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Mass Spectrum of the Two Dimensional $\lambda\phi^4 - \frac{1}{4}\phi^2 - \mu\phi$ Quantum Field Model

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Abstract. It is shown that r-particle irreducible kernels in the two-dimensional $\lambda\phi^4 - \frac{1}{4}\phi^2 - \mu\phi$ quantum field theory have $(r+1)$-particle decay for $|\mu| \leqq \lambda^2 \ll 1$. As a consequence there is an upper mass gap and, in the subspace of two-particle states, a bound state. The proof extends Spencer's expansion [20] to handle fluctuations between the two wells of the classical potential. A new method for resumming the low temperature cluster expansion is introduced.

Introduction

Much progress has been made recently in describing in detail pure phases of quantum field models in low temperature regions of coupling. Glimm et al. [16] developed a convergent expansion for the Schwinger functions of the $\lambda\phi^4 - \frac{1}{4}\phi^2 - \mu\phi$ model in two dimensions (with $|\mu| \leqq \lambda^2 \ll 1$), establishing also the mass gap of the theory. Subsequently their expansion technique has been applied to some ϕ_2^6 models with three minima [22, 23], to the two-dimensional pseudo-scalar Yukawa model in the two-phase region [1], and to the Coulomb gas in the sine-Gordon representation [2, 3]. Investigators have concentrated on proving the cluster property of correlations and the mass gap, leaving the higher spectrum unexplored.

A wealth of information is known about the spectrum of single phase $\lambda P(\phi)_2$ theories with λ small. The n-particle cluster expansion [14] was used to establish the existence of isolated one-particle states and to show that for $\lambda < \lambda(n, \varepsilon)$, n field operators are sufficient to generate all states of energy less than $(n+1)m(1-\varepsilon)$, where m is the single particle mass. Spencer [20] introduced an expansion for r-particle irreducible kernels, proving $(r+1)$-particle decay. For even theories this information was used to analyze the mass spectrum below $3m-\varepsilon$ [8, 9, 21], with results including asymptotic expansions for bound state masses and scattering amplitudes, and asymptotic completeness in this energy region. Burnap [5] showed (without resorting to the n-particle cluster expansion) that in general

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circumstances the upper mass gap follows from decay properties of one-particle irreducible kernels. He applied the result to the $\lambda\phi_3^4$ theory [4]. For $\lambda P(\phi)_2$ theories with odd powers of ϕ in the interaction, Glimm and Jaffe [13] adapted the n-particle cluster expansion [14] to establish discreteness of spectrum below $2m$ and Koch [17] used irreducible kernels to analyze bound states in detail below $2m$.

In this paper some of the above results on mass spectra are established for the two-dimensional $\mathscr{P}(\phi)=\lambda\phi^4-\frac{1}{4}\phi^2-\mu\phi-E_c$ model with $|\mu|\leqq\lambda^2\ll 1$. The external field will vary with λ according to the relation $\mu=\lambda^2\mu'(1+\sqrt{18}\gamma+4\gamma^2)$, where $\gamma=\lambda^{5/2}\ \mu'$ and $\mu'\in[0, 1)$ is fixed. E_c is adjusted so that $\inf\mathscr{P}=0$. The classical polynomial $\mathscr{P}$ has an absolute minimum at $\xi_+=(8\lambda)^{-1/2}+\lambda^2\mu'$ and a relative minimum at $\xi_-\cong-\xi_+$. The model is defined in a finite volume Λ as in [16]. In terms of the variable $\varphi=\phi-\xi_+$, the polynomial becomes $\mathscr{P}(\varphi+\xi_+)=\lambda\varphi^4+(2\lambda)^{1/2}(1+\sqrt{8}\gamma)\varphi^3+\sqrt{18}\gamma(1+\sqrt{2}\gamma)\varphi^2+\frac{1}{2}\varphi^2$. Denote the free Gaussian measure for the Euclidean field $\varphi(x)$ by $d\varphi$, where the covariance is $(-\Delta+1)^{-1}$. The interacting, finite volume expectation with + boundary conditions is

$$\langle R\rangle=\frac{\int Re^{-V}d\varphi}{\int e^{-V}d\varphi}, \tag{1.1}$$

where

$$V=\int_\Lambda :\lambda\varphi(x)^4+(2\lambda)^{1/2}(1+\sqrt{8}\gamma)\,\varphi(x)^3+\sqrt{18}\gamma(1+\sqrt{2}\gamma)\,\varphi(x)^2:dx\,. \tag{1.2}$$

Wick ordering will always be defined using the free covariance. Dependence on Λ will usually be suppressed. Truncated expectations are defined in the usual way and are written with semicolons, for example $\langle R_1;R_2\rangle=\langle R_1R_2\rangle-\langle R_1\rangle\langle R_2\rangle$.

With these boundary conditions, the infinite volume limit of the model is a pure phase with exponential clustering [16]. $(r+1)$-particle decay of r-particle irreducible kernels will be established in this paper. This leads to the following result.

Theorem 1.1. *Given an $\varepsilon>0$, let λ be sufficiently small and positive. The spectrum of the mass operator M consists solely of two eigenvalues, zero and $m(\lambda)=1+O(\lambda)$, in the interval $[0, 2m(\lambda)-\varepsilon)$. When restricted to the subspace of states generated by up to two field operators, there is exactly one eigenvalue $m_B(\lambda)$ of M in the interval $[2m(\lambda)-\varepsilon, 2m(\lambda))$. The binding energy $2m(\lambda)-m_B(\lambda)$ is equal to $36\lambda^2+O(\lambda^{5/2})$.*

The proof of Theorem 1.1, assuming decay properties of irreducible kernels, is contained in the literature [5, 7–10, 17, 21]. The expansion of this paper applies to irreducible kernels and to their $\lambda^{1/2}$-derivatives. Hence perturbation theory for the kernels considered in [17] yields the estimates on $m(\lambda)$ and $m_B(\lambda)$ that are in Theorem 1.1. The reader is referred to [17] for a complete discussion of the mass spectrum on the subspace of one- and two-particle states. The absence of any spectrum other than $m(\lambda)$ in the interval $(0, 2m(\lambda)-\varepsilon)$ follows from [5]. The effect of three or more field operators on the spectrum between $2m(\lambda)-\varepsilon$ and $2m(\lambda)$ is not considered here, because the results of the n-particle cluster expansion are lacking for this model.

The decay estimates on irreducible kernels are proven by using an expansion in decoupling parameters and in parameters which control large fluctuations of the field. When the measure is completely decoupled, r-particle irreducible kernels vanish to $(r+1)^{\text{th}}$ order in the decoupling parameters, leading to $(r+1)$-particle decay. The expansion in decoupling parameters is based on Spencer's expansion [20] and his methods are used to evaluate and bound derivatives. As in [16], convergence from decoupling lines may be obtained only in regions far from fluctuations between minima. However, as these large fluctuations have very small contribution to the measure, they may be eliminated with only a small error. Once an appropriate region free of large fluctuations has been isolated, Spencer lines [20] may be introduced to exhibit multiparticle decay.

All derivatives are bounded by means of analyticity [12, 20]. To establish bounds uniformly in large domains of complex parameter space (and in Λ) the low temperature cluster expansion of [16] is applied. The method for removing large fluctuations perturbs the theory sufficiently weakly to allow the use of an inequality on partition functions that was proven in [16]. The inequality is incorporated into a new resummation of the expansion using some ideas of Pirogov and Sinai [18]. Constraints on the resummed expansion are handled with some techniques of Bałaban and Gawędzki [1].

The remainder of this paper is organized into five sections. In Sect. 2, the interpolating measures are defined and conditions are derived for the vanishing of irreducible kernels and their derivatives. In the next section, the expansion is generated and a resummation is performed. In Sect. 4, analyticity techniques are used to bound individual terms of the expansion. This reduces the problem to a proof of a uniform upper bound with clustering for generalized measures. The low temperature cluster expansion is performed in Sect. 5, and its convergence is proven assuming bounds on individual terms. These bounds are proven in the final section.

Remark. After submitting this article for publication, we received a preprint of Koch [24] establishing similar results for $\mu=0$. In contrast to his work, our method is suited to handle the case of a nonzero external field.

2. Irreducible Kernels

Following [16], we insert a partition of unity into the measure to make a decomposition according to whether the field in a unit square is on average in the plus well or in the minus well. Let Δ_i^1 denote the unit square in $\mathbb{R}^2$ with lower left corner at $i=(i_0, i_1)\in\mathbb{Z}^2$. Define the average field in Δ_i^1 by

$$\phi(\Delta_i^1)=\int_{\Delta_i^1}\phi(x)dx \tag{2.1}$$

and let $\sigma_i=\pm 1$ be an Ising spin variable in Δ_i^1. We introduce approximate characteristic functions of $[0,\infty)$ and $(-\infty,0]$:

$$\chi_+(\xi)=\pi^{-1/2}\int_0^\infty e^{-(\xi-z)^2}dz\,, \tag{2.2}$$

$$\chi_-(\xi)=\chi_+(-\xi)\,. \tag{2.3}$$

Note that $\chi_+ + \chi_- = 1$. Then set $\chi_\Sigma = \prod_{\Delta_i^1 \subseteq \Lambda} \chi_{\sigma_i}(\phi(\Delta_i^1))$ where Σ denotes a spin configuration, that is, a function on the squares in Λ taking values ± 1. We take Λ to be a large square composed of unit lattice squares. We use the identity $1 = \sum_\Sigma \chi_\Sigma$ to expand the measure according to spin configurations:

$$e^{-V} d\varphi = \sum_\Sigma \chi_\Sigma e^{-V} d\varphi . \tag{2.4}$$

Here the sum runs over all spin configurations Σ in Λ. It is worth recalling the one-to-one correspondence between spin configurations in Λ and sets of Peierls contours which mark boundaries between seas of aligned spins. We set $\sigma_i = +$ for $\Delta_i^1 \nsubseteq \Lambda$, so all spins outside the outermost contours are $+$.

Two additional length scales will be needed. We choose $l \cong |\log\lambda|^{1/4} \gg 1$ to exhibit the approximately Gaussian character of the measure far from phase boundaries. We choose $L \cong |\log\lambda|^2 \gg l$ to define what regions are far from phase boundaries. For convenience, take l and L/l to be integers.

Let b denote a bond of the lattice $l\mathbb{Z}^2$. For each b introduce two parameters u_b and r_b, each taking values in $[0, 1]$. The u-parameters are introduced to remove those terms of $\sum_\Sigma$ with phase boundaries within L of particular bonds of $l\mathbb{Z}^2$. When the unwanted terms have been removed, the r-parameters introduce Dirichlet data on bonds of $l\mathbb{Z}^2$. Expanding in the u-parameters yields a sum over phase-boundary-free regions of $\mathbb{R}^2$. The expansion in r-parameters is used to control this sum. It is a two-dimensional analog of Spencer's expansion.

The u-dependent measure is

$$\sum_\Sigma \prod_{b:\mathrm{dist}(b,\Sigma) \leq L} u_b \chi_\Sigma e^{-V} d\varphi . \tag{2.5}$$

When $u_b = 0$, there are no phase boundaries within L of b. We introduce Dirichlet data into $d\varphi$ in the standard way [15]. The measure $d\varphi(r)$ has zero Dirichlet data on b when $r_b = 0$; free boundary conditions on b when $r_b = 1$. We shall never use the r-parameters to place Dirichlet data within L of Σ or in seas of minus spins. This is enforced by allowing $r_b \neq 1$ only if $u_b = 0$ and $\Sigma = +$ near b.

We introduce Spencer lines [20] in order to exhibit multiparticle decay. Let $\bar{\mathcal{L}}_i$ be the lines $x_0 = il$ in $\mathbb{R}^2$, for $i \in \mathbb{Z}$. For each i, fix $\mathcal{L}_i \subseteq \bar{\mathcal{L}}_i$ to be some finite union of bonds of the $l\mathbb{Z}^2$ lattice. Introduce Dirichlet data on $\mathcal{L}_i$ with the parameter $t_i \in [0, 1]$. Denote the resulting measure by $d\varphi(r, t)$. We allow $t_i \neq 1$ only if $\mathcal{L}_i$ is in a sea of $+$ spins. The set $\{\mathcal{L}_i\}$ will vary from term to term in the expansion in the u- and r-parameters in order to satisfy this constraint.

We state the above restrictions in terms of a condition that must be satisfied at all times.

Condition A. *For all b, if $r_b \neq 1$ then $u_b = 0$ and enough u's are zero so that all the nonvanishing terms of $\sum_\Sigma$ in (2.5) have $\sigma_i = +$ within L of b. Similarly, for all i, if $t_i \neq 1$ then $u_b = 0$ for all $b \subseteq \mathcal{L}_i$ and all nonvanishing terms have $\sigma_i = +$ within L of $\mathcal{L}_i$.*

We may now define an expectation which depends on u, r, t, $\{\mathcal{L}_i\}$, and Λ.

$$\langle R \rangle = \frac{\sum_\Sigma \prod_{b:\mathrm{dist}(b,\Sigma) \leq L} u_b \int R \chi_\Sigma e^{-V} d\varphi(r,t)}{\sum_\Sigma \prod_{b:\mathrm{dist}(b,\Sigma) \leq L} u_b \int \chi_\Sigma e^{-V} d\varphi(r,t)} . \tag{2.6}$$

We allow for R's containing derivatives. This expectation is used to construct the irreducible kernels and to give them appropriate dependence on the parameters. As an example, consider the one-particle irreducible kernel $k(x,y)$ [20]. Suppressing dependence on u, r, etc., we define the following kernels when Condition A holds.

$$S(x,y)=\langle\varphi(x);\varphi(y)\rangle, \tag{2.7}$$

$$\Gamma(x,y)=(S^{-1})(x,y), \tag{2.8}$$

$$k(x,y)=(\Gamma-C^{-1})(x,y). \tag{2.9}$$

Here we use operator inverses and C is the covariance of $d\varphi(r,t)$.

We need to express $k(x,y)$ as a Neumann series of connected expectations as in [20]. In Sect. 4 this representation will be used to introduce dependence on additional parameters $h(\alpha)$. We follow [20] in this calculation except that we leave derivatives $\frac{\delta}{\delta\varphi}$ as such and do not explicitly differentiate the interaction. The presence of the χ-factors in (2.6) make it awkward to differentiate the interaction when integrating by parts. We obtain

$$\begin{aligned} C_x^{-1}\langle\varphi(x);\varphi(y)\rangle &= \delta(x-y)+\left\langle\varphi(y);\frac{\delta}{\delta\varphi(x)}\right\rangle \\ &\equiv \mathbb{1}(x,y)+A(x,y) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} C_y^{-1}A(x,y) &= \left\langle\frac{\delta}{\delta\varphi(y)};\frac{\delta}{\delta\varphi(x)}\right\rangle \\ &\equiv B(x,y). \end{aligned} \tag{2.11}$$

Note that these derivatives act on the χ-factors in (2.6) as well as on e^{-V}. In both cases there will result an overall factor at least as small as $\lambda^{1/2}$. We express k in terms of the operators A and B:

$$\begin{aligned} k &= \Gamma-C^{-1} \\ &= (\mathbb{1}+A)^{-1}C^{-1}-C^{-1} \\ &= -(\mathbb{1}+A)^{-1}B. \end{aligned} \tag{2.12}$$

Or,

$$k=-(\mathbb{1}+BC)^{-1}B. \tag{2.13}$$

We refer ahead to Theorem 4.1 for the estimates that guarantee that the Neumann series in (2.12) and (2.13) converge for small λ.

