Commun. Math. Phys. 78, 363–371 (1981)

Communications in Mathematical Physics

Lattice Systems with a Continuous Symmetry

II. Decay of Correlations

Jean Bricmont[1★+], Jean–Raymond Fontaine[2★+], Joel L. Lebowitz[2★], and Thomas Spencer[2★]

1 Department of Mathematics, Princeton University, Princeton, NJ 08540, USA
2 Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA

Abstract. We consider perturbations of a massless Gaussian lattice field on $\mathbb{Z}^d, d \geqq 3$, which preserves the continuous symmetry of the Hamiltonian, e.g.,

$$-H = \sum_{\langle x,y \rangle} (\phi_x - \phi_y)^2 + T(\phi_x - \phi_y)^4, \ \phi_x \in \mathbb{R}.$$

It is known that for all $T > 0$ the correlation functions in this model do not decay exponentially. We derive a power law upper bound for all (truncated) correlation functions. Our method is based on a combination of the log concavity inequalities of Brascamp and Lieb, reflection positivity and the Fortuin, Kasteleyn and Ginibre (F.K.G.) inequalities.

I. Introduction

In this paper, we consider the same model of an anharmonic crystal as in [5] (part I of this series):

$$-\beta H = \sum_{\langle x,y \rangle} [(\phi_x - \phi_y)^2 + T(\phi_x - \phi_y)^4]$$

where $\langle x, y \rangle$ indicates that sum is over nearest neighbors in $\mathbb{Z}^d$. For $T = 0$, this is a massless Gaussian model and it is known that the correlation functions $\langle \phi_0, \phi_x \rangle$ and $|\langle \nabla_0^e \phi \nabla_x^e \phi \rangle|$ are not summable over the lattice (where $\nabla_x^e \phi = \phi_{x+e} - \phi_x$, e is a unit vector).

The question that we try to answer is: what is the decay of the correlations when $T > 0$?

Using the Brascamp and Lieb inequalities and some refinements of them,

★ (J. B.) Supported by NSF Grant N^r MCS78-01885
(J. L. L. and J. R. F.) Supported by NSF Grant N^r PHY78-15320
(T. S.) Supported by NSF Grant N^r DMR73-04355
+ On leave from: Institut de Physique Theorique, Universite de Louvain, Belgium

0010-3616/81/0078/0363/$01.80

we obtain suitable bounds on the Fourier transforms of several correlation functions in terms of the corresponding Gaussian ones (see Theorem 1).

Using the transfer matrix formalism for reflection positive (nearest neighbor) interactions, this gives a power law upper bound on the decay of these correlation functions (see Theorem 2). Correlation inequalities, when available, lead to stronger results (see [15]).

Using ideas of Park [12], Fröhlich and Spencer [10], one shows, as a converse to Theorem 2 that, $\langle \phi_0 \phi_x \rangle$ and $|\langle \nabla_0^e \phi \nabla_x^e \phi \rangle|$ are not summable over the lattice for all T (see Remarks in Sect. IV).

In Theorem 3 we extend the bound of Theorem 2 to a larger class of correlation functions. This follows from a kind of "domination" by the two-point function, which uses only reflection positivity.

Finally in Sect. V, we bound all truncated correlation functions in terms of the two point function, by a suitable modification of the arguments of [11, 14] based on F.K.G. inequality [6].

II. The Model

Let, ϕ_x, $x \in \mathbb{Z}^d$, be a real random variable. We consider the following Hamiltonian with periodic boundary conditions on the parallelipiped Λ (c.f. [5]):

$$-\beta H = \sum_{\langle x,y \rangle \subset \Lambda} (\phi_x - \phi_y)^2 + T \sum_{x,y \in \Lambda} J(x-y)(\phi_x - \phi_y)^4 \tag{1}$$

$J(x-y) \geqq 0$ has finite range D. If $D=1$ we call it a nearest neighbor interaction.

