}_0$ such that

$$\varphi(A) = (\phi, A\phi)$$

for all $A \in \mathfrak{A}(S_0)'$ [7, 2.7.9]. Define $W_0 : \mathscr{H} \to \mathscr{H}_0$ by

$$W_0 \pi(A) \xi = A\phi$$

for $A \in \mathfrak{A}(S_0)'$. W_0 is an isometry, since ξ is cyclic for $\pi_\varphi(\mathfrak{A}(S_0)')$, and $E = W_0 W_0^* \in \mathfrak{A}(S_0)''$. Using methods of [8], one can show [4, I] that to any nonzero projection $E \in \mathfrak{A}(S_0)''$ there is some isometry $W \in \mathfrak{A}(S_0)''$ such that $E = WW^*$. Hence $U = W^* W_0$ is a unitary mapping from $\mathscr{H}$ onto $\mathscr{H}_0$ and we have the desired relation

$$U\pi(A) = AU$$

for all $A \in \mathfrak{A}(S_1')$.

(iv) We have

$$\pi(\alpha_x(A)) = \pi_\varphi(U_0(x) U_\varrho(-x) \beta_x(A) U_\varrho(x) U_0(-x)).$$

Here U_0 denotes the representation of the translation group in the vacuum sector. Since $U_0(x) U_\varrho(-x)$ is an intertwiner from ϱ to the translated morphism $\varrho_x = \alpha_x \varrho \alpha_{-x}$, it commutes with $\mathfrak{A}(S') \cap \mathfrak{A}(S'+x)$ due to (1.1) and is therefore contained in $\mathscr{B}_{S_0}$. So setting

$$U_\pi(x) = \pi_\varphi(U_0(x)\, U_\varrho(-x))\, U_\varphi(x) \tag{2.24}$$

we fulfil (2.23). The group relation is easily verified. The strong continuity of U_π comes from the fact, mentioned in the proof of (i), that π_φ, restricted to $\varrho(\mathfrak{A}(S_0 + y))'$ for some y, is a W^*-representation.

(v) The spectrum of U_φ contains the sum of the spectra of U_π and U_ϱ, since $\pi_\varphi \circ \varrho$ can be taken as product of π and ϱ [1, II, Theorem 5.2]. In fact, for $B \in \mathfrak{A}$ and $\Psi \in \mathscr{H}$ we have

$$U_\varphi(x) \pi_\varphi(B) \Psi = \pi_\varphi(U_\varrho(x) B U_0(-x))\, U_\pi(x) \Psi. \tag{2.25}$$

Hence the spectral support w.r.t. U_φ of the vector $\pi_\varphi(C)\Psi$, where $C = \int d^4x\, f(x) U_\varrho(x) B U_0(-x)$ with some integrable function f, is contained in the sum of the spectral support of Ψ w.r.t. U_π and the support of the Fourier transform of f. On the other hand

$$\|\pi_\varphi(C)\, U_\pi(y) \Psi\|^2 = (\Psi, \pi_\varphi(\alpha_{-y}(C^*C))\Psi) \to \|\Psi\|^2\, \|C\Omega\|^2 \tag{2.26}$$

if $(-y)$ goes to infinity in S. Thus to any neighbourhood of some point in the sum of the spectra of U_π and U_ϱ there are nonzero vectors in $\mathscr{H}$, the spectral support of which w.r.t. U_φ is contained in this neighbourhood. Then the positivity condition on the spectrum of U_φ [Lemma 2.3 (ii)] implies the positivity of the spectrum of U_π. q.e.d.

3. Finiteness of Statistics and the Mass Spectrum in the Conjugate Sector

If the particle, from which we started, would be finitely localizable in the sense of [1], it would follow from the analysis in [1] that this particle has finite statistics and that the mass spectrum in the conjugate sector is equivalent to the spectrum in the particle sector. Similar to the analysis of multiparticle states in [4, II], in the general case the results of [1] remain valid, but the arguments have to be modified.

The complications arising from the fact that the weaker localization properties of ϱ do not imply $\varrho(\mathfrak{A})\subset\mathfrak{A}$, can be circumvented as discussed in [4, II]. For the purpose of this paper, we remark that morphisms with the localization property (1.1) possess a unique extension to $\mathscr{B}_{S_0}$ which is weakly continuous on $\mathfrak{A}(S_0+x)'$ for any x; and if the morphism is localized in the spacelike complement of S_0+x for some x, then $\mathscr{B}_{S_0}$ is invariant under the application of this morphism. In the following we shall use this extension.