The next proposition gives conditions for the vanishing of $k(x,y)$ and $\frac{\partial}{\partial t_i}k(x,y)$.

Proposition 2.1. *Suppose Condition A holds and there exists a complete contour Γ of $r=0$ bonds or $t=0$ lines separating x from y. Then $k(x,y)=0$ and $\frac{\partial}{\partial t_i}k(x,y)=0$.*

Proof. Denote the interior of Γ by Y. The measure used in (2.6) factorizes across Γ:

$$\begin{aligned}
&\int \sum_{\Sigma} \prod_{b:\mathrm{dist}(b,\Sigma)\leqq L} u_b R_Y R_{\sim Y} \chi_\Sigma e^{-V} d\varphi(r,t) \\
&= \Bigg(\int \sum_{\substack{\Sigma\cap Y \\ \Sigma=+\text{ on }\Gamma}} \prod_{\substack{b\subseteq Y \\ \mathrm{dist}(b,\Sigma\cap Y)\leqq L}} u_b R_Y \chi_{\Sigma\cap Y} e^{-V(Y)} d\varphi(r,t)\Bigg) \\
&\quad \cdot \Bigg(\int \sum_{\substack{\Sigma\cap\sim Y \\ \Sigma=+\text{ on }\Gamma}} \prod_{\substack{b\subseteq\sim Y \\ \mathrm{dist}(b,\Sigma\cap\sim Y)\leqq L}} u_b R_{\sim Y} \chi_{\Sigma\cap\sim Y} e^{-V(\sim Y)} d\varphi(r,t)\Bigg).
\end{aligned} \tag{2.14}$$

Here $\Sigma\cap Y$ is the restriction of the spin configuration to Y and $\chi_{\Sigma\cap Y} = \prod_{\Delta_\varepsilon^1\subseteq Y\cap\Lambda} \chi_{\sigma_\varepsilon}(\phi(\Delta_\varepsilon^1))$. R_Y and $R_{\sim Y}$ are supported in Y and $\sim Y$, respectively. The u-factors may be distributed as in (2.14) because $\mathrm{dist}(\Sigma\cap Y, \sim Y) > L$. Factorization implies that connected expectations between Y and $\sim Y$ vanish. The covariance $C(r,t)$ also vanishes between Y and $\sim Y$. Therefore Γ, C^{-1}, and k vanish between Y and $\sim Y$.

The following calculation is as in [20]. The formula for differentiating expectations

$$\frac{\partial}{\partial t_i}\langle\varphi(x);\varphi(y)\rangle = -\tfrac{1}{2}\int\langle\varphi(x);\varphi(y);\varphi(z_1)\varphi(z_2)\rangle\left(\frac{\partial}{\partial t_i}C^{-1}\right)(z_1,z_2)dz \tag{2.15}$$

holds for the expectation (2.6). By factorization, if x and y are separated by Γ, then (2.15) becomes

$$\frac{\partial}{\partial t_i}S = -S\left(\frac{\partial}{\partial t_i}C^{-1}\right)S. \tag{2.16}$$

The factor of 1/2 is cancelled by the two possibilities for positioning z_1 and z_2 relative to Γ. We use (2.16) to calculate $\frac{\partial}{\partial t_i}k$:

$$\begin{aligned}
\frac{\partial}{\partial t_i}k(x,y) &= \left(\frac{\partial}{\partial t_i}(\Gamma - C^{-1})\right)(x,y) \\
&= \left(-\Gamma\left(\frac{\partial}{\partial t_i}S\right)\Gamma - \frac{\partial}{\partial t_i}C^{-1}\right)(x,y) \\
&= \left(\Gamma S\left(\frac{\partial}{\partial t_i}C^{-1}\right)S\Gamma - \frac{\partial}{\partial t_i}C^{-1}\right)(x,y) = 0.
\end{aligned} \tag{2.17}$$

Proposition 2.1 is proven.

For other irreducible kernels, similar constructions and proofs apply. For R in (2.6), we will consider generalized derivatives whose actions may be restricted in various ways. For example, the analog of the expectation $\left\langle\prod_i :P^{(k)}(x_i):\right\rangle$ in $P(\phi)_2$ theories is constructed in section four. ($P^{(k)}$ denotes the k^{th} derivative of P.) These expectations may be used to construct the kernels considered in [17]. We also consider derivatives $\partial_\chi(x)$ which are the same as $\frac{\delta}{\delta\varphi(x)}$ except that they act only on χ-factors or their derivatives.

The first r derivatives with respect to t vanish at $t=0$ for r-particle irreducible kernels constructed in this way. The proofs depend only on factorization, integration by parts, and analogs of (2.15). These properties hold for the expectation (2.6), as we saw in the case of $k(x, y)$. We omit further details.

In order to handle $\lambda^{1/2}$-derivatives of irreducible kernels, we prove that the generalized expectations are C^∞ in $\lambda^{1/2}$. Since

$$\partial_{\lambda^{1/2}}\chi_\pm^{(\,)}(\varphi(\Delta^1)+\xi_+)=(\partial_{\lambda^{1/2}}\xi_+)\partial_\chi(\Delta^1)\chi_\pm^{(\,)}(\varphi(\Delta^1)+\xi_+)\,\text{in}\int\cdot d\varphi(r, t),$$

we see that $\lambda^{1/2}$-derivatives of generalized expectations yield sums of other generalized expectations. On these we perform an asymptotic expansion in $\lambda^{1/2}$ as in [16]. Each integration by parts produces derivatives of χ-factors as well as the usual perturbation expansion. We establish in Sect. 4 that expectations of $\partial_\chi(x)$'s are exponentially small in $\lambda^{1/2}$ and that truncated expectations cluster. Hence the derivatives of χ-factors contribute only to the remainder in the asymptotic expansion, and the $\lambda^{1/2}$-derivatives are bounded as $\lambda \to 0$. Using their representation in terms of convergent Neumann series of expectations, irreducible kernels and t-derivatives of irreducible kernels are also C^∞ in $\lambda^{1/2}$. As in [17], we commute $\lambda^{1/2}$-derivatives with t-derivatives to see that $\lambda^{1/2}$-derivatives of irreducible kernels vanish to the appropriate order in t at $t=0$.

For the sake of definiteness, we consider only $k(x, y)$ in the remainder of this paper. From the above remarks, we see that the expansion applies as well to other r-particle irreducible kernels considered in [5, 6, 17, 20]. However, λ must be taken sufficiently small, depending on r.

3. An Expansion for Irreducible Kernels

In this section we prove two-particle decay of $k(x, y)$, assuming some bounds on derivatives of k with respect to u, r, and t. These bounds will be proven in Sect. 4.

Theorem 3.1. *Let $\varepsilon > 0$ be given. Then for λ sufficiently small and $u=r=t=1$,*

$$\int k(z_1, z_2) f(z_1, z_2) dz \leqq \|f\|_{L^\infty} e^{-2(1-\varepsilon)|x_0 - y_0|}. \tag{3.1}$$

Here $f \in C^0(\Delta_x \times \Delta_y)$, where Δ_x and Δ_y are l-lattice squares with lower left corners at x and y, respectively.

Proof. For simplicity, write simply k for the left hand side of (3.1). For every b within L of Λ we apply the identity

$$k(u_b=1)=k(u_b=0)+\int_0^1 \frac{\partial}{\partial u_b} k(u_b) du_b. \tag{3.2}$$

The result is the following expansion for k:

$$k(u=1)=\sum_{\Gamma_u} \int du\, \partial_u^{\Gamma_u} k(u, r). \tag{3.3}$$

The sum runs over all possible configurations of $u=0$ bonds and $\frac{\partial}{\partial u}$ bonds. For each configuration we let Γ_u be the union of the $\frac{\partial}{\partial u}$ bonds. The remaining bonds have $u=0$. We use the notation $\partial_u^{\Gamma_u} = \prod_{b\in\Gamma_u} \frac{\partial}{\partial u_b}$. The integration runs over the differentiated u_b's.

We begin to control the sum by performing a resummation. Resumming a bond b means fixing the state $\left(u=0 \text{ or } \frac{\partial}{\partial u}\right)$ of all bonds but b and applying (3.2) to b in reverse. We define for each term of (3.3) the set of bonds to be resummed. Clearly this definition must be invariant under changes in state of bonds to be resummed, for otherwise both terms on the right hand side of (3.2) would not be present. When the definition is invariant, we say the resummation is without constraint.

We say a set of bonds is a $\frac{\partial}{\partial u}$ contour if it is a simple closed contour of $\frac{\partial}{\partial u}$ bonds. We resum all bonds that are interior to some $\frac{\partial}{\partial u}$ contour. Changing a configuration inside a $\frac{\partial}{\partial u}$ contour cannot alter the fact that bonds inside the contour are resummed. Hence the resummation is without constraint. After resummation, every configuration consists of a collection of $\frac{\partial}{\partial u}$ contours, with $u=1$ inside the contours. Bonds outside all contours may be either $\frac{\partial}{\partial u}$ bonds or $u=0$ bonds.

We expand in the r-parameters for bonds with $u_b=0$. These bonds $\left(\text{as well as } \frac{\partial}{\partial u} \text{ bonds}\right)$ have no $\frac{\partial}{\partial u}$ contours surrounding them, by construction. Therefore every term in $\sum_{\Sigma}$ with a phase boundary around b is multiplied by $0=\prod_{b':\mathrm{dist}(b',\Sigma)\leqq L} u_{b'}$. Thus the spins near b agree in sign with the plus phase that exists in $\sim\Lambda$. This property of the resummed u-expansion is what will allow the use of inequalities on partition functions in section five. The inequalities are crucial for handling the case $\mu\neq 0$. In addition, Condition A will not be violated by introducing Dirichlet data on $u=0$ bonds outside $\frac{\partial}{\partial u}$ contours. For each such bond the identity

$$k(r_b=1)=k(r_b=0)+\int_0^1 \frac{\partial}{\partial r_b} k(r_b)\,dr_b \tag{3.4}$$

is applied. The result is

$$k(u=r=1)=\sum_{\Gamma_u,\Gamma_r} \int du\,dr\,\partial_u^{\Gamma_u}\partial_r^{\Gamma_r} k(u,r)\,. \tag{3.5}$$

Here Γ_r is the union of the $\frac{\partial}{\partial r}$ bonds and $\partial_r^{\Gamma_r}=\prod_{b\in\Gamma_r}\frac{\partial}{\partial r_b}$. The sum runs over all configurations of $\frac{\partial}{\partial r}$ bonds, $\frac{\partial}{\partial u}$ bonds, $r=0$ bonds, and $u=1$ bonds that may be obtained by the above construction. Thus every bond inside a $\frac{\partial}{\partial u}$ contour has $u=1$, while every bond outside all $\frac{\partial}{\partial u}$ contours is either $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial r}$, or $r=0$.

For each term of (3.5), let W be the closure of the connected component of $\mathbb{R}^2\backslash\{r=0 \text{ bonds}\}$ that contains Δ_x. By Proposition 2.1, the term vanishes unless also $\Delta_y \subseteqq W$. The measure factorizes across ∂W, so connected expectations vanish between W and $\sim W$. Using the representation of k as a convergent Neumann

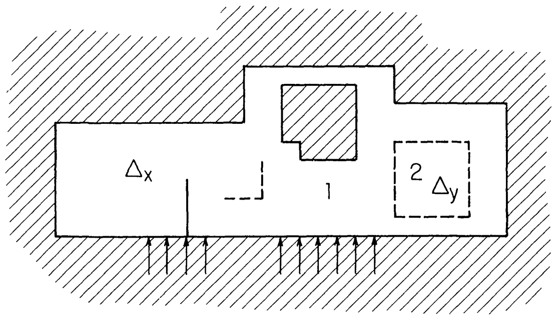

Fig. 1. A typical term in the expansion for k. Solid lines indicate Dirichlet contours; dashed lines indicate $\frac{\partial}{\partial u}$ bonds. Shaded regions have $u=r=0$, while region 2 has $u=r=1$. All other bonds are $\frac{\partial}{\partial r}$ bonds. W consists of regions 1 and 2, while V consists of region 1 only. The positions of Δ_x and Δ_y are as indicated. Spencer lines (not shown) are inserted above the arrows

series of connected expectations (2.12), we see that k does not depend on the status of the bonds of $\sim W$. Thus $\frac{\partial k}{\partial u_b} = \frac{\partial k}{\partial r_b} = 0$ for b outside W, and the only nonvanishing terms have $u=r=0$ outside W. Therefore, we need only sum over W and its bond configuration.

Let V be the set obtained by deleting the interiors of $\frac{\partial}{\partial u}$ contours from W. All bonds of $W \setminus V$ have $u=r=1$. V is a connected union of lattice squares because W is, and because the components of $W \setminus V$ are open sets whose boundaries $\left(\frac{\partial}{\partial u}\text{contours}\right)$ meet ∂W or each other at isolated points or not at all (see Fig. 1).

We next introduce Spencer lines. Let I be the set of integers i such that $\overline{\mathscr{L}}_i$ separates Δ_x from Δ_y and such that $\overline{\mathscr{L}}_i$ never meets a $\frac{\partial}{\partial u}$ bond. Define the Spencer line at $i \in I$ to be $\mathscr{L}_i = \overline{\mathscr{L}}_i \cap W$. $\mathscr{L}_i$ does not leave V because it may not cross $\frac{\partial}{\partial u}$ contours. Thus the bonds of $\mathscr{L}_i$ are either $r=0$ bonds or $\frac{\partial}{\partial r}$ bonds. Condition A is not violated by introducing these Spencer lines.

With $t_i = 0$, there is a complete contour of $r=0$ bonds or $t=0$ lines separating Δ_x from Δ_y. From Proposition 2.1 we conclude that $k(t_i=0) = \frac{\partial}{\partial t_i} k(t_i=0) = 0$. Therefore we may apply Taylor's formula with remainder to obtain

$$k(t=1) = \int \prod_{i \in I} \left[(1-t_i) \frac{\partial^2}{\partial t_i^2} \right] k(t) dt . \tag{3.6}$$

The integration extends over $t_i \in [0, 1]$ for $i \in I$. We evaluate u- and r-derivatives of k by differentiating the right hand side of (3.6).

The next proposition gives the bounds on derivatives of k that we use to control the expansion and obtain two-particle decay.

Proposition 3.2. *Let $k = \int k(z_1, z_2) f(z_1, z_2) dz$ be as in Theorem 3.1 and let $\varepsilon > 0$ be given. Consider terms of the above expansion. There exist positive constants a, c such that for λ sufficiently small*

$$\left| \partial_u^{\Gamma_u} \partial_r^{\Gamma_r} \prod_{i \in I} \frac{\partial^2}{\partial t_i^2} k \right| \leqq \|f\|_{L^\infty} e^{-a\lambda^{-1/2} l^2 |\Gamma_u| / L^2} e^{-cl|\Gamma_r|} e^{-2(1-\varepsilon) l |I|} . \tag{3.7}$$

Here $|\Gamma_u|$ *and* $|\Gamma_r|$ *are the number of bonds in* Γ_u *and* Γ_r, *respectively, and* $|I|$ *is the number of integers in* I.