For $d \geqq 3$, we let $\langle \quad \rangle$ denote some limiting state defined on $\pi_x \phi_x^{n_x}$ and obtained by first adding a mass term $m^2 \sum_{x \in \Lambda} \phi_x^2$ to (1) and then letting $\Lambda \to \mathbb{Z}^d$ and $m \to 0$. For $d = 1, 2$, the expectation values $\langle \quad \rangle$ are defined only for products of gradiants like $\nabla_x^e \phi = \phi_{x+e} - \phi_x$. They are obtained by setting $\phi_{x_0} = 0$ in (1) for some $x_0 \in \mathbb{Z}^d$ and letting $\Lambda \to \mathbb{Z}^d$. See [5] for more details.

Remark. All the results below will be valid for all T in (1). One may consider also any convex polynomial instead of a quartic for the perturbation. Furthermore, if the interaction is nearest-neighbor one may replace in Theorems 1, *a*, *b* and 2, *a*, *b*, and in other results $(\phi_x - \phi_y)^4$ by any polynomial in $(\phi_x - \phi_y)$ semi-bounded from below (see [5], Sect. IV).

III. Bounds on the Fourier Transform

We use the shorthand notation: $\langle A ; B \rangle = \langle AB \rangle - \langle A \rangle \langle B \rangle$. We define the following Fourier transforms:

$$S_T^{(1)}(p) = \sum_{x \in \mathbb{Z}^d} \langle \phi_0 \phi_x \rangle e^{ipx}$$

$$S_T^{(2)}(p) = \sum_{x \in \mathbb{Z}^d} \langle \nabla_0^e \phi \nabla_x^e \phi \rangle e^{ipx}$$

$$S_T^{(3)}(p) = \sum_{x \in \mathbb{Z}^d} \langle \phi_0^2 ; \phi_x^2 \rangle e^{ipx}$$

$$S_T^{(4)}(p) = \sum_{x \in \mathbb{Z}^d} \langle (\nabla_0^e \phi)^2 ; (\Delta_x^e \phi)^2 \rangle e^{ipx}$$

$S_0^{(i)}(P)$ being the Fourier transforms for the harmonic crystal which are explicitly computable.

Since the series above do not converge absolutely, we only know that the definitions make sense because of the following Theorem (for more details see the proof of Theorem 3.1 in [7]).

Theorem 1. *For all T and all $p \neq 0$,*

a) $S_T^{(1)}(p) \leqq S_0^{(1)}(p)$ *for* $d \geqq 3$
b) $S_T^{(2)}(p) \leqq S_0^{(2)}(p)$ *for all* d
c) $S_T^{(3)}(p) \leqq S_0^{(3)}(p)$ *for* $d \geqq 3$
d) $S_T^{(4)} \leqq S_0^{(4)}(p)$, *bounded for all* d.

Proof. a) is simply the Brascamp–Lieb inequalities [2, 3, 4] and b) follows from a) because $S_T^{(2)}(p) = 2(1 - \cos p_e) S_T^{(1)}(p)$. c) and d) both follow from Theorem 2.3 of [4]. The proof is based on the application of the Brascamp–Lieb inequalities to the distribution of the variables $\psi_x^{\pm} = (\phi_x \pm \phi'_x)/\sqrt{2}$ where the $\{\phi'_x\}$ are 'duplicate' independent variables with the same distribution as the $\{\phi_x\}$.

Let f be a function on $\mathbb{Z}^d$ of finite support. Then

$$\langle \phi^2(f); \phi^2(f) \rangle = \sum_{x,y} \langle \psi_x^+ \psi_x^- \psi_y^+ \psi_y^- \rangle f(x) f(y).$$

One first application of the Brascamp–Lieb inequalities gives

$$\langle \phi^2(f) \phi^2(f) \rangle \leqq \sum_{x,y} c_{xy} f(x) f(y) \langle \phi_x \phi_y \rangle,$$

where $c_{xy} = \langle \phi_x \phi_y \rangle_{T=0}$. This corresponds to choosing $L = \delta_{xy} f(x)$ in Theorem 2.3 of [4]. Using again Brascamp–Lieb, the Fourier transform of $\langle \phi_x \phi_y \rangle$ is bounded by $S_0^{(1)}(p)$. So

$$\langle \phi^2(f); \phi^2(f) \rangle \leqq \int |\tilde{f}(p)|^2 S_0^1(p) * S_0^1(p) d^d p.$$

The proof of d) is similar.