Now consider the expression

$$(\pi_\varphi(X)\xi, U_\pi(x)\pi_\varphi(Y)\xi), \tag{3.1}$$

where $X, Y\in\mathscr{B}_{S_0}$. We shall see that (3.1) is the analog of the two-point function for an interpolating field between vacuum and conjugate sector. According to the definition of U_π (2.21) and of the left inverse ϕ (2.14), (3.1) equals

$$(\pi_\varphi(X)\xi, \pi_\varphi(U_0(x)YU_\varrho(-x))\xi)=(\Omega, \phi(X^*U_0(x)YU_\varrho(-x))\Omega). \tag{3.2}$$

Now assume that there exist morphisms ϱ_1, $\varrho_2\simeq\varrho$ with unitary intertwiners U_i from ϱ to ϱ_i, $i=1,2$ such that ϱ_1 and U_1X^* are localized in some spacelike cone S_1 and ϱ_2 and U_2Y^* are localized in some spacelike cone $S_2\subset S_1'$. Denoting $U_0(x)YU_\varrho(-x)$ by $Y(x)$ and $U_0(x)U_2U_\varrho(-x)$ by $U_2(x)$ and keeping in mind that $U_2(x)$ intertwines from ϱ to $(\varrho_2)_x=\alpha_x\varrho_2\alpha_{-x}$, we have for $x\in(S_1-S_2)'$:

$$\begin{aligned}X^*Y(x)&=U_1^*(U_1X^*)\alpha_x(YU_2^*)U_2(x)\\&=U_1^*\alpha_x(YU_2^*)(U_1X^*)U_2(x)\\&=U_1^*\varrho_1(\alpha_x(YU_2^*))(\varrho_2)_x(U_1X^*)U_2(x)\\&=\varrho(\alpha_x(YU_2^*))U_1^*U_2(x)\varrho(U_1X^*)\\&=\varrho(Y(x))\{\varrho(U_2(x)^*)U_1^*U_2(x)\varrho(U_1)\}\varrho(X)^*.\end{aligned} \tag{3.3}$$

The operator in brackets is the statistics operator ε_ϱ [1, I, Lemma 2.6]. This operator can be thought of as the unobservable operation of permuting two relatively spacelike localized identical particles. ε_ϱ does not depend on the choice of the morphisms ϱ_1 and $(\varrho_2)_x$, so long as their localization regions are relatively spacelike. ε_ϱ commutes with all observables in the two-particle representation ϱ^2. Therefore $\phi(\varepsilon_\varrho)$ commutes with $\varrho(\mathfrak{A})$, so $\phi(\varepsilon_\varrho)$ is a multiple of the identity:

$$\phi(\varepsilon_\varrho)=\lambda_\varrho\mathbb{1}. \tag{3.4}$$

As shown in [1, I], the possible values of λ_ϱ are 0, $\pm\frac{1}{d}$, $d\in\mathbb{N}$. $\lambda_\varrho=\pm\frac{1}{d}$ means that the particle carries para-Bose- resp. para-Fermi statistics of order d. The case $\lambda_\varrho=0$ is called infinite statistics and was thought to be pathological. We shall show that the occurrence of infinite statistics is indeed incompatible with the spectral properties of particle states.

Using the defining properties of a left inverse (2.15), we get from (3.1)–(3.4) for $x\in(S_1-S_2)'$

$$(\pi_\varphi(X)\xi, U_\pi(x)\pi_\varphi(Y)\xi)=\lambda_\varrho(Y^*\Omega, U_\varrho(-x)X^*\Omega). \tag{3.5}$$

Now assume that λ_ϱ vanishes. Then from the positivity condition on the spectrum of U_π [Theorem 2.4 (v)] we conclude via the Edge of the Wedge-theorem that the left hand side of (3.5) vanishes identically in x, if $(S_1-S_2)'$ contains some nonvoid open set. So setting $X=AU_1$ and $Y=BU_2$ with $A, B\in\mathfrak{A}$, we get (3.7)

$$(\pi(A)\pi_\varphi(U_1)\xi, \pi(B)\pi_\varphi(U_2)\xi)=0 \tag{3.6}$$

for all $A, B\in\mathfrak{A}$ which is a contradiction to the irreducibility of π.

Having established the finiteness of statistics, we can use (3.5) to analyze the energy-momentum spectrum in the conjugate sector. We choose spacelike cones S_1 and S_2 such that there is some spacelike cone S_3 with $S_3\supset S_1-S_2$. Then precisely as in [1, II, Theorem 6.3] we can apply the techniques of the Jost-Lehmann-Dyson representation, to conclude for $x\in S_3'$

$$(\pi_\varphi(X)\xi, U_\pi(x)E_\pi(\Delta)\pi_\varphi(Y)\xi)=\lambda_\varrho(Y^*\Omega, U_\varrho(-x)E_\varrho(\Delta)X^*\Omega), \tag{3.7}$$

where $E_\pi(\Delta)$ and $E_\varrho(\Delta)$ denote the spectral projections onto some Borel set Δ for the mass operators belonging to U_π and U_ϱ, respectively. Thus the mass spectrum in the conjugate sector equals the mass spectrum in the sector of ϱ; especially there are particles in the conjugate sector with the same mass as the particle described by ϱ.

Acknowledgements. This paper is a byproduct of my collaboration with D. Buchholz and would be inconceivable without the many ideas which he contributed to our joint work. I gratefully acknowledge helpful discussions with R. Haag and I also would like to thank K. Pohlmeyer for his continuing interest and encouragement.

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Communicated by R. Haag

Received September 24, 1980