The proof of Proposition 3.2 is deferred to section four. Note that the case $x_0 = y_0$ of Theorem 3.1 is a special case of this proposition. In the rest of this section we consider only $x_0 \neq y_0$.

We showed above that nonvanishing terms in (3.5) are determined by W and its configuration. These are in turn uniquely determined by V and its configuration because the bonds of $W \setminus V$ have all been resummed to $u = r = 1$. V does not necessarily contain Δ_x or Δ_y but it must at least surround them. Since V is connected, the number of V's with a given number of lattice squares $|V|$ is bounded by $e^{O(1)|V|}$. Each square has no more than 3^4 possible configurations $\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial r}, \text{or } r=0 \text{ on each of four bonds}\right)$. Thus the total number of terms with a given $|V|$ is bounded by $e^{O(1)|V|}$.

To every square of V we may associate at least half of a derivative bond $\left(\frac{\partial}{\partial u} \text{ or } \frac{\partial}{\partial r}\right)$. Otherwise W would be divided into parts by $r=0$ bonds, contradicting the construction of W. (It is impossible to have $|W|=1$ because $x_0 \neq y_0$.) Therefore,

$$|\Gamma_u| + |\Gamma_r| \geqq \tfrac{1}{2}|V|.$$

For every $i \in (x_0/l, y_0/l]$ (or $(y_0/l, x_0/l]$), either $i \in I$ or $\mathscr{L}_i$ meets a $\frac{\partial}{\partial u}$ bond of W. No more than two $\mathscr{L}_i$'s may intersect one $\frac{\partial}{\partial u}$ bond. Therefore $|I| + 2|\Gamma_u| \geqq |x_0 - y_0|/l$. We split the convergence associated with $\frac{\partial}{\partial u}$ bonds into two parts: one to control $|V|$, the other to assist in the two-particle decay. Taking λ small enough so that

$$\tfrac{1}{2}a\lambda^{-1/2}l^2/L^2 \geqq cl \quad \text{and} \quad \tfrac{1}{4}a\lambda^{-1/2}l^2/L^2 \geqq 2l$$

we may bound each term using (3.7):

$$\left|\partial_u^{\Gamma_u}\partial_r^{\Gamma_r}\prod_{i\in I}\frac{\partial^2}{\partial t_i^2}k\right| \leqq \|f\|_{L^\infty}e^{-cl|V|/2}e^{-2(1-\varepsilon)|x_0-y_0|}. \tag{3.8}$$

Taking the supremum over the region of urt-integration, we have for λ small enough $\sum_{|V|=2}^{\infty} e^{O(1)|V|}e^{-cl|V|/2} \leqq 1$ so that the sum of all terms is less than $\|f\|_{L^\infty}e^{-2(1-\varepsilon)|x_0-y_0|}$. This completes the proof of Theorem 3.1.

4. Analyticity Bounds

In this section we prove Proposition 3.2 using analyticity techniques [12, 20]. Derivatives of k are bounded by obtaining bounds on k uniform in large domains of complex parameter space and by applying Cauchy's formula. The chief advantage of these techniques is the avoidance of factorial growth in the number of derivatives. We shall establish large analyticity domains for u. For r- and t-derivatives, however, we must follow Spencer [20] by introducing dependence on additional parameters $h(\alpha)$ which possess large domains of analyticity. The r- and t-derivatives are then expressed in terms of h-derivatives.

Let $I^{(2)}$ be the disjoint union of two copies of I and let β be the union of $I^{(2)}$ with the set of bonds in Γ_r. For any $\alpha \subseteqq \beta$ let $\partial^{\alpha}_{rt} = \prod_{b\in\alpha} \frac{\partial}{\partial r_b} \prod_{i\in\alpha} \frac{\partial}{\partial t_i}$ and denote the set of partitions of β by $\mathscr{P}(\beta)$. The basic formula for r- and t-derivatives is [15]

$$\begin{aligned} &\partial^{\beta}_{rt} \int R\chi_\Sigma e^{-V} d\varphi(r,t) \\ &\quad = \sum_{\pi\in\mathscr{P}(\beta)} \int \prod_{\alpha\in\pi} [\tfrac{1}{2}\partial^{\alpha}_{rt} C \cdot \Delta_\varphi] R\chi_\Sigma e^{-V} d\varphi(r,t) . \end{aligned} \tag{4.1}$$

We localize the derivatives of covariances by expressing $C(z_1, z_2)$ as a sum

$$C(z_1,z_2) = \sum_{j\in\mathbb{Z}^4} C_j(z_1,z_2), \tag{4.2}$$

where $j=(j_1,j_2)$ and

$$C_j(z_1,z_2) = \chi_{j_1}(z_1) C(z_1,z_2) \chi_{j_2}(z_2). \tag{4.3}$$

Here χ_{j_1}, χ_{j_2} are the characteristic functions of the l-lattice squares with lower left corners at lj_1, lj_2, respectively.

We now express the right-hand side of (4.1) in terms of new parameters $h(\alpha)$:

$$\begin{aligned} &\partial^{\beta}_{rt} \int R\chi_\Sigma e^{-V} d\varphi(r,t) \\ &\quad = \sum_{\pi\in\mathscr{P}(\beta)} \left[\prod_{\alpha\in\pi} \frac{\partial}{\partial h(\alpha)}\right] \int \prod_{\alpha\in\pi} \prod_{j\in\mathbb{Z}^4} (1 + \tfrac{1}{2} h(\alpha) \partial^{\alpha}_{rt} C_j \cdot \Delta_\varphi) R\chi_\Sigma e^{-V} d\varphi(r,t)|_{h=0} . \end{aligned} \tag{4.4}$$

Let $\delta^{\beta}_h = \sum_{\pi\in\mathscr{P}(\beta)} \prod_{\alpha\in\pi} \frac{\partial}{\partial h(\alpha)}$. Define the h-dependent expectation by inserting the $1 + h\Delta$ factors of (4.4) into (2.6)

$$\langle R \rangle = \frac{\sum_\Sigma \prod_{b:\mathrm{dist}(b,\Sigma)\leqq L} u_b \int \prod_{\alpha\subseteqq\beta} \prod_{j\in\mathbb{Z}^4} (1 + \tfrac{1}{2} h(\alpha) \partial^{\alpha}_{rt} C_j \cdot \Delta_\varphi) R\chi_\Sigma e^{-V} d\varphi(r,t)}{\sum_\Sigma \prod_{b:\mathrm{dist}(b,\Sigma)\leqq L} u_b \int \prod_{\alpha\subseteqq\beta} \prod_{j\in\mathbb{Z}^4} (1 + \tfrac{1}{2} h(\alpha) \partial^{\alpha}_{rt} C_j \cdot \Delta_\varphi) \chi_\Sigma e^{-V} d\varphi(r,t)} . \tag{4.5}$$

We use these new expectations to define $k(h)$ from the formula (2.13). The covariance $C(r,t)$ is given the following h-dependence:

$$C(r,t,h,x,y) = \int \prod_{\alpha\subseteqq\beta} \prod_{j\in\mathbb{Z}^4} (1 + \tfrac{1}{2} h(\alpha) \partial^{\alpha}_{rt} C_j \cdot \Delta_\varphi) \varphi(x)\varphi(y) d\varphi(r,t) . \tag{4.6}$$

Convergence of the Neumann series in appropriate domains of the parameters will be guaranteed by Theorem 4.1. We remark that h-dependence is introduced only after integrations by parts and other constructions have been applied to express kernels and their $\lambda^{1/2}$-derivatives in terms of sums of products of expectations.

Let F_R denote the numerator of (4.5). From (4.4) we have $\partial_{rt}^{\beta} F_R|_{h=0} = \delta_h^{\beta} F_R|_{h=0}$ and similarly for $C(h)$. The operator δ_h^{β} behaves just like ∂_{rt}^{β} with respect to products and quotients. This implies that

$$\begin{aligned} \partial_{rt}^{\beta}\langle R\rangle|_{h=0} &= \partial_{rt}^{\beta}(F_R/F_{\emptyset})|_{h=0} = \delta_h^{\beta}\langle R\rangle|_{h=0} \\ \partial_{rt}^{\beta} k|_{h=0} &= \delta_h^{\beta} k|_{h=0}\,. \end{aligned} \tag{4.7}$$

See Lemmas 3.1 and 3.2 of [20] for proofs of these facts.

We need to consider R's with derivatives $\frac{\delta}{\delta\varphi}$ so that we may be able to estimate expectations such as B in (2.11). We must first isolate the δ-function contributions to expectations with derivatives. For example, when two derivatives act on a function of $\int :\varphi(x)^n: dx$, there are two terms:

$$\begin{aligned} &\frac{\delta}{\delta\varphi(x)}\,\frac{\delta}{\delta\varphi(y)}\,\Omega\left(\int :\varphi(x)^n: dx\right) \\ &\quad = n(n-1) :\varphi(x)^{n-2}: \Omega'\left(\int :\varphi(x)^n: dx\right)\delta(x-y) \\ &\qquad + n^2 :\varphi(x)^{n-1}: :\varphi(y)^{n-1}: \Omega''\left(\int :\varphi(x)^n: dx\right). \end{aligned} \tag{4.8}$$

We denote the first term by $\partial_{\varphi}^2(x)\Omega\delta(x-y)$, and the second by $\partial_{\varphi}(x)\partial_{\varphi}(y)\Omega$. Similarly $\partial_{\varphi}^k(x)$ is defined as the term that results when $k-1$ derivatives all act on the $:\varphi(x)^{n-1}:$ factor in $\frac{\delta}{\delta\varphi(x)}\Omega$. In general, when several operators $\partial_{\varphi}^{k_i}(x_i)$ act on Ω we use the formula

$$\left[\prod_i \partial_{\varphi}^{k_i}(x_i)\right]\Omega = \left[\prod_i \left(\frac{n!}{(n-k_i)!} :\varphi(x_i)^{n-k_i}:\right)\right]\Omega^{(\sum_i k_i)}. \tag{4.9}$$

Note that the $\partial_{\varphi}^{k_i}$'s do not act on each other. If any $k_i > n$, we get zero. The generalization of (4.9) to functions of arbitrary $:\varphi(f)^n:$ is straightforward. For products, we have a Leibniz rule:

$$\left[\prod_{i\in\mathcal{I}} \partial_{\varphi}^{k_i}(x_i)\right]\Omega_1\Omega_2 = \sum_{\mathcal{J}\subseteqq\mathcal{I}} \left(\left[\prod_{i\in\mathcal{J}} \partial_{\varphi}^{k_i}(x_i)\right]\Omega_1\right)\left(\left[\prod_{i\in\mathcal{I}\setminus\mathcal{J}} \partial_{\varphi}^{k_i}(x_i)\right]\Omega_2\right). \tag{4.10}$$

Ordinary derivatives may be expressed in terms of the $\partial_{\varphi}^k(x)$'s using the formula

$$\left[\prod_{i\in\mathcal{I}} \frac{\delta}{\delta\varphi(x_i)}\right]\Omega = \sum_{p\in\mathcal{P}(\mathcal{I})} \left[\prod_{\mathcal{J}\in p}\left(\partial_{\varphi}^{|\mathcal{J}|}(x_{i_{\mathcal{J}}}) \prod_{i\in\mathcal{J},\, i\neq i_{\mathcal{J}}} \delta(x_i - x_{i_{\mathcal{J}}})\right)\right]\Omega. \tag{4.11}$$

Here $i_{\mathcal{J}}$ is the smallest integer in $\mathcal{J}\subseteqq\mathcal{I}$.

The R's we consider in the next theorem are of the form

$$R = \int w(x) \prod_{i=1}^{n_1} :\varphi(x_i)^{p_i}: \prod_{i=n_1+1}^{n_2} \partial_{\varphi}^{k_i}(x_i) \prod_{i=n_2+1}^{n} \partial_{\chi}(x_i)\, dx, \tag{4.12}$$

where $w(x_1, \ldots, x_n) \in L^p\left(\prod_{i=1}^{n} \Delta_i\right)$ is supported in a product of l-lattice squares, and

$p > 1$ is fixed. $\partial_\chi(x)$ is the same as $\dfrac{\delta}{\delta\varphi(x)}$ except that it operates only on χ-factors or their derivatives. For the truncated expectation $\langle R_1 ; R_2 \rangle$ we let w be a function of both sets of variables and let the integration be implicit in the notation. Infinite sums of products of expectations of R's of this form are sufficient to construct the irreducible kernels and their $\lambda^{1/2}$-derivatives. When we write $\langle R_1 ; R_2 \rangle$ it is understood that the derivatives of R_1 do not act on the monomials of R_2. We define $\deg R = \sum_{i=1}^{n_1} p_i + 4(n - n_1)$ and assume all $k_i \leqq 4$. Let $D(R_1, R_2)$ be the distance between the supports of R_1 and R_2. We define $\delta(R) = n_2 - n_1$ to be the number of ∂_φ^k's in R and $\delta_\chi(R)$ to be the number of ∂_χ's.

We make a number of definitions which will enable us to describe domains of analyticity in the next theorem. Let $\Gamma_\alpha \subseteqq \Gamma_r$ be the union of the bonds in α and let $I_\alpha = I^{(2)} \cap \alpha$ be the integers in α. Define $d(\alpha) = \sup_{i, j \in I_\alpha} |i - j|$ if $I_\alpha \neq \emptyset$ has no duplications; $d(\alpha) = -1$ if $I_\alpha = \emptyset$; otherwise $d(\alpha) = \infty$. Define $\delta(\alpha) = \sup_{b, i \in \alpha} \operatorname{dist}(b, \overline{\mathscr{L}}_i)/l$; if α contains only bonds or only integers then set $\delta(\alpha) = 0$. As in [15], let $L(\Gamma_\alpha)$ denote the set of linear orderings of the bonds in Γ_α and define $\mathscr{L}(\Gamma_\alpha) = \bigcup_{\Gamma' \subseteqq \Gamma_\alpha} L(\Gamma')$. For each $o \in \mathscr{L}(\Gamma_\alpha)$ we define a length $|o|$ in units of l that arises in estimating derivatives of covariances. If $o = (b_1, \ldots, b_n)$, define $b_1' = b_1$ and b_j' inductively as the first bond after b_{j-1}' not touching b_{j-1}'. Then set $|o| = \sum_{j \geqq 2} \operatorname{dist}(b_j', b_{j-1}')/l$. If b_2 does not exist we set $|o| = 0$. By convention, $\{\emptyset\} \in L(\emptyset)$.