Remark. One may replace in c) ϕ_0^2 by $\phi_z \phi_{z'}$ and ϕ_x^2 by $\phi_{z+x} \phi_{z'+x}$ for any two points $z, z' \in \mathbb{Z}^d$ and the results still hold.

IV. Bounds on the Correlation Functions

In this section, we restrict ourselves to nearest-neighbor interactions. As was noticed in [8, 9], the model is then reflection positive with respect to planes containing the sites (we take periodic boundary conditions). Using this we prove

Theorem 2. *There exist a constant c such that, for all $x \in \mathbb{Z}^d$,*

a) $0 \leqq \langle \phi_0 \phi_x \rangle \leqq c \dfrac{\ln x}{|x|} \leqq \dfrac{c}{e(1-k)} |x|^{-k} (0 < k < 1)$ *for* $d = 3$

$0 \leqq \langle \phi_0 \phi_x \rangle \leqq c |x|^{-1}$ *for* $d \geqq 4$

b) $|\langle \nabla_0^e \phi \nabla_x^{e'} \phi \rangle| \leqq c |x|^{-1}$ *for all* d.

c) $|\langle \phi_0^2 ; \phi_x^2 \rangle| \leqq c |x|^{-1}$ *for* $d \geqq 3$.

Proof. We start with a) for $d=3$; the positivity follows from the F.K.G. inequalities (see Sect. IV).

Define $f_L(x)$: by $f_L(x)=1$ if $x=(2z_1,0,0)$, $-L\leqq z_1\leqq L, f_L(x)=0$ otherwise, and write $s(x-y)=\langle\phi_x\phi_y\rangle$. By the Brascamp–Lieb inequalities [2, 3, 4].

$$\sum_{x,y} f_L(x)f_L(y)s(x-y)\leqq\sum_{x,y} f_L(x)f_L(y)c_{xy}$$
$$=\int_{-\pi}^{\pi}|f_L(p)|^2\left[\sum_{\xi}(1-\cos p_\xi)\right]^{-1}\prod_{i=1}^{d}dp_i$$

where $f_L(p)=\sum_{y=-L}^{L} e^{-i2yp_1}$, p_1 is the component along the e_1 axis. Clearly,

$$|f_L(p)|^2\leqq 4\frac{(1-\cos 2p_1L)}{(1-\cos 2p_1)}.$$

So we have

$$\sum_{x,y} s(x-y)f_L(x)f_L(y)\leqq 4\int_{-\pi}^{\pi}\frac{(1-\cos 2p_1L)d^dp}{(1-\cos 2p_1)\left(\sum_{\xi}(1-\cos p_\xi)\right)}.$$

The integral has three singularities: at $p_\xi=0\,\forall\xi$, and at $p_1=\pm\pi$.

Let us consider the singularity at 0 and restrict the integration from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$. Setting $Lp_\xi=p'_\xi$ the integral is bounded by

$$CL^{4-d}\int_{-(1/2)\pi L}^{(1/2)\pi L}\frac{1-\cos 2p'_1}{p'^2_1}\frac{d^dp'}{p'^2} \tag{2}$$

because $1-\cos x\geqq\frac{2}{\pi^2}x^2, x\in[-\pi,\pi]$. In the (worst) case, $d=3$, this is less than

$$CL\int_{-(1/2)\pi L}^{(1/2)\pi L}\frac{1-\cos p'_1}{p'^2_1}dp'_1\int_{-(1/2)\pi L}^{(1/2)\pi L}\frac{d^2p'}{p'^2_2+p'^2_3}. \tag{3}$$

The integral in p'_1 is uniformly bounded in L and the one in p'_2, p'_3 diverges as $\ln L$.