Theorem 4.1. *Let $\varepsilon > 0$ and $p > 1$ be given, and let λ be sufficiently small. Consider bond configurations occurring in the final form of the expansion of section three. There exist positive constants a, c, g, d, and K such that $\langle R_1 ; R_2 \rangle$ is analytic in u and h and*

$$|\langle R_1 ; R_2 \rangle| \leqq \|w\|_{L^p} M(\deg R_1 R_2) \lambda^{\delta(R_1 R_2)/2} e^{Kl \deg R_1 R_2} e^{-gD(R_1, R_2)} \tag{4.13}$$

for $u, h \in \mathscr{D}(\pi)$. Here π is any element of $\mathscr{P}(\beta)$ and $\mathscr{D}(\pi)$ is the complex domain defined by

$$\begin{aligned}
&|u_b| \leqq e^{a\lambda^{-1/2} l^2 / L^2}, \; b \text{ a } \frac{\partial}{\partial u} \text{ bond} \\
&u_b = 0, \; b \text{ a } u = 0 \text{ bond} \\
&u_b = 1, \; b \text{ a } u = 1 \text{ bond} \\
&|h(\alpha)| \leqq \Big(\sum_{o \in L(\Gamma_\alpha)} e^{-cl|o|} \Big)^{-1} e^{cl\delta(\alpha)} e^{2cl|\Gamma_\alpha|} e^{(1 - \varepsilon/2) l (d(\alpha) + 1)}, \quad \alpha \in \pi \\
&h(\alpha) = 0, \; \alpha \notin \pi .
\end{aligned} \tag{4.14}$$

If $\delta_\chi(R_1 R_2) > 0$, a factor $e^{-d\lambda^{-1/2}}$ can be included on the right-hand side of (4.13). *The bound* (4.13) *holds uniformly in* $\deg R_1 R_2$ *if M is also allowed to depend on λ. An analogous bound holds without clustering for untruncated expectations.*

This theorem will be proven in Sects. 5 and 6 with a low temperature cluster expansion. We remark that bounds uniform in $\deg R_1 R_2$ are needed to prove for fixed λ that one-particle irreducible expectations of arbitrarily high degree have two-particle decay, as required by [5]. We now prove a lemma for Proposition 3.2.

Lemma 4.2. *Under the hypotheses of Theorem 4.1,* $k=\int k(z_1,z_2)f(z_1,z_2)dz$ *is analytic and uniformly bounded by* $\|f\|_{L^\infty}$ *for* u, h *in* $\mathscr{D}(\pi)$, *for any* $f\in C^0(\Delta_x\times\Delta_y)$.

Proof. We express the B-operator of (2.11) in terms of expectations of ∂^k_φ's:

$$B(x,y)=\left\langle\frac{\delta}{\delta\varphi(x)};\frac{\delta}{\delta\varphi(y)}\right\rangle=\langle\partial^2_\varphi(x)\rangle\delta(x-y)+\langle\partial_\varphi(x);\partial_\varphi(y)\rangle\,. \tag{4.15}$$

Theorem 4.1 guarantees that $\langle\partial^2_\varphi(x)\rangle$ and $\langle\partial_\varphi(x);\partial_\varphi(y)\rangle$ are locally in L^q, $q<\infty$. Also, $\langle\partial_\varphi(x);\partial_\varphi(y)\rangle$ decays exponentially in $|x-y|$. Moreover, both expectations involve at least one derivative so there is an explicit factor of $\lambda^{1/2}$ in (4.13) which more than dominates the diverging factor $e^{Kl\deg R_1R_2}$. Also, for any function $g\in L^q(\Delta)$, we have $C(h)g\in C^0\cap L^\infty(\mathbb{R}^2)$. Together these facts imply that $(\mathbb{1}+BC)^{-1}$ is a convergent Neumann series for λ small enough. For details see [17]. Using again the factor $\lambda^{1/2}$, we have

$$\begin{aligned}\int k(z_1,z_2)f(z_1,z_2)dz&=\int-((\mathbb{1}+BC)^{-1}B)(z_1,z_2)f(z_1,z_2)dz\\&\leqq\|f\|_{L^\infty}\end{aligned} \tag{4.16}$$

for λ small.

Proof of Proposition 3.2. We apply Cauchy's integral formula to evaluate derivatives of k. From (4.7) we have

$$\begin{aligned}\partial_u^{\Gamma_u}\partial_r^{\Gamma_r}\prod_{i\in I}\frac{\partial^2}{\partial t_i^2}k&=\partial_u^{\Gamma_u}\delta_h^{\beta}k|_{h=0}\\&=\sum_{\pi\in\mathscr{P}(\beta)}\partial_u^{\Gamma_u}\prod_{\alpha\in\pi}\frac{\partial}{\partial h(\alpha)}k|_{h=0}\\&=\sum_{\pi\in\mathscr{P}(\beta)}\oint k\prod_{b\in\Gamma_u}\left[\frac{(u_b'-u_b)^{-2}}{2\pi i}du_b'\right]\prod_{\alpha\in\pi}\left[\frac{h(\alpha)^{-2}}{2\pi i}dh(\alpha)\right],\end{aligned} \tag{4.17}$$

where $\beta=\Gamma_r\cup I^{(2)}$. The parameters u_b, r_b, t_i are all in $[0,1]$. The contours of integration are the largest ones allowed by (4.14):

$$\begin{aligned}|u_b'|&=e^{a\lambda^{-1/2}l^2/L^2}\\|h(\alpha)|&=\left(\sum_{o\in L(\Gamma_\alpha)}e^{-cl|o|}\right)^{-1}e^{cl\delta(\alpha)}e^{2cl|\Gamma_\alpha|}e^{(1-\varepsilon/2)l(d(\alpha)+1)}.\end{aligned} \tag{4.18}$$

This yields the bound

$$\begin{aligned}\left|\partial_u^{\Gamma_u}\partial_r^{\Gamma_r}\prod_{i\in I}\frac{\partial^2}{\partial t_i^2}k\right|&\leqq\|f\|_{L^\infty}e^{-a\lambda^{-1/2}l^2|\Gamma_u|/L^2}e^{-2(1-\varepsilon)l|I|}e^{-2cl|\Gamma_r|}\\&\quad\cdot e^{-\varepsilon l|I|/2}\sum_{\pi\in\mathscr{P}(\beta)}\prod_{\alpha\in\pi}\sum_{o\in L(\Gamma_\alpha)}e^{-cl|o|}e^{-cl\delta(\alpha)}e^{-\varepsilon l(d(\alpha)+1)/4}.\end{aligned} \tag{4.19}$$

Here we have used the fact that $\sum_{\alpha\in\pi}|\Gamma_\alpha|=|\Gamma_r|$ and $\sum_{\alpha\in\pi}(d(\alpha)+1)\geqq2|I|$. Hence (3.7) will be proven if we can show that

$$e^{-cl|\Gamma_r|}e^{-\varepsilon l|I|/2}\sum_{\pi\in\mathscr{P}(\beta)}\prod_{\alpha\in\pi}\sum_{o\in L(\Gamma_\alpha)}e^{-cl|o|}e^{-cl\delta(\alpha)}e^{-\varepsilon l(d(\alpha)+1)/4}\leqq1\,. \tag{4.20}$$

We use some counting arguments from [15].

We start by expanding the left hand side of (4.20) into a sum.

$$e^{-cl|\Gamma_r|}e^{-\varepsilon l|I|/2}\sum_{\{(o_i,\alpha_i)\}}\prod_i\left[e^{-cl|o_i|}e^{-cl\delta(\alpha_i)}e^{-\varepsilon l(d(\alpha_i)+1)/4}\right]. \tag{4.21}$$

Here $\{(o_i, \alpha_i)\}$ is a set of ordered pairs (o_i, α_i) with $o_i \in \mathscr{L}(\Gamma_r)$, $o_i = \Gamma_{\alpha_i}$, and $\{\alpha_i\} \in \mathscr{P}(\beta)$. We allow $o_i = \emptyset$ and we occasionally ignore the distinction between o_i and the set of bonds ordered by o_i. The sum in (4.21) is bounded as follows:

$$\begin{aligned}
&\sum_{\{(o_i,\alpha_i)\}}\prod_i\left[e^{-cl|o_i|}e^{-cl\delta(\alpha_i)}e^{-\varepsilon l(d(\alpha)+1)/4}\right]\\
&\quad\leqq \prod_{(o,\alpha)}\left(1+e^{-cl|o|}e^{-cl\delta(\alpha)}e^{-\varepsilon l(d(\alpha)+1)/4}\right)\\
&\quad\leqq \exp\left(\sum_{(o,\alpha)} e^{-cl|o|}e^{-cl\delta(\alpha)}e^{-\varepsilon l(d(\alpha)+1)/4}\right)\\
&\quad= \exp\left(\sum_{\substack{(o,\alpha)\\ o\neq\emptyset}} e^{-cl|o|}e^{-cl\delta(\alpha)}e^{-\varepsilon l(d(\alpha)+1)/4}\right)\exp\left(\sum_{\alpha\subseteqq I^{(2)}} e^{-\varepsilon l(d(\alpha)+1)/4}\right).
\end{aligned} \tag{4.22}$$

In the second exponential, there are $|I|$ choices of $\min\alpha$, the smallest integer in α. Fixing $\min\alpha$, there are at most 2^{2C} choices of α with $d(\alpha)+1=C$. This combinatoric factor is dominated by $e^{-\varepsilon lC/4}$, so the second exponential is bounded by $e^{O(1)|I|}$. In the first exponential, fix o. The sum over $\min\alpha$ and the sum over α are controlled by the factors $e^{-cl\delta(\alpha)}$ and $e^{-\varepsilon l(d(\alpha)+1)/4}$, respectively. The number of $o_i \in \mathscr{L}(\Gamma_r)$ with $|o_i| \leqq m$ is bounded by $|\Gamma_r|e^{O(1)(m+1)}$ [15]. Thus the second exponential is bounded by $e^{O(1)|\Gamma_r|}$. Putting these bounds into (4.21), we obtain (4.20). This completes the proof.

5. The Cluster Expansion

Theorem 4.1 is proven in this section with the use of the low temperature cluster expansion of [16]. We introduce a new way of organizing the expansion that is related to some ideas of Pirogov and Sinai [18]. Since we consider $\mu \neq 0$, we do not have a symmetry $\phi \to -\phi$. However, an inequality on partition functions that was proven with correlation inequalities in [16] is available for use. The ability to make use of this inequality is the main advantage of the method of removing phase boundaries given in section three. It allows us to handle a nonzero external field. The idea of the resummation technique is to multiply every factor in the cluster expansion by an appropriate ratio of partition functions. The ratio is bounded by 1 by the inequality on partition functions.

The resummation transforms the cluster expansion into a form in which the techniques of Bałaban and Gawędzki [1] are applicable. Their ideas originate in the work of Kunz and Souillard and are related to the formalism of [11]. The notion of connectedness that we need is more complex than that of [1]. Nevertheless, explicit division by the partition function is possible and the Kirkwood-Salzburg equations of [1] may be used to prove convergence of the expansion.

We start by generating the basic expansion and establishing bounds needed for convergence. Some technical estimates are deferred to Sect. 6. In the second half of this section we define the resummation and prove convergence of the expansion.

The first step is the expansion in phase boundaries. In accordance with (4.5), we have

$$\begin{aligned}\langle R\rangle &= F_R/F_\emptyset\\ F_R &= \sum_{\Sigma} F_{R,\Sigma},\end{aligned} \tag{5.1}$$

where

$$F_{R,\Sigma}=\prod_{b:\operatorname{dist}(b,\Sigma)\leqq L} u_b \int \prod_{\alpha\subsetneqq\beta}\prod_{j\in\mathbb{Z}^4}\left(1+\tfrac{1}{2}h(\alpha)\partial_{rt}^{\alpha}C_j\cdot\Delta_\varphi\right)R\chi_\Sigma e^{-V}d\varphi(r,t). \tag{5.2}$$

We translate φ as in [16] by a function on $\mathbb{R}^2$ that depends on Σ. The new field $\psi(x)$ has a mean that behaves roughly like $\sigma_\iota\xi_+$ and that is exactly $\sigma_\iota\xi_+$ farther than $L/2$ from Σ. Choose a C^∞ bump function $\zeta(x)$ on $\mathbb{R}^2$ satisfying

$$\begin{aligned}&0\leqq\zeta(x)\leqq 1\\ &\zeta(x)=0 \quad \text{for} \quad |x|>\tfrac{1}{2}\\ &\zeta(x)=1 \quad \text{for} \quad |x|\leqq\tfrac{1}{4}.\end{aligned} \tag{5.3}$$

Let η be a small constant (independent of λ), and define the new field by

$$\psi(x)=\varphi(x)+\xi_+-\frac{\int(-\Delta+\eta)^{-1}(x-y)\zeta((x-y)/L)\sigma(y)\xi_+dy}{\int(-\Delta+\eta)^{-1}(y)\zeta(y/L)dy}. \tag{5.4}$$

The meaning of ψ depends on Σ, though the dependence is not explicit in the notation. Let $d\psi(r,t)$ denote the Gaussian measure in which $\psi(x)$ has mean zero and covariance equal to that of $d\varphi(r,t)$. We define $Q(\Sigma)$, the translated interaction for the spin configuration Σ by the equation

$$e^{-V}d\varphi(r,t)=e^{-Q(\Sigma)}d\psi(r,t). \tag{5.5}$$

See [16] for an explicit formula for $Q(\Sigma)$. Note that $\psi(x)-\varphi(x)=0$ wherever $r\neq 1$ or $t\neq 1$, by Condition A.

In each $F_{R,\Sigma}$ we introduce Dirichlet data into $d\psi(r,t)$ on those bonds of the $l\mathbb{Z}^2$ lattice that are farther than L from Σ. Denote the set of such bonds by $\mathscr{B}(\Sigma)$. The covariance in the $\partial_{rt}^{\alpha}C_j$-factors and in the measure now depends on a new set of parameters $\{s_b\}_{b\in\mathscr{B}(\Sigma)}$ which interpolate between zero Dirichlet data ($s_b=0$) and absence of Dirichlet data ($s_b=1$). We perform a cluster expansion in these new parameters:

$$F_{R,\Sigma}(s=1)=\sum_{\Gamma_s\subseteqq\mathscr{B}(\Sigma)}\int ds\,\partial_s^{\Gamma_s}F_{R,\Sigma}(s). \tag{5.6}$$

Here Γ_s denotes a subset of $\mathscr{B}(\Sigma)$ and $\partial_s^{\Gamma_s}=\prod_{b\in\Gamma_s}\dfrac{\partial}{\partial s_b}$. The integral runs over the range $[0,1]$ for each s_b, $b\in\Gamma_s$. For $b\in\mathscr{B}(\Sigma)\backslash\Gamma_s$ we set $s_b=0$ in (5.6).