The singularities at $\pm\pi$ give a contribution bounded by CL. So,

$$\sum_{x,y} s(x-y)f_L(x)f_L(y)\leqq CL\ln L. \tag{4}$$

By reflection positivity, each term in (4) is positive. (We could also use F.K.G. inequalities, see Sect. V, but this would not work for case b)). Moreover, using the transfer matrix formalism [13]

$$\langle\phi_0\phi_{2x}\rangle=\langle\phi_0T^{2x}\phi_0\rangle \tag{5}$$

when x is along a coordinate axis, say e_1 and T is the transfer matrix associated with the e_1 direction. T is known to be self-adjoint with respect to the scalar product $(A,B)=\langle\theta(A)B\rangle$ and $\|T\|\leqq 1$. By the spectral theorem, with $d\mu$ the spectral measure,

$$\langle \phi_0 \phi_{2x} \rangle = \int_{-1}^{+1} \lambda^{2|x_1|} d\mu(\lambda) \leqq \int_{-1}^{+1} \lambda^{2|x_1|-2} d\mu(\lambda) = \langle \phi_0 \phi_{2x-2} \rangle \tag{6}$$

which shows that $s(2x)$ is monotone decreasing in $|x|, x = (x_1, 0, 0)$. Using (6) one has,

$$L \sum_{0 \leqq x_1 \leqq 2L} s(2x) \leqq \sum_{x,y} s(x-y) f_L(x) f_L(y)$$

and by (4),

$$\sum_{0 \leqq x_1 \leqq 2L} s(2x) \leqq C \ln L .$$

Finally,

$$s(2x) \leqq \frac{1}{|x_1|} \sum_{y=0}^{x_1} s(2y) \leqq C \frac{\ln |x_1|}{|x_1|} .$$

This proves our Proposition for x along a coordinate axis with even coordinate. If the coordinate x_1 is odd, we make a reflection through the plane perpendicular to the axis e, at coordinate $\frac{|x_1|}{2}$. Then we have

$$|\langle \phi_0 \phi_x \rangle| \leqq \langle \phi_0 \phi_{x-1} \rangle^{1/2} \langle \phi_0 \phi_{x+1} \rangle^{1/2} \leqq c \frac{\ln |x_1|}{|x_1|} .$$

If x does not lie on a coordinate axis, we suppose that x_1 is the largest coordinate of x. Then,

$$|x_1| \geqq \frac{|x|}{d} = \frac{1}{d} \sum_{i=1}^{d} |x_i| .$$

Making the same reflection as before, we obtain our result if we use also

$$\frac{\ln x}{x^{\varepsilon}} \leqq \frac{1}{e\varepsilon} .$$

For $d \geqq 4$ the integral (3) is of order L and this finishes the proof of case a).

For case b) we first use reflections to reduce ourselves to the case where both gradients $\nabla^e, \nabla^{e'}$ are in the same direction, i.e., $e = e'$. Then, using Theorem 1, b), one shows that

$$\sum_{x,y} f_L(x) f_L(y) |\langle \nabla_x^e \phi \nabla_x^e \phi \rangle|$$
$$\leqq c \int_{-\pi}^{+\pi} |f_L(p)|^2 \prod_{i=1}^{d} dp_i \leqq cL \tag{7}$$

To prove the monotonicity in x, along a coordinate axis, we write

$$|\langle \nabla_0^e \phi \nabla_x^e \phi \rangle| = |\langle \phi_0 \phi_{x+e} \rangle + \langle \phi_0 \phi_{x-e} \rangle - 2 \langle \phi_0 \phi_x \rangle|$$

Using the spectral representation (6) we see that for $|x|$ *odd* $|\langle \nabla_0^e \phi \nabla_x^e \phi \rangle|$ is monotone decreasing in $|x|$. Combining this with (7) we see that for $|x|$ odd

$$|\langle \nabla_0^e \phi \nabla_x^e \phi \rangle| \leqq \frac{\ln |x|}{|x|}.$$

If $|x|$ is even, then making a reflection through $\frac{|x|}{2}$ we get

$$\begin{aligned} |\langle \nabla_0^e \phi \nabla_x^e \phi \rangle \leqq & |\langle \nabla_0^e \phi \nabla_{x-1}^e \phi \rangle|^{1/2} \\ & |\langle \nabla_0^e \phi \nabla_{x+1}^e \phi \rangle|^{1/2} \end{aligned}$$

which finishes case b); for x not on a coordinate axis, we do as in a). Case c) is similar to a) for $d \geqq 4$.