Let Z_κ denote the closures of the connected components of $\mathbb{R}^2\backslash\{s=0 \text{ bonds}\}$. For simplicity we defer the integration against the test function w of (4.12) and take

R to be a product of: $\varphi(x_i)^{p_i}$:'s, $\partial_\varphi^{k_i}(x_i)$'s, and $\partial_\chi(x_i)$'s. We claim that $F_{R,\Sigma}$ may be written as a product of F's associated with each Z_κ:

$$F_{R,\Sigma}(s) = \prod_\kappa F_{R,\Sigma,Z_\kappa}(s)$$

$$F_{R,\Sigma,Z_\kappa}(s) = \prod_{\substack{b \subseteq Z_\kappa \\ \mathrm{dist}(b,\Sigma\cap Z_\kappa) \leq L}} u_b \tag{5.7}$$

$$\cdot \int \prod_{\alpha \subseteq \beta} \prod_{\substack{j \in \mathbb{Z}^4 \\ \Delta_{j_1} \cup \Delta_{j_2} \subseteq Z_\kappa}} (1 + \tfrac{1}{2}h(\alpha)\partial_{rt}^\alpha C_j \cdot \Delta_\varphi) R_{Z_\kappa} \chi_{\Sigma\cap Z_\kappa} e^{-Q(\Sigma\cap Z_\kappa, Z_\kappa)} d\psi(r,t,s).$$

In this formula, it is understood that derivatives ∂_φ^k and $\frac{\delta}{\delta\varphi}$ in R_{Z_κ} and Δ_φ have been applied in the integrand as it was before the translation to the ψ-field. When this convention is not in effect, we will use the letter ψ rather than φ to denote derivatives, e.g., Δ_ψ instead of Δ_φ. The bonds of Z_κ are by construction farther than L from any $\Sigma\cap Z_{\kappa'}$. Therefore the u-factors may be distributed as in (5.7). The factor $R\chi e^{-Q}$ may be written as a product, as in [16]. Moreover,

$$\partial_{rt}^\alpha C_j(r,t,s) = 0 \tag{5.8}$$

unless both Δ_{j_1} and Δ_{j_2} are in the same Z_κ. Thus (5.7) is valid. In fact, $\prod_{\alpha \subseteq \beta}$ runs only over α such that for all b, $i \in \alpha$, $b \cap \mathrm{int}\, Z_\kappa \neq \emptyset \neq \mathscr{L}_i \cap \mathrm{int}\, Z_\kappa$. For the other α's we have $\partial_{rt}^\alpha C_j = 0$.

The quantity F_{R,Σ,Z_κ} depends only on $\Sigma \cap Z_\kappa$ and s_b, $b \in Z_\kappa$. We may write (5.1) and (5.7) as

$$F_R = \sum_\Sigma \sum_{\Gamma_s \subseteq \mathscr{B}(\Sigma)} \prod_\kappa \int ds\, \partial_s^{\Gamma_s \cap Z_\kappa} F_{R,\Sigma,Z_\kappa}(s). \tag{5.9}$$

Each boundary segment of Z_κ is either in a sea of $+$ spins or in a sea of $-$ spins. Let ∂Z_κ^+ be the $+$ boundary of Z_κ and ∂Z^- be the $-$ boundary. Denote the triple $(Z_\kappa, \partial Z_\kappa^+, \partial Z_\kappa^-)$ by $\mathbb{Z}_\kappa$. We call such a triple a cluster. We reorder the summations in (5.9) by summing first over all terms compatible with a given $\{\mathbb{Z}_\kappa\}$ and then summing over all possible $\{\mathbb{Z}_\kappa\}$'s. With

$$\varrho(\mathbb{Z}_\kappa) = \sum_{\substack{\Sigma\cap Z_\kappa, \Gamma_s \cap Z_\kappa \\ \text{compatible with } \mathbb{Z}_\kappa \text{ connected}}} \int ds\, \partial_s^{\Gamma_s \cap Z_\kappa} F_{R,\Sigma\cap Z_\kappa, Z_\kappa}(s) \tag{5.10}$$

we have

$$F_R = \sum_{\{\mathbb{Z}_\kappa\} \text{ admissible}} \prod_\kappa \varrho(\mathbb{Z}_\kappa). \tag{5.11}$$

$\{\mathbb{Z}_\kappa\}$ is admissible if $\bigcup_\kappa Z_\kappa = \mathbb{R}^2$ and the $\mathbb{Z}_\kappa$'s agree on boundary signs. This step is possible because $\sum_{\Sigma,\Gamma_s}$ in (5.6) factors into independent sums for each Z_κ.

We also define partition function type objects $\Omega(\mathbb{V})$. In contrast to $\mathbb{Z}$, $\mathbb{V}$ need not be connected.

$$\Omega(\mathbb{V}) = \sum_{\{\mathbb{Z}_\kappa\} \text{ admissible in } \mathbb{V}} \prod_\kappa \varrho^0(\mathbb{Z}_\kappa). \tag{5.12}$$

Here $\varrho^0(\mathbb{Z}_\kappa)$ is the same as $\varrho(\mathbb{Z}_\kappa)$ except $R=\emptyset$ for every term in (5.10). $\{\mathbb{Z}_\kappa\}$ is admissible in $\mathbb{V}$ if $\bigcup_\kappa Z_\kappa = V$ and the $\mathbb{Z}_\kappa$'s agree with each other and with $\mathbb{V}$ on boundary signs.

We now formulate bounds on $\varrho(\mathbb{Z}_\kappa)$ and $\Omega(\mathbb{V})$ which yield convergence of the expansion. Denote by $|Z|$ the number of l-lattice squares of Z.

Lemma 5.1. *Under the conditions of Theorem 4.1, there exists a constant $c_1>0$ such that for λ sufficiently small and $\partial Z^- = \emptyset$,*

$$\|\varrho(\mathbb{Z})\|_{L^q} \leqq M_1(\deg R_Z)\lambda^{\delta(R_Z)/2} e^{Kl\deg R_Z} e^{-c_1 l(|Z|-1)}. \tag{5.13}$$

If $\partial Z^- \neq \emptyset$ the factor $\lambda^{\delta(R_Z)/2}$ is replaced by $\lambda^{-\deg R_Z/2}$. If $\partial Z^- \neq \emptyset \neq \partial Z^+$ or if $\delta_\chi(R_Z)>0$ we include factors of $e^{-d\lambda^{-1/2}}$. This bound is independent of $\deg R_Z$ *if $\lambda^{\delta(R_Z)/2}$ is replaced by $\lambda^{-\deg R_Z/2}$. Here q, p are dual Hölder exponents and the L^q norm is with respect to the product of the l-lattice squares in which the uncontracted variables of R_Z lie. If $R_Z=\emptyset$ we replace the L^q norm with absolute value signs.*

With $R=\emptyset$ and Δ any l-lattice square, we have

$$|\varrho(\Delta, \partial\Delta, \emptyset)|^{-1} \leqq e^{c_2\lambda^{1/2} l^2}. \tag{5.14}$$

for some constant c_2. Finally, if $u=r=t=1$ and $h=0$ in V, we have

$$\Omega(\mathbb{V}) \leqq \Omega(\mathbb{V}^+), \tag{5.15}$$

where $\mathbb{V}^+ = (V, \partial V, \emptyset)$.

Proof. For $\varrho(\Delta, \partial\Delta, \emptyset)$ there is only one term in (5.10) that contributes:

$$\varrho(\Delta, \partial\Delta, \emptyset) = F_{R,\Sigma\equiv +,\Delta}. \tag{5.16}$$

For $\Delta_{j_1} = \Delta_{j_2} = \Delta$ we have $\partial^\alpha_{rt} C_j = 0$ so that the $(1+h\partial C\Delta)$ factors are absent in (5.7). The product of u's in (5.7) is empty. With $R=\emptyset$ we are left with

$$\varrho(\Delta, \partial\Delta, \emptyset) = \int \chi_{\Sigma\equiv +} e^{-Q(\Sigma\equiv +,\Delta)} d\psi(s=0 \text{ on } \partial\Delta). \tag{5.17}$$

The right-hand side is bounded below by $e^{-c_2\lambda^{1/2}l^2}$ in [16, Lemma 4.2.2]. This proves (5.14).

With $u=r=t=1$ and $h=0$ in V, $\Omega(\mathbb{V})$ and $\Omega(\mathbb{V}^+)$ are partition functions of the type considered in [16]. The inequality (5.15) is proven using correlation inequalities in [16, Lemma 4.2.3].

We now establish (5.13) with the use of the following lemma, proven in Sect. 6.

Lemma 5.2. *Under the conditions of Theorem 4.1, if $\Sigma \equiv +$ then*

$$\|\partial_s^{\Gamma_s} F_{R,\Sigma,Z}\|_{L^q} \leqq M_1(\deg R_Z)\lambda^{\delta(R_Z)/2} e^{Kl\deg R_Z} e^{-cl|\Gamma_s|} \cdot e^{-d_1\lambda^{-1/2}|\Sigma|} e^{O(1)|Z|} \tag{5.18}$$

for some $d_1>0$. $|\Sigma|$ denotes the length of the phase boundary Σ. For a $\deg R_Z$ *independent bound, or if Σ is not identically $+$, we replace the factor $\lambda^{\delta(R_Z)/2}$ with $\lambda^{-\deg R_Z/2}$. If $\delta_\chi(R_Z)>0$, we include a factor $e^{-d_1\lambda^{-1/2}}$. Here $\Gamma_s \subseteqq \operatorname{int} Z$, $\Sigma = \Sigma \cap Z$, $s=0$ on ∂Z, and $\|\cdot\|_{L^q}$ is as in Lemma 5.1.*

If $|Z|=1$, (5.13) follows immediately. Consider the case $|Z|>1$. The factor $\lambda^{\delta(R_Z)/2}$ in (5.13) may be obtained from $e^{-d_1\lambda^{-1/2}|\Sigma|}$ in (5.18) with a decrease in d_1 if Σ is not identically $+$ and $\partial Z^- = \emptyset$. We may extract a factor $e^{-d\lambda^{-1/2}}$ with a decrease in d_1 if $\partial Z^- \neq \emptyset \neq \partial Z^+$ because $|\Sigma|>1$ for all terms of the sum (5.10).

The number of Σ's with a given $|\Sigma|$ is bounded by $\binom{2l^2|Z|}{|\Sigma|}$. Therefore

$$\sum_{\substack{\Sigma \text{ compatible}\\ \text{with } \mathbb{Z}}} e^{-d_2\lambda^{-1/2}|\Sigma|} \leqq (1+e^{-d_2\lambda^{-1/2}})^{2l^2|Z|} \leqq e^{O(1)|Z|}. \tag{5.19}$$

The number of Γ_s's in Z is bounded by $e^{O(1)|Z|}$. $Z \backslash \Gamma_s$ cannot have more than one component. Thus in order for Σ, Γ_s to be compatible with $\mathbb{Z}$, there must be a certain density of either phase boundaries or bonds of Γ_s. This is expressed in the inequality

$$4L^2|\Sigma|/l^2 + 2|\Gamma_s| \geqq |Z|. \tag{5.20}$$

Therefore,

$$e^{-cl|\Gamma_s|}e^{-d_3\lambda^{-1/2}|\Sigma|} \leqq e^{-cl|Z|/2} \tag{5.21}$$

for λ sufficiently small. The factor $e^{-cl|Z|/2}$ dominates the $e^{O(1)|Z|}$ factors and establishes (5.13). This completes the proof.

Proof of Theorem 4.1. We rewrite the expansions (5.11) and (5.12) in terms of new objects $\varrho(\mathbb{Z})\Omega(\mathbb{V})/\Omega(\mathbb{V}^+)$. Consider first (5.12) with $\partial V^- = \emptyset$. For a term in $\sum_{\{\mathbb{Z}_\kappa\}}$, let $\{\mathbb{Z}_i\}$ be the clusters that are not surrounded by minus loops of any cluster. For each $\mathbb{Z}_i$, let $\ell_{i,\alpha}$ be its minus loops. Note that the $\mathbb{Z}_i$'s are positive, i.e., they have plus external loops. Therefore $\operatorname{int}\ell_{i,\alpha} \cap Z_i = \emptyset$. We resum the expansion (5.12) inside all $\ell_{i,\alpha}$'s. As in Sect. 3, this resummation is without constraint. Let $V_i = V \cap \bigcup_\alpha \overline{\operatorname{int}\ell_{i,\alpha}}$ and let $\mathbb{V}_i = (V_i, \partial V_i \cap \partial V^+, \bigcup_\alpha \ell_{i,\alpha})$. The resummation inside the $\ell_{i,\alpha}$'s yields $\prod_i \Omega(\mathbb{V}_i)$. Thus (5.12) becomes

$$\Omega(\mathbb{V}) = \sum_{\substack{\{\mathbb{Z}_i\} \text{ positive}\\ \text{restricted}}} \prod_i [\varrho^0(\mathbb{Z}_i)\Omega(\mathbb{V}_i)]. \tag{5.22}$$

Restricted means that no $\mathbb{Z}_i$ is surrounded by a minus loop of $\{\mathbb{Z}_i\}$.

We claim that the following formula holds for any $\mathbb{V}$ with $\partial V^- = \emptyset$:

$$\Omega(\mathbb{V}) = \sum_{\substack{\{\mathbb{Z}_\kappa\} \text{ positive}\\ \text{unrestricted in } \mathbb{V}}} \prod_\kappa \left[\varrho^0(\mathbb{Z}_\kappa)\frac{\Omega(\mathbb{V}_\kappa)}{\Omega(\mathbb{V}_\kappa^+)}\right]. \tag{5.23}$$

Here $\sum_{\{\mathbb{Z}_\kappa\}}$ is over $\mathbb{Z}_\kappa$'s that may be surrounded by minus loops. The $\mathbb{Z}_\kappa$'s may disagree with each other on boundary signs but they may not disagree with $\mathbb{V}$. They may not overlap and their union must equal V. We prove (5.23) by induction on $|V|$. If $|V|=1$, then $\Omega(\mathbb{V}) = \varrho^0(\mathbb{V})$ so that (5.23) holds. Suppose (5.23) holds for

$|V|<N$. Rewrite (5.22) as

$$\Omega(\mathbb{V})=\sum_{\substack{\{\mathbb{Z}_i\}\text{ positive}\\ \text{restricted}}}\prod_i\left[\varrho^0(\mathbb{Z}_i)\frac{\Omega(\mathbb{V}_i)}{\Omega(\mathbb{V}_i^+)}\right]\prod_i\Omega(\mathbb{V}_i^+) \tag{5.24}$$

and substitute (5.23) for $\Omega(\mathbb{V}_i^+)$. Since $|V_i|<|V|=N$, this is a valid operation. The result is

$$\Omega(\mathbb{V})=\sum_{\substack{\{\mathbb{Z}_i\}\text{ positive}\\ \text{restricted}}}\prod_i\left[\varrho^0(\mathbb{Z}_i)\frac{\Omega(\mathbb{V}_i)}{\Omega(\mathbb{V}_i^+)}\right]\cdot\prod_i\left\{\sum_{\substack{\{\mathbb{Z}_{j_i}\}\text{ positive}\\ \text{unrestricted in }\mathbb{V}_i^+}}\prod_{j_i}\left[\varrho^0(\mathbb{Z}_{j_i})\frac{\Omega(\mathbb{V}_{j_i})}{\Omega(\mathbb{V}_{j_i}^+)}\right]\right\}. \tag{5.25}$$

The unrestricted sums over $\{\mathbb{Z}_{j_i}\}$ relieve the restriction on $\sum_{\{\mathbb{Z}_i\}}$ so that the sum over $\{\mathbb{Z}_\kappa\}=\{\mathbb{Z}_i,\mathbb{Z}_{j_i}\}$ is unrestricted. This proves (5.23) for arbitrary $\mathbb{V}$ with $\partial V^-=\emptyset$.