Remarks. 1. Using a Mermin–Wagner type of argument, one shows [16, 17] that for all T:

$$S_T^{(1)}(p) \geqq (2 + 12T < (\phi_0 - \phi_1)^2 >)^{-1} S_0^{(1)}(p).$$

As noticed by Park [12] and Fröhlich and Spencer [10] this, combined with Theorem 1 shows that for some constants c_1, c_2

$$c_2 \frac{1 - \cos p_e}{\sum_\xi (1 - \cos p_\xi)} \leqq S_T^{(1)}(p) |1 - e^{ip_e}|^2 \leqq c_1 \frac{1 - \cos p_e}{\sum_\xi (1 - \cos p_\xi)}.$$

If we let $p \to 0$ (for $d > 1$) parallel to e or perpendicular to it we see that $S_T^{(1)}(p)|1 - e^{ip_e}|^2$ is not continuous at $p = 0$. By the Riemann–Lebesgue lemma, this implies that

$$\sum_x |\langle \phi_0 \phi_x \rangle| \quad \text{and} \quad \sum_x |\langle \nabla_0^e \phi \nabla_x^e \phi \rangle|$$

diverge which is, in a sense, a converse to Theorem 2.

2. Of course one would expect a better decay than the one given by Theorem 2. We could obtain this if instead of summing over a line, we were summing over a square or a cube. But then we need to know that $\langle \phi_0 \phi_x \rangle$ reaches its minimum for x in the corners of the cube. However, by using correlation inequalities, one can show this for Ising and plane rotator models. This was noticed and used in [15] to yield $s(x) \leqq C|x|^{2-d}$.

We have the following extension of Theorem 2, which is a simple application of reflection positivity.

Theorem 3

a) For $d \geqq 3$ *let* $f = \prod_x \phi_x^{n_x}$ *where* $x_1 \leqq 0$ *if* $n_x \neq 0$ *and let* $y = (y, 0, \ldots, 0), y \geqq 0$. *Then*

$$|\langle f \phi_y \rangle| \leqq \langle f^2 \rangle^{1/2} \langle \phi_0 \phi_{2y} \rangle^{1/2} \leqq \langle f^2 \rangle^{1/2} \left(\frac{\ln |y|}{|y|} \right)^{1/2}$$

$$\begin{aligned} |\langle f; \phi_y^2 \rangle| & \leqq \langle f^2 \rangle^{1/2} \langle \phi_0^2; \phi_{2y}^2 \rangle^{1/2} \\ & \leqq \langle f^2 \rangle^{1/2} |y|^{-1/2} \end{aligned}$$

b) For all d, let $f=\prod_{\langle x,y\rangle}(\phi_x-\phi_y)^{n_{xy}}$ *where* $x_1, y_1 \leqq 0$ *if* $n_{xy}\neq 0$ *and* $z=(z,0,\ldots,0)$, $z\geqq 0$*; then*

$$|\langle f\nabla_z^{e_1}\phi\rangle| \leqq \langle f^2\rangle^{1/2}|\langle \nabla_0^{e_1}\phi\nabla_{2z+1}^{e_1}\phi\rangle|^{1/2}$$
$$\leqq \langle f^2\rangle^{1/2}|z|^{-1/2}.$$

Proof. This follows immediately from the Schwartz inequality deduced from reflection positivity [8] and the translation invariance of $\langle\quad\rangle$. We also use the ordinary Schwartz inequality to bound $\langle f\theta f\rangle^{1/2}\leqq\langle f^2\rangle^{1/2}$.

Remark. More generally, Theorem 2 shows that, for reflection positive models, if $\langle A;\tau^x A\rangle$ clusters then for any B, $\langle B;\tau^x A\rangle$ clusters.

V. F.K.G. Inequalities

In [1] it was proven that the F.K.G. inequalities hold whenever

$$\frac{d^2H}{d\phi_x d\phi_y}\geqq 0 \quad \forall x,y$$

which is the case of (1).

We say that a function $f(\{\phi_x\})$ is *increasing* if $f(\{\phi_x\})\geqq f(\{\phi'_x\})$ whenever $\phi_x\geqq\phi'_x$ for all x.

If f and g are increasing, F.K.G. inequalities give

$$\langle fg\rangle\geqq\langle f\rangle\langle f\rangle \tag{10}$$

provided the expectation values are well defined.

Let, for $\lambda\geqq 0, \sigma_{x,\lambda}$ be defined by

$$\sigma_{x,\lambda}(\phi_x)=\begin{cases}\phi_x & \text{if } |\phi_x|\leqq\lambda\\ \lambda\,\text{sign}\,\phi_x, & \text{if } |\phi_x|\geqq\lambda\end{cases}$$
$$\tilde\sigma_{x,\lambda}=\lambda^{-1}\sigma_{x,\lambda}.$$

Given a function $n=(n_x)$ from $\mathbb{Z}^d$ into $\mathbb{N}$ of finite support, we let

$$|n|=\sum_x n_x,\quad \underline{n}=\{x|n_x\neq 0),$$
$$\sigma_n=\prod_x\sigma_{x,\lambda}^{n_x},\quad \tilde\sigma_n=\prod_x\tilde\sigma_{x,\lambda}^{n_x}$$
$$\tilde\Sigma_n=\sum_x n_x\tilde\sigma_{x,\lambda}.$$

The following lemma simplifies the argument of [11] because we do not have to use the gas variable. We drop the index λ in what follows.

Lemma

a) For any two functions n, m from $\mathbb{Z}^d$ *into* $\mathbb{N}$ *of finite support,*

$$|\langle\sigma_n\sigma_m\rangle-\langle\sigma_n\rangle\langle\sigma_m\rangle|\leqq\lambda^{(|n|+|m|-2)}\sum_{x,y}n_xn_y\langle\sigma_x\sigma_y\rangle$$

b) $\langle\sigma_x\sigma_y\rangle\leqq\langle\varphi_x\varphi_y\rangle$.

Proof
a) is the statement that

$$\pm(\langle\tilde{\sigma}_n\tilde{\sigma}_m\rangle-\langle\tilde{\sigma}_n\rangle\langle\tilde{\sigma}_m\rangle)\leqq\langle\tilde{\Sigma}_n\tilde{\Sigma}_m\rangle. \tag{11}$$

But, since $|\tilde{\sigma}_n|\leqq 1$, $\tilde{\Sigma}_n\pm\tilde{\sigma}_n$ are increasing for all n. Therefore (11) follows by adding the two F.K.G. inequalities:

$$\langle(\tilde{\Sigma}_n+\tilde{\sigma}_n)(\tilde{\Sigma}_m\mp\tilde{\sigma}_m)\rangle\geqq\langle\tilde{\Sigma}_n+\tilde{\sigma}_n\rangle\langle\tilde{\Sigma}_m\mp\tilde{\sigma}_m\rangle$$
$$\langle(\tilde{\Sigma}_n-\tilde{\sigma}_n)(\tilde{\Sigma}_m\pm\tilde{\sigma}_m)\rangle\geqq\langle\tilde{\Sigma}_n-\tilde{\sigma}_n\rangle\langle\tilde{\Sigma}_m\pm\tilde{\sigma}_m\rangle$$

($\langle\tilde{\Sigma}_m\rangle=0$ by symmetry here.)
b) follows from F.K.G. because $\phi_x-\sigma_x$ is increasing and we write

$$\langle\phi_x\phi_y\rangle=\langle(\phi_x-\sigma_x)\phi_y\rangle+\langle\sigma_x(\phi_y-\sigma_y)\rangle+\langle\sigma_x\sigma_y\rangle$$

The first two terms are positive ($\langle\phi_y\rangle=\langle\sigma_x\rangle=0$) which proves b).