An analogous formula holds for $\tilde{\Omega}(\mathbb{V})=\Omega(\mathbb{V})\Big/\prod_{\Delta\subsetneq V}\varrho^0(\Delta,\partial\Delta,\emptyset)$ and $\tilde{\varrho}^{(0)}(\mathbb{Z})=\varrho^{(0)}(\mathbb{Z})\Big/\prod_{\Delta\subsetneq Z}\varrho^0(\Delta,\partial\Delta,\emptyset)$:

$$\tilde{\Omega}(\mathbb{V})=\sum_{\substack{\{\mathbb{Z}_\kappa\}\text{ positive}\\ \text{unrestricted in }\mathbb{V}}}\prod_\kappa\left[\tilde{\varrho}^0(\mathbb{Z}_\kappa)\frac{\Omega(\mathbb{V}_\kappa)}{\Omega(\mathbb{V}_\kappa^+)}\right]. \tag{5.26}$$

We need not consider $\mathbb{Z}_\kappa=(\Delta,\partial\Delta,\emptyset)$ in (5.26) because $\tilde{\varrho}^0(\Delta,\partial\Delta,\emptyset)=1$. With $\tilde{F}_R=F_R\Big/\prod_{\Delta\subsetneq\mathbb{R}^2}\varrho^0(\Delta,\partial\Delta,\emptyset)$, (5.26) holds for $\tilde{F}_\emptyset=\tilde{\Omega}(\mathbb{R}^2)$. (Note that $\varrho^0(\Delta,\partial\Delta,\emptyset)=1$ if $\Delta\cap\Lambda=\emptyset$.)

Equations (5.22)–(5.26) are closely related to some equations in [18]. We desire a generalization of these equations for $R\neq\emptyset$. Multiplication by ratios of partition functions containing R-factors must be avoided. This entails a consideration of $\mathbb{V}$'s with $\partial V^-\neq\emptyset$.

The basic objects we need to consider will be denoted $\Xi_{\mathbb{D},p}(\mathbb{Z})$. $\mathbb{Z}$ is a connected cluster, but $\mathbb{D}$ need not be. $\mathbb{D}$ may not even make sense as a cluster. We will have $\partial D^+\cup\partial D^-\subsetneq\partial D$, but no other relation between D, ∂D^+, and ∂D^- is assumed. The subscript p is either 0 or 1. If the outer boundary of $\mathbb{Z}$ is minus, then $\Xi_{\mathbb{D},p}(\mathbb{Z})$ is defined to be zero. If $\mathbb{Z}$ is positive, let $\{\ell_\alpha\}$ be its minus loops. We can assume that $D=\bigcup_\alpha D_\alpha$ and $D_\alpha\subsetneq\overline{\mathrm{int}\,\ell_\alpha}$. Define $\hat{\mathbb{D}}=\mathbb{D}^+$ if $p=0$; $\hat{\mathbb{D}}=\mathbb{D}$ if $p=1$. If $\hat{\mathbb{D}}$ disagrees with $\mathbb{Z}$ on boundary signs, then $\Xi_{\mathbb{D},p}(\mathbb{Z})=0$. Let $\mathbb{L}=\left(\bigcup_\alpha\overline{\mathrm{int}\,\ell_\alpha},\emptyset,\bigcup_\alpha\ell_\alpha\right)$ and set $\mathbb{V}=\mathbb{L}\setminus\hat{\mathbb{D}}\equiv(L\setminus\hat{D},\ \partial(L\setminus\hat{D})\cap\partial\hat{D}^+,\ \partial(L\setminus\hat{D})\cap(\partial\hat{D}^-\cup\partial L^-))$. If $\mathbb{V}$ does not make sense as a cluster with every component of ∂V given a unique sign or if $\Omega(\mathbb{V})=0$, then $\Xi_{\mathbb{D},p}(\mathbb{Z})=0$. Finally, for $\mathbb{Z}$ positive and $\mathbb{V}$ a sensible cluster with $\Omega(\mathbb{V})\neq0$, we define

$$\Xi_{\mathbb{D},p}(\mathbb{Z})=\tilde{\varrho}(\mathbb{Z})\frac{\Omega(\mathbb{V})}{\Omega(\mathbb{V}^+)}. \tag{5.27}$$

$\Xi_{\mathbb{D},p}(\mathbb{Z})$ is a dressed-up version of the objects appearing in (5.26). The purpose of the definition is to have $\mathbb{D}$ indicate regions with R-factors that cannot be included in $\mathbb{V}$. The subscript p determines whether it is appropriate to delete $\mathbb{D}$ or $\mathbb{D}^+$ from $\mathbb{V}$. $\Xi^0_{\mathbb{D},p}(\mathbb{Z})$ is defined similarly but with $\tilde{\varrho}^0$ replacing $\tilde{\varrho}$ in (5.27).

We next introduce operations that change the subscripts $\mathbb{D}$, p on Ξ. Using these operations, the expansion will be converted to a form very close to that considered in [1]. The operations are different from the ones used in [1] and are more complicated. We say a cluster is a vacuum cluster if it contains no R-factors; otherwise it is a nonvacuum cluster. For the first type we use the letter $\mathbb{X}$; for the second we use the letter $\mathbb{Y}$. $\mathbb{Z}$ may refer to either type. We consider arbitrary $\mathbb{X}$'s but only positive $\mathbb{Y}$'s in defining the operations. We use the notation $\mathbb{D}\cup\mathbb{X} = (D\cup X,\ \partial(D\cup X)\cap(\partial D^+\cup\partial X^+),\ \partial(D\cup X)\cap(\partial D^-\cup\partial X^-))$

$$U(\mathbb{Y}_1,\mathbb{Y}_2)=\begin{cases}0 \text{ if } \mathbb{Y}_1 \text{ and } \mathbb{Y}_2 \text{ overlap}\\ \text{changes } \Xi^0_{\mathbb{D},p}(\mathbb{Y}_1) \text{ to } \Xi^0_{\mathbb{D},0}(\mathbb{Y}_1) \text{ if a minus loop of } \mathbb{Y}_2 \text{ surrounds } \mathbb{Y}_1\\ \text{changes } \Xi^0_{\mathbb{D},p}(\mathbb{Y}_2) \text{ to } \Xi^0_{\mathbb{D},0}(\mathbb{Y}_2) \text{ if a minus loop of } \mathbb{Y}_1 \text{ surrounds } \mathbb{Y}_2\\ 1 \quad \text{otherwise,}\end{cases}$$

$$U(\mathbb{X},\mathbb{Y})=\begin{cases}0 \text{ if } \mathbb{X} \text{ and } \mathbb{Y} \text{ overlap}\\ \text{changes } \Xi^0_{\mathbb{D},p}(\mathbb{Y}) \text{ to } \Xi^0_{\mathbb{D},0}(\mathbb{Y}) \text{ if a minus loop of } \mathbb{X}\\ \quad \text{surrounds } \mathbb{Y} \text{ and } \mathbb{X} \text{ is positive}\\ \text{changes } \Xi_{\mathbb{D},p}(\mathbb{X}) \text{ to } \tilde{\varrho}(\mathbb{X}) \text{ if a minus loop of } \mathbb{Y} \text{ surrounds } \mathbb{X}\\ [\text{leaves } \tilde{\varrho}(\mathbb{X}) \text{ alone if } \mathbb{X} \text{ and } \mathbb{Y} \text{ do not overlap}]\\ \text{changes } \Xi^0_{\mathbb{D},p}(\mathbb{Y}) \text{ to } \Xi^0_{\mathbb{D}\cup\mathbb{X},p}(\mathbb{Y}) \text{ if a minus loop of } \mathbb{Y} \text{ surrounds } \mathbb{X}\\ 1 \quad \text{otherwise,}\end{cases}$$

$$U(\mathbb{X}_1,\mathbb{X}_2)=\begin{cases}0 \text{ if } \mathbb{X}_1 \text{ and } \mathbb{X}_2 \text{ overlap, or if they disagree}\\ \quad \text{on the sign of a common boundary}\\ \text{changes } \Xi_{\mathbb{D},p}(\mathbb{X}_1) \text{ to } \tilde{\varrho}(\mathbb{X}_1) \text{ if a minus loop of } \mathbb{X}_2\\ \quad \text{surrounds } \mathbb{X}_1 \text{ and } \mathbb{X}_2 \text{ is positive}\\ \text{changes } \Xi_{\mathbb{D},p}(\mathbb{X}_2) \text{ to } \tilde{\varrho}(\mathbb{X}_2) \text{ if a minus loop of } \mathbb{X}_1\\ \quad \text{surrounds } \mathbb{X}_2 \text{ and } \mathbb{X}_1 \text{ is positive}\\ [\text{leaves } \tilde{\varrho}(\mathbb{X}_1) \text{ or } \tilde{\varrho}(\mathbb{X}_2) \text{ alone if } \mathbb{X}_1 \text{ and } \mathbb{X}_2 \text{ do not overlap}\\ \quad \text{or disagree}]\\ \text{changes } \Xi_{\mathbb{D},p}(\mathbb{X}_1) \text{ to } \Xi_{\mathbb{D}\cup\mathbb{X}_2,p}(\mathbb{X}_1) \text{ if a minus loop of } \mathbb{X}_1\\ \quad \text{surrounds } \mathbb{X}_2 \text{ and } \mathbb{X}_1 \text{ is positive}\\ \text{changes } \Xi_{\mathbb{D},p}(\mathbb{X}_2) \text{ to } \Xi_{\mathbb{D}\cup\mathbb{X}_1,p}(\mathbb{X}_2) \text{ if a minus loop of } \mathbb{X}_2\\ \quad \text{surrounds } \mathbb{X}_1 \text{ and } \mathbb{X}_2 \text{ is positive}\\ 1 \quad \text{otherwise.}\end{cases}$$

We resum the expansion (5.11) by summing over all $\{\mathbb{Y}_s\}$'s consistent with a given $\{\mathbb{X}_r\}$ and then summing over $\{\mathbb{X}_r\}$. $\bigcup_r X_r$ must contain all the squares in which R-factors appear. We also divide by $\prod_{\Delta} \varrho^0(\Delta, \partial\Delta, \emptyset)$

$$\tilde{F}_R = \sum_{\{\mathbb{X}_r\}} \prod_r \tilde{\varrho}(\mathbb{X}_r) \sum_{\{\mathbb{Y}_s\}} \prod_s \tilde{\varrho}^0(\mathbb{Y}_s). \tag{5.28}$$

Here $\mathbb{Y}_s \neq (\Delta, \partial\Delta, \emptyset)$, but by adding such squares to $\{\mathbb{X}_r, \mathbb{Y}_s\}$, it must be possible to obtain an admissible $\{\mathbb{Z}_\kappa\}$, in the sense of (5.11).

As in (5.22), let $\{\mathbb{Z}_i\}$ be the clusters that are not surrounded by the minus loops of any cluster. $\mathbb{Z}_i$ can be either a $\mathbb{Y}_s$ or an $\mathbb{X}_r$. For each $\mathbb{Z}_i$, let $\ell_{i,\alpha}$ be its minus loops. As before, the $\mathbb{Z}_i$'s are positive. We fix $\{\mathbb{Z}_i\}$ and $\{\mathbb{X}_r\}$ and resum the rest of the expansion. The summation inside $\bigcup_\alpha \ell_{i,\alpha}$ yields a partition function $\tilde{\Omega}(\mathbb{V}_i)$ as before. In this case, however, the presence of $\mathbb{X}_r$'s inside $\ell_{i,\alpha}$ means that the $\mathbb{X}_r$'s must be deleted from $\mathbb{V}$. Setting $\mathbb{V}_i = \mathbb{L}_i \setminus \bigcup_r \mathbb{X}_r$, where $\mathbb{L}_i = \left(\bigcup_\alpha \overline{\operatorname{int}\ell_{i,\alpha}}, \emptyset, \bigcup_\alpha \ell_{i,\alpha}\right)$ we obtain

$$\tilde{F}_R = \sum_{\{\mathbb{X}_r\}} \sum_{\substack{\{\mathbb{Z}_i\}\text{ positive}\\ \text{restricted}}} \prod_{\substack{r'\\ \mathbb{X}_{r'} \notin \{\mathbb{Z}_i\}}} \tilde{\varrho}(\mathbb{X}_{r'}) \prod_i [\tilde{\varrho}^{(0)}(\mathbb{Z}_i)\tilde{\Omega}(\mathbb{V}_i)]. \tag{5.29}$$

Expressing $\tilde{\varrho}^{(0)}(\mathbb{Z}_i)\tilde{\Omega}(\mathbb{V}_i)$ in terms of Ξ's, (5.29) becomes

$$\tilde{F}_R = \sum_{\{\mathbb{X}_r\}} \sum_{\substack{\{\mathbb{Z}_i\}\text{ positive}\\ \text{restricted}}} \prod_{\substack{r'\\ \mathbb{X}_{r'} \notin \{\mathbb{Z}_i\}}} \tilde{\varrho}(\mathbb{X}_{r'}) \prod_i [\Xi^{(0)}_{\mathbb{D}_i,1}(\mathbb{Z}_i)\tilde{\Omega}(\mathbb{V}_i^+)], \tag{5.30}$$

where $\mathbb{D}_i = \bigcup_{r' : X_{r'} \subseteq L_i} \mathbb{X}_{r'}$.

We rewrite (5.26) in terms of Ξ's to obtain

$$\tilde{\Omega}(\mathbb{V}_i^+) = \sum_{\substack{\{\mathbb{Y}_{\kappa_i}\}\text{ positive}\\ \text{unrestricted in }\mathbb{V}_i^+}} \prod_{\kappa_i} \Xi^0_{\mathbb{D}_{\kappa_i},0}(\mathbb{Y}_{\kappa_i}), \tag{5.31}$$

where $\mathbb{D}_{\kappa_i} = \bigcup_{r' : X_{r'} \subseteq L_{\kappa_i}} \mathbb{X}_{r'}$, $L_{\kappa_i} = \bigcup_\alpha \overline{\operatorname{int}\ell_{\kappa_i,\alpha}}$, and $\ell_{\kappa_i,\alpha}$ are the minus loops of $\mathbb{Y}_{\kappa_i}$. In deriving this step, we have matched the way $\mathbb{V}_\kappa$ was defined for (5.25) with the way $\mathbb{V}$ was defined for (5.27). We used $p=0$ in (5.31) because $\mathbb{V}_i^+$ appears on the left-hand side, not $\mathbb{V}_i$. With $\mathbb{V}_i^+$ appearing, any minus loops in $\mathbb{X}_{r'}$'s going into $\mathbb{D}_{\kappa_i}$ had to be changed to plus by using $p=0$. We may relax the condition that $\{\mathbb{Y}_{\kappa_i}\}$ can be supplemented with $(\Delta, \partial\Delta, \emptyset)$'s to agree with $\mathbb{V}_i^+$ on boundaries, because terms violating the condition have $\Xi^0_{\mathbb{D}_{\kappa_i},0}(\mathbb{Y}_{\kappa_i}) = 0$ for some $\mathbb{Y}_{\kappa_i}$. Inserting (5.31) into (5.30) yields

$$\tilde{F}_R = \sum_{\{\mathbb{X}_r\}} \sum_{\substack{\{\mathbb{Z}_i\}\text{ positive}\\ \text{restricted}}} \prod_{\substack{r'\\ \mathbb{X}_{r'} \notin \{\mathbb{Z}_i\}}} \tilde{\varrho}(\mathbb{X}_{r'}) \prod_i \left[\Xi^{(0)}_{\mathbb{D}_i,1}(\mathbb{Z}_i) \sum_{\substack{\{\mathbb{Y}_{\kappa_i}\}\text{ positive}\\ \text{unrestricted in }V_i}} \prod_{\kappa_i} \Xi^0_{\mathbb{D}_{\kappa_i},0}(\mathbb{Y}_{\kappa_i})\right]. \tag{5.32}$$

We rewrite this expression in terms of sums of products of $\Xi^{(0)}_{\emptyset,1}(\mathbb{Z})$'s. The differences in subscripts on Ξ's are handled with the U-operations

$$\tilde{F}_R = \sum_{\{\mathbb{X}_r\}} \sum_{\substack{\{\mathbb{Z}_i\}\text{ positive}\\ \text{restricted}}} \sum_{\substack{\{\mathbb{Y}_\kappa\}\text{ positive}\\ \text{unrestricted in }\cup_i V_i}} \prod_{r_1 < r_2} U(\mathbb{X}_{r_1}, \mathbb{X}_{r_2})$$

$$\cdot \prod_{r,s} U(\mathbb{X}_r, \mathbb{Y}_s) \prod_{s_1 < s_2} U(\mathbb{Y}_{s_1}, \mathbb{Y}_{s_2}) \prod_r \Xi_{\emptyset,1}(\mathbb{X}_r) \prod_s \Xi^0_{\emptyset,1}(\mathbb{Y}_s). \tag{5.33}$$

The U-operations commute, because the effect of a U cannot be undone by another U. Every $\mathbb{X}_r$ that is surrounded by a minus loop has its Ξ converted to a $\tilde{\varrho}$. This yields the $\prod_{r':\mathbb{X}_{r'}\notin\{\mathbb{Z}_i\}} \tilde{\varrho}(\mathbb{X}_{r'})$ in (5.32). Every $\mathbb{Y}_s$ that is surrounded by a minus loop has its Ξ_1 converted to Ξ_0. $\mathbb{Y}_s$'s and $\mathbb{X}_r$'s with minus loops surrounding $\mathbb{X}_{r'}$'s have the $\mathbb{X}_{r'}$'s joined with their $\mathbb{D}$ subscripts. Thus the subscripts on Ξ's in (5.32) are achieved.