Theorem 4. *Let $d\geqq 3$. For any $a,b\in\mathbb{N}$, there exists a constant $c(a,b)$ such that, for all n,m with $|n|=a$, $|m|=b$.*

$$\left|\left\langle\prod_x\phi_x^{n_x}\prod_y\phi_y^{m_y}\right\rangle-\left\langle\prod_x\phi_x^{n_x}\right\rangle\left\langle\prod_y\phi_y^{m_y}\right\rangle\right|$$
$$\leqq c(a,b)F_{a,b}\left(\max_{\substack{x\in \underline{n}\\ y\in \underline{m}}}\langle\phi_x\phi_y\rangle\right) \tag{12}$$

where

$$F_{a,b}(z)=(\log z)^{a+b-2}z.$$

Proof. For each factor ϕ_x in the expectation value of the l.h.s. of (12) we write $\phi_x=(\phi_x-\sigma_x)+\sigma_x$ and expand the products over x and y into a sum. There is one term in that sum of the form $\langle\sigma_n\sigma_m\rangle-\langle\sigma_n\rangle\langle\sigma_m\rangle$ that we bound via the Lemma. All other terms are of the form $\left\langle\prod_x(\phi_x-\sigma_x)^{\tilde{n}_x}\prod_y\phi_y^{\tilde{m}_y}\right\rangle$ with at least one $n_x\neq 0$. But the Brascamp–Lieb inequalities tell us that the distribution of ϕ_x is of the form $e^{-\alpha\phi^2}G(\phi_x)$ with G log concave and $\alpha\neq 0$ (for $d\geqq 3$). Since $|\phi_x-\sigma_x|\leqq\phi_x\chi(|\phi_x|\geqq\lambda)$, we have (using Schwartz's inequality) that each of the terms with $(\phi_x-\sigma_{x,\lambda})$ is bounded by (Const)$e^{-\lambda}$. We choose

$$\lambda=-\log\left(\max_{\substack{x\in \underline{n}\\ y\in \underline{m}}}\langle\phi_x\phi_y\rangle\right)$$

and this finishes the proof.

Using Proposition 2, we have

Corollary. *Let the interaction be nearest-neighbor. For each $k<1$ and each $a,b\in\mathbb{N}$, there exists a constant $c(k,a,b)$ such that for all n,m with $|n|=a$, $|m|=b$,*

$$\left\langle\prod_x\phi_x^{n_x}\prod_y\phi_y^{m_y}\right\rangle-\left\langle\prod_x\phi_x^{n_x}\right\rangle\left\langle\prod_y\phi_y^{m_y}\right\rangle$$
$$\leqq c(k,a,b)[\operatorname{dist}(n,m)]^{-k}.$$

References

1. Battle, G., Rosen, L. : J. Stat. Phys. **22** 128 (1980)
2. Brascamp, H. J., Lieb, E. H. : J. Funct. Anal. **22**, 366 (1976)
3. Brascamp, H. J., Lieb, E. H., Lebowitz, J. L. : Bull. Int. Statist. Inst. **46**, Invited Paper No. 62 (1975)
4. Brascamp, H. J., Lieb, E. H. : Lecture given at the Conference on Functional Integration, Cumberland Lodge, England, April 2–4, 1974.
5. Bricmont, J., Fontaine, J.R., Lebowitz, J.L., Spencer, T.: Commun. Math. Phys. **78**, 281–302 (1980)
6. Fortuin, C., Kasteleyn, P., Ginibre, J. : Commun. Math. Phys. **22**, 89 (1971)
7. Fröhlich, J., Simon, B., Spencer, T. : Commun. Math. Phys. **50**, 79 (1976)
8. Fröhlich, J., Lieb, E. H. : Commun. Math. Phys. **60**, 233 (1978)
9. Fröhlich, J., Israël, R., Lieb, E. H., Simon, B. : Commun. Math. Phys. **62**, 1 (1978)
10. Fröhlich, J., Spencer, T. : On the statistical mechanics of classical goulomb and dipole gases. Preprint, IHES, 80 (to appear in J. Stat. Phys.)
11. Lebowitz, J. : Commun. Math. Phys. **28**, 313 (1972)
12. Park, Y. M. : Commun. Math. Phys. **70**, 161 (1979)
13. Schor, R. : Commun. Math. Phys. **53**, 213 (1978)
14. Simon, B. : The $p(\phi)_2$ Euclidean (quantum) field theory. Princeton, N. J. : Princeton University Press, 1974.
15. Sokal, A. : In preparation
16. Mermin, N. D., Wagner, H. : Phys. Rev. Lett. **17**, 1133 (1966)
17. Hohenberg, P. C. : Phys. Rev. **158**, 383 (1967)

Communicated by A. Jaffe

Received May 7, 1980