We make use of the $U=0$ and $\Xi=0$ cases to remove the restrictions in (5.33). In the sum $\sum_{\{\mathbb{X}_r,\mathbb{Z}_i\}}$, $\mathbb{X}_r$'s with minus outer boundaries cannot appear unless surrounded by a minus contour of some $\mathbb{Z}_i$. But $\Xi(\mathbb{X}_r)=0$ for $\mathbb{X}_r$'s with minus outer boundaries, and $\Xi(\mathbb{X}_r)$ is not converted to $\tilde{\varrho}(\mathbb{X}_r)$ if $\mathbb{X}_r$ is not surrounded by a minus contour of a $\mathbb{Z}_i$. Therefore we may lift the restriction. The restriction that the $\mathbb{Y}_s$'s in $\{\mathbb{Z}_i\}$ not disagree with $\mathbb{X}_r$'s on common boundaries may be lifted because for such terms $\Xi^0_{\mathbb{D},1}(\mathbb{Y}_s)=0$. As in the passage from (5.25) to (5.23), the sum over $\{\mathbb{Z}_i\}$ and $\{\mathbb{Y}_\kappa\}$ becomes an unrestricted sum over $\{\mathbb{Y}_s\}$ positive in $\mathbb{R}^2\setminus\bigcup_r X_r$. We may extend this to an unrestricted sum in $\mathbb{R}^2$, because the extra terms have overlapping $\mathbb{X}_r$'s and $\mathbb{Y}_s$'s so that $U=0$. We change from $\sum_{\{\mathbb{Y}_s\}}$ to $\sum_k \frac{1}{k!}\sum_{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)}$ and then extend $\sum_{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)}$ to overlapping $\mathbb{Y}$'s. Here $(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)$ is an ordered family of $\mathbb{Y}$'s. Finally, we extend $\sum_{\{\mathbb{X}_r\}}$ to all sets of $\mathbb{X}_r$'s, including overlapping $\mathbb{X}_r$'s and $\mathbb{X}_r$'s that disagree on common boundaries. The extra terms have $U=0$. The result is

$$\tilde{F}_R=\sum_{\{\mathbb{X}_r\}}\sum_k\frac{1}{k!}\sum_{\substack{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)\\ \text{positive}}}\prod_{r_1<r_2}U(\mathbb{X}_{r_1},\mathbb{X}_{r_2})\prod_{r,s}U(\mathbb{X}_r,\mathbb{Y}_s)$$

$$\cdot\prod_{s_1<s_2}U(\mathbb{Y}_{s_1},\mathbb{Y}_{s_2})\prod_r\Xi_{\emptyset,1}(\mathbb{X}_r)\prod_{s=1}^k\Xi^0_{\emptyset,1}(\mathbb{Y}_s). \tag{5.34}$$

Every R-factor must be contained in some $\mathbb{X}_r$. The sums are unrestricted in all other respects. The $\mathbb{X}_r$'s need not be positive. The expansion now has the same form as the one in [1].

We follow [1] and obtain an explicit factor of $\tilde{F}_\emptyset$ from (5.34). Define $A(\mathbb{Z},\mathbb{Y})$ by $U=1+A$ and expand the products of U's in (5.34).

$$\tilde{F}_R=\sum_{\{\mathbb{X}_r\}}\sum_k\frac{1}{k!}\sum_{(\mathbb{Y}_1,\ldots,\mathbb{Y}_k)}\prod_{r_1<r_2}U(\mathbb{X}_{r_1},\mathbb{X}_{r_2})\prod_{\mathscr{L}\in G}A(\mathscr{L})$$

$$\cdot\prod_r\Xi_{\emptyset,1}(\mathbb{X}_r)\prod_s\Xi^0_{\emptyset,1}(\mathbb{Y}_s). \tag{5.35}$$

Here G is a graph of unordered pairs $\{\mathbb{X}_r,\mathbb{Y}_s\}$ or $\{\mathbb{Y}_{s_1},\mathbb{Y}_{s_2}\}$ (called lines $\mathscr{L}$). Let G_c be the part of G that contains lines connected directly or indirectly to some $\mathbb{X}_r$. Let $G_0=G\setminus G_c$. G is said to be connected with respect to $\{\mathbb{X}_r\}$ if $G_0=\emptyset$. We sum separately over the $\mathbb{Y}$'s in G_c and the $\mathbb{Y}$'s in G_0. This yields

$$\tilde{F}_R = \left(\sum_{\{\mathbb{X}_r\}} \sum_{k_c} \frac{1}{k_c!} \sum_{(\mathbb{Y}'_1, \ldots, \mathbb{Y}'_{k_c})} \sum_{G_c} \prod_{r_1 < r_2} U(\mathbb{X}_{r_1}, \mathbb{X}_{r_2}) \prod_{\mathscr{L} \in G_c} A(\mathscr{L}) \right.$$
$$\left. \cdot \prod_r \Xi_{\emptyset,1}(\mathbb{X}_r) \prod_{s=1}^{k_c} \Xi^0_{\emptyset,1}(\mathbb{Y}'_s) \right)$$
$$\cdot \left(\sum_{k_0} \frac{1}{k_0!} \sum_{(\mathbb{Y}''_1, \ldots, \mathbb{Y}''_{k_0})} \sum_{G_0} \prod_{\mathscr{L} \in G_0} A(\mathscr{L}) \prod_{s=1}^{k_0} \Xi^0_{\emptyset,1}(\mathbb{Y}''_s) \right). \tag{5.36}$$

The second factor is $\tilde{F}_\emptyset$. Hence we may give the final form of the expansion:

$$\langle R \rangle = \tilde{F}_R / \tilde{F}_\emptyset = \sum_j \sum_{\{\mathbb{X}_1, \ldots, \mathbb{X}_j\}} \sum_k \frac{1}{k!} \sum_{(\mathbb{Y}_1, \ldots, \mathbb{Y}_k)} \Phi(\mathbb{X}_1, \ldots, \mathbb{X}_j; \mathbb{Y}_1, \ldots, \mathbb{Y}_k). \tag{5.37}$$

Here Φ is defined by

$$\Phi(\mathbb{Z}_1, \ldots, \mathbb{Z}_j; \mathbb{Y}_1, \ldots, \mathbb{Y}_k) = \sum_{G_c} \prod_{r_1 < r_2} U(\mathbb{Z}_{r_1}, \mathbb{Z}_{r_2}) \prod_{\mathscr{L} \in G_c} A(\mathscr{L})$$
$$\cdot \prod_{r=1}^{j} \Xi^{(0)}_{\emptyset,1}(\mathbb{Z}_r) \prod_{s=1}^{k} \Xi^0_{\emptyset,1}(\mathbb{Y}_s) \tag{5.38}$$

and G_c is connected with respect to $\{\mathbb{Z}_1, \ldots, \mathbb{Z}_j\}$.

The Kirkwood-Salzburg type equations of [1] may be applied to prove convergence of the expansion. The equation expresses Φ as a sum of terms involving Φ's with fewer clusters and with some subset Ω of $\{\mathbb{Y}_1, \ldots, \mathbb{Y}_k\}$ moved across the semicolon:

$$\Phi(\mathbb{Z}_1, \ldots, \mathbb{Z}_j; \mathbb{Y}_1, \ldots, \mathbb{Y}_k) = \sum_{\Omega} \prod_{r=2}^{j} U(\mathbb{Z}_1, \mathbb{Z}_r) \prod_{s \in \Omega} A(\mathbb{Z}_1, \mathbb{Y}_s)$$
$$\cdot \Xi^{(0)}_{\emptyset,1}(\mathbb{Z}_1) \Phi(\mathbb{Z}_2, \ldots, \mathbb{Z}_j, (\mathbb{Y}_s)_{s \in \Omega}; (\mathbb{Y}_s)_{s \notin \Omega}). \tag{5.39}$$

In order to prove convergence of the sum over $(\mathbb{Y}_1, \ldots, \mathbb{Y}_k)$ in (5.37), we assume by induction that the sum converges for smaller $j+k$ and with U's acting on clusters left of the semicolon. Substituting (5.39) into (5.37), the induction hypothesis applies to control sums over $(\mathbb{Y}_s)_{s \notin \Omega}$.

Two sources of convergence control the sums over $(\mathbb{Y}_s)_{s \in \Omega}$. The first is that a term vanishes unless all $A(\mathbb{Z}_1, \mathbb{Y}_s) \neq 0$ for $s \in \Omega$. In [1] this meant that all $\mathbb{Y}_s$, $s \in \Omega$ had to overlap or surround $\mathbb{Z}_1$. Here one cannot always infer that $\mathbb{Y}_s$'s overlap or surround $\mathbb{Z}_1$ because $U(\mathbb{Z}_1, \mathbb{Y}_s) \neq 1$ is possible for $\mathbb{Y}_s$'s surrounded by a minus loop of $\mathbb{Z}_1$. However, by a judicious choice of $\mathbb{Z}_1$ from $\{\mathbb{Z}_r\}$ we can arrange for $A(\mathbb{Z}_1, \mathbb{Y}_s)$ to be zero for $\mathbb{Y}_s$'s not overlapping or surrounding $\mathbb{Z}_1$. We choose $\mathbb{Z}_1 \in \{\mathbb{Z}_r\}$ to be any $\mathbb{X}$ or $\mathbb{Y}$ whose minus loops do not surround any other $\mathbb{X}$ in $\{\mathbb{Z}_r\}$. Then for a $\mathbb{Y}$ surrounded by a minus loop of $\mathbb{Z}_1$ we have $\mathbb{D} = \emptyset$ in $\Xi^0_{\mathbb{D},p}(\mathbb{Y})$. When $\mathbb{D} = \emptyset$, changing p from 1 to 0 has no effect. Therefore $U(\mathbb{Z}_1, \mathbb{Y}_s) = 1$ and $A(\mathbb{Z}_1, \mathbb{Y}_s) = 0$.

The number of $\mathbb{Y}_s$'s with $|Y_s| = N$ that overlap or surround $\mathbb{Z}_1$ is bounded by $|Z_1| e^{O(1)N}$. This combinatoric factor and the sum over $\{\mathbb{X}_r\}$ in (5.37) are controlled by the second source of convergence: exponential decay of $\Xi^{(0)}_{\mathbb{D},p}(\mathbb{Z})$ with $l|Z|$. We need the bound (5.13) with $\Xi^{(0)}_{\mathbb{D},p}(\mathbb{Z})$ replacing $\varrho(\mathbb{Z})$. Using (5.14), we see that the bound holds for $\tilde{\varrho}(\mathbb{Z})$. The ratio $\Omega(\mathbb{V})/\Omega(\mathbb{V}^+)$ in (5.27) is bounded by 1 by (5.15) if $u=r=t=1$ and $h=0$ in V.

For cases when $u\neq 1$, $r\neq 1$, $t\neq 1$, or $h\neq 0$ in V, we do not have a bound on $\Omega(\mathbb{V})/\Omega(\mathbb{V}^+)$. However, the expansion of Sect. 3 was designed so that u, r, and t differ from 1 only in regions not surrounded by minus contours. More precisely, terms in $\sum_{\Sigma}$ with minus contours around regions with $u\neq 1$, $r\neq 1$, or $t\neq 1$ are multiplied by $0=\prod_{b:\mathrm{dist}(b,\Sigma)\leqq L} u_b$. We need $h\neq 0$ only in regions with $\frac{\partial}{\partial r}$ bonds or $\frac{\partial}{\partial t}$ lines. Therefore, we may take $\varrho(\mathbb{Z})=0$ without affecting the expansion if $\mathbb{Z}$ has a minus contour surrounding regions with $u\neq 1$, $r\neq 1$, $t\neq 1$, or $h\neq 0$. This implies that $\Xi^{(0)}_{\mathbb{D},p}(\mathbb{Z})=0$ whenever the bound on $\Omega(\mathbb{V})/\Omega(\mathbb{V}^+)$ is not available.

In all cases the bound (5.13) holds for $\Xi^{(0)}_{\mathbb{D},p}(\mathbb{Z})$. Convergence of the expansion (5.37) and Theorem 4.1 now follow as in [1]. The factors in (5.13) associated with R and the L^q regularity accumulate in the product of Ξ's in (5.37) and (5.38). For fixed $\deg R$, any missing factors of $\lambda^{\delta(R_Z)/2}$ or extra factors of $\lambda^{-\deg R_Z/2}$ coming from $\mathbb{X}$'s with $\partial X^-\neq\emptyset$ are compensated by factors of $e^{-d\lambda^{-1/2}}$ from associated $\mathbb{Z}$'s with $\partial Z^-\neq\emptyset\neq\partial Z^+$. After contracting with w, one obtains the bound

$$\langle R\rangle \leqq \|w\|_{L^p} M(\deg R)\lambda^{\delta(R)/2} e^{Kl\deg R} \tag{5.40}$$

uniform in λ for fixed $\deg R$, or uniform in $\deg R$ for M depending on λ. If $\delta_\chi(R)>0$ we can include a factor $e^{-d\lambda^{-1/2}}$. For the truncated expectation $\langle R_1; R_2\rangle$, we may extract an extra decay $e^{-gD(R_1,R_2)}$, as in [1]. This establishes the bound (4.13) of Theorem 4.1. As Λ is finite, analyticity in u and h is immediate. Theorem 4.1 is proven.

6. Bounds on Terms of the Cluster Expansion

This section is devoted to the proof of Lemma 5.2. We follow [20] in much of this section. The basic estimates on integrals such as $\int \chi_\Sigma e^{-Q(\Sigma)}d\psi$ come from [16].

Let B_0 be the set of pairs (α, j) that enter the product $\prod_\alpha \prod_j$ in (5.7). We expand the product into a sum

$$\sum_{B\subseteq B_0}\ \prod_{\sigma=(\alpha,j)\in B} \tfrac{1}{2}h(\alpha)\partial^\alpha_{rt}C_j\cdot\Delta_\varphi\,. \tag{6.1}$$

We define some distances (in units of l) which will be used to control the combinatorics. Let γ be a union of l-lattice bonds, and let $i(\alpha)$ be the least integer in α, if α contains integers. Define

$$d(j,\gamma)=\sup_{b\subseteq\gamma}(\mathrm{dist}(\Delta_{j_1},b)+\mathrm{dist}(\Delta_{j_2},b))/l$$

$$d(\sigma)=(\mathrm{dist}(\Delta_{j_1},\Delta_{j_2})+\mathrm{dist}(\Delta_{j_1},\bar{\mathscr{L}}_{i(\alpha)})+\mathrm{dist}(\Delta_{j_2},\bar{\mathscr{L}}_{i(\alpha)}))/l+d(j,\Gamma_\alpha)\,. \tag{6.2}$$

We intend to prove that

$$\begin{aligned}
&\partial_s^{\Gamma_s}\int\prod_{\sigma\in B}[\tfrac{1}{2}h(\alpha)\partial^\alpha_{rt}C_j\cdot\Delta_\varphi]R_Z\chi_{\Sigma\cap Z}e^{-Q(\Sigma\cap Z,Z)}d\psi(r,t,s)\\
&\leqq \|w\|_{L^p}M_1(\deg R_Z)\lambda^{\delta(R_Z)/2}e^{Kl\deg R_Z}e^{-cl|\Gamma_s|}\\
&\quad\cdot e^{-d_4\lambda^{-1/2}|\Sigma|_+}e^{O(1)|Z|}\prod_{\sigma\in B}e^{-cld(\sigma)}\,,
\end{aligned}\tag{6.3}$$

where $|\Sigma|_+=|\Sigma|+1$ if $\delta_\chi(R_Z)>0$ and $|\Sigma|_+=|\Sigma|$ if $\delta_\chi(R_Z)=0$. The factor $\lambda^{\delta(R_Z)/2}$ is replaced by $\lambda^{-\deg R_Z/2}$ if Σ is not identically $+$ or if (6.3) is to be uniform in $\deg R_Z$. Here we have reintroduced a test function w into R_Z. Lemma 5.2 follows from (6.3) because

$$\begin{aligned}\sum_{B\subseteq B_0}\prod_{\sigma\in B}e^{-cld(\sigma)}&=\prod_{\sigma\in B_0}(1+e^{-cld(\sigma)})\\&\leqq\prod_{\sigma\in B_0}\exp(e^{-cld(\sigma)})\leqq e^{O(1)|Z|}.\end{aligned}\tag{6.4}$$

We have used the fact that

$$\begin{aligned}\sum_{\sigma\in B_0}e^{-cld(\sigma)}&\leqq\sum_{j_1}\sum_{j_2}O(1)e^{-cld(\sigma)/2}\\&\leqq\sum_{j_1}O(1)\leqq O(1)|Z|.\end{aligned}\tag{6.5}$$

The u-factors of (5.7) can be handled with a decrease in d_4 in (6.3) because

$$\prod_{\substack{b\\ \operatorname{dist}(b,\Sigma)\leqq L}}u_b\leqq e^{8a\lambda^{-1/2}|\Sigma|}.\tag{6.6}$$

The number of bonds within L of Σ is less than $8|\Sigma|L^2/l^2$, hence (6.6).

We apply the derivatives $\partial_s^{\Gamma_s}$ in (6.3). In each term introduced by the differentiation let Γ_1 be the set of bonds b such that $\frac{\partial}{\partial s_b}$ acts on some $\partial_{rt}^\alpha C_j$, and let $\Gamma_2=\Gamma_s\backslash\Gamma_1$. There are at most $2^{|\Gamma|}$ choices of Γ_1 and Γ_2. This combinatoric factor may be absorbed into $e^{O(1)|Z|}$, so we fix Γ_1,Γ_2 from now on. For each $\sigma=(\alpha,j)$, let γ_σ be the set of bonds such that $\frac{\partial}{\partial s_b}$ acts on $\partial_{rt}^\alpha C_j$. We are reduced to bounding

$$\begin{aligned}\int&\Bigg[\sum_{\not{p}\in\mathscr{P}(\Gamma_2)}\sum_{\{j_\gamma\}_{\gamma\in\not{p}}}\prod_{\gamma\in\not{p}}\partial_s^\gamma C_{j_\gamma}\cdot\Delta_\psi\Bigg]\Bigg[\sum_{\{\gamma_\sigma\}}\prod_{\sigma\in B}h(\alpha)\partial_s^{\gamma_\sigma}\partial_{rt}^\alpha C_j\cdot\Delta_\varphi\Bigg]\\&\cdot R_Z\chi_{\Sigma\cap Z}e^{-Q(\Sigma\cap Z,Z)}d\psi(r,t,s)\end{aligned}\tag{6.7}$$

by the right hand side of (6.3). Here we have applied (4.1) for $\partial_s^{\Gamma_2}$ and expanded $\partial_s^\gamma C$ into its localizations j_γ.

Let θ index the terms in the sums in (6.7), that is, $\theta=\{\{\gamma_\sigma\},\not{p},\{j_\gamma\}_{\gamma\in\not{p}}\}$. Each time a derivative acts on $\chi_{\Sigma\cap Z}$, l^2 terms result. We take the supremum and include a factor of l^2 for each such derivative. Let $T(\theta)$ be the number of terms that result from applying the derivatives in (6.7). We take the supremum over these terms and let W_θ be the resulting polynomial in ψ that multiplies $\chi'_\Sigma e^{-Q}d\psi$ in (6.7), where χ'_Σ is a possible derivative of χ_Σ. We bound the θ-term of (6.7) by

$$\begin{aligned}&T(\theta)\int W_\theta\chi'_{\Sigma\cap Z}e^{-Q(\Sigma\cap Z,Z)}d\psi(r,t,s)\\&\quad\leqq T(\theta)\|W_\theta\|_{L^{p_1}(d\psi)}\|\chi'_{\Sigma\cap Z}e^{-Q(\Sigma\cap Z,Z)}\|_{L^{p_2}(d\psi)},\end{aligned}\tag{6.8}$$

where p_1, p_2 are dual Hölder exponents. We take p_1 large enough so that the following bound of [16] applies:

$$\|\chi'_{\Sigma\cap Z}e^{-Q(\Sigma\cap Z,Z)}\|_{L^{p_2}(d\psi)}\leqq\left(\prod_{\Delta^1}n(\Delta^1)!\right)e^{-d_5\lambda^{-1/2}(|\Sigma|+|\chi'|)}e^{O(1)|Z|}.\tag{6.9}$$

Here $n(\Delta^1)$ is the number of differentiations of χ_Σ in the unit square Δ^1 and $|\chi'|$ is the number of $\Delta^1 \subseteq Z$ such that $n(\Delta^1) \geqq 1$.

The following estimate on derivatives of covariances will be used to bound $\|W_\theta\|_{L^{p_1}(d\psi)}$. Recall the definitions of $L(\Gamma_\alpha)$, $|o|$, $\delta(\alpha)$, and $d(\alpha)$ given in Sect. 4.

Lemma 6.1. *Given $\varepsilon > 0$ and $q \in [1, \infty)$, there exist positive constants c, K_1 such that for λ sufficiently small,*

$$\begin{aligned}\|\partial_s^{\gamma_\sigma}\partial_{rt}^\alpha C_j\|_{L^q} \leqq e^{K_1 l}&\Big(\sum_{o\in L(\Gamma_\alpha)} e^{-cl|o|}\Big) e^{-cl\delta(\alpha)} e^{-2cl|\Gamma_\alpha|} e^{-(1-\varepsilon/2)l(d(\alpha)+1)} \\ &\cdot e^{-4cld(\sigma)} e^{-2cld(j,\gamma_\sigma)} \Big(\sum_{o\in L(\gamma_\sigma)} e^{-cl|o|}\Big) e^{-cl|\gamma_\sigma|}.\end{aligned} \tag{6.10}$$

Here the L^q norm is either $L^q(\Delta_{j_1} \times \Delta_{j_2})$ or, if $j_1 = j_2$ and $\gamma_\sigma \cup \alpha \neq \emptyset$, $L^q(\Delta_{j_1})$.

Proof. We begin by scaling distances down by a factor of l, so that Dirichlet data is on a unit lattice and the mass is increased to l. The L^q norm decreases by a factor of $l^{4/q}$ or $l^{2/q}$, which may be absorbed into $e^{K_1 l}$. We multiply several bounds together to obtain (6.10). Each bound is produced by differentiating C_j with respect to a subset of $\gamma_\sigma \cup \alpha$ and using the Weiner path representation for derivatives of C. See [15, 19, 20] for details. With $i, i' \in \alpha$ and $|i-i'| = d(\alpha)$,

$$\|\partial_s^{\gamma_\sigma}\partial_{rt}^\alpha C_j\|_{L^q}^{1-\varepsilon/4} \leqq \|\partial_{t_i}\partial_{t_{i'}} C_j\|_{L^q}^{1-\varepsilon/4} \leqq e^{-(1-\varepsilon/2)ld(\alpha)}. \tag{6.11}$$

If $d(\alpha) = \infty$ (that is, α has duplications) then $\partial_{rt}^\alpha C_j = 0$ because C is linear in the t_i's. Similarly,

$$\|\partial_s^{\gamma_\sigma}\partial_{rt}^\alpha C_j\|_{L^q}^{\varepsilon/12} \leqq e^{-cl\delta(\alpha)} e^{-4cld(\sigma)} e^{-2cld(j,\gamma_\sigma)} \tag{6.12}$$

and

$$\begin{aligned}\|\partial_s^{\gamma_\sigma}\partial_{rt}^\alpha C_j\|_{L^q}^{\varepsilon/12} &\leqq \|\partial_r^{\Gamma_\alpha} C_j\|_{L^q}^{\varepsilon/12} \leqq O(1) \sum_{o\in L(\Gamma_\alpha)} e^{-3cl|o|} \\ &\leqq e^{O(1)l} \Big(\sum_{o\in L(\alpha)} e^{-cl|o|}\Big) e^{-2cl|\Gamma_\alpha|}\end{aligned} \tag{6.13}$$

for some $c > 0$. Here we have used the fact that $|o| \geqq O(1)|\Gamma_\alpha| - O(1)$. We may replace Γ_α with γ_σ in (6.13). Adding 1 to $d(\alpha)$ with an increase in K_1, we obtain (6.10).

We next estimate the coefficients in W_θ. Recall the difference between $\frac{\delta}{\delta\psi}$ and $\frac{\delta}{\delta\varphi}$: ψ-derivatives act as usual, while φ-derivatives act with $V(\varphi)$ replacing $Q(\psi)$ and then are translated to the ψ-field. Let $N_\Sigma(\theta)$ be the number of ψ-differentiations of e^{-Q} within $L/2$ of Σ plus the number of φ-differentiations (other than in R) of e^{-V} within $L/2$ of a minus spin. Such differentiations introduce at most factors of $O(1)\lambda^{-1}$. [The coefficients in $V(\varphi)$ are $O(\lambda^{1/2})$ but translation produces factors of $O(\lambda^{-3/2})$.] Recall that R_Z contains ∂_φ^k and ∂_χ factors, each of which we count as one derivative. Let $N_\chi(\theta)$ be the number of differentiations of $\chi_{\Sigma\cap Z}$ (or of its derivatives). The l^2 factor introduced above for each such derivative is absorbed into a factor $e^{K_2 l}$ that we will associate to every derivative. Let $N_R(\theta)$ be the number of differentiations of monomials or derivatives in R_Z or of factors produced by such differentiations. If $\Sigma \equiv +$, the coefficients of R are $O(1)$ because then $\psi = \varphi$. Otherwise they may be $O(1)\lambda^{-\deg R_Z/2}$ because of the translation from φ to ψ. Define $N_0(\theta)$ to be the number of ψ-differentiations of e^{-Q} farther than $L/2$ from Σ plus the number of φ-differentiations of e^{-V} that are farther than $L/2$ from any minus spin or that are in R. Each such differentiation introduces a factor $O(1)\lambda^{1/2}$. All other derivatives act on factors produced by differentiations of e^{-Q} or e^{-V}, hence they introduce only $O(1)$ coefficients. Let the total number of derivatives be $N(\theta)$.

We now bound $\|W_\theta\|_{L^{p_1}(d\psi)}$ using Lemma 9.4 of [15]. Expanding in unit localizations produces a factor of l^2 at each vertex. Applying Lemma 9.2 of [15] for l-lattice squares introduces some additional factors of $l^{2/q'}$. These factors are absorbed into $e^{K_2 l}$. Note that $h(\alpha)$ cancels the first four convergence factors in (6.10). Let $N(\Delta)$ be the degree of W_θ in l-lattice square Δ. Then

$$\begin{aligned}\|W_\theta\|_{L^{p_1}(d\psi)} \leqq \|w\|_{L^p} e^{K_2 l(N(\theta)+\deg R_Z)} \lambda^{-N_\Sigma(\theta)} \lambda^{N_0(\theta)/2} \lambda^{-\deg R_Z/2} \\ \cdot \prod_{\sigma\in B}\left[e^{-4cld(\sigma)} e^{-cld(j,\gamma_\sigma)}\left(\sum_{o\in L(\gamma_\sigma)} e^{-cl|o|}\right) e^{-cl|\gamma_\sigma|}\right] \\ \cdot \prod_{\gamma\in \not p}\left[e^{-2cld(j_\gamma,\gamma)}\left(\sum_{o\in L(\gamma)} e^{-cl|o|}\right) e^{-cl|\gamma|}\right] \prod_{\Delta} (p_1 N(\Delta))!^{1/p_1} e^{K_3 N(\Delta)}.\end{aligned} \tag{6.14}$$

The factor $\lambda^{-\deg R_Z/2}$ is absent if $\Sigma \equiv +$.

We estimate $\sum\limits_{\{\gamma_\sigma\}}$ and $\sum\limits_{\not p\in\mathscr{P}(\Gamma_2)}$ in (6.7).

Lemma 6.2. *For λ small,*

$$\sum_{\{\gamma_\sigma\}} \prod_{\sigma\in B}\left[e^{-cld(\sigma)} e^{-cld(j,\gamma_\sigma)}\left(\sum_{o\in L(\gamma_\sigma)} e^{-cl|o|}\right)\right] \leqq e^{O(1)|Z|} \tag{6.15}$$

and

$$\sum_{\not p\in\mathscr{P}(\Gamma_2)} \prod_{\gamma\in\not p} \sum_{o\in L(\gamma)} e^{-cl|o|} \leqq e^{O(1)|Z|}. \tag{6.16}$$

Proof. The second bound is Proposition 8.2 of [15]. We follow [20] for the first bound. The left hand side of (6.15) is bounded by

$$\begin{aligned}&\prod_{\sigma\in B} e^{-cld(\sigma)} \sum_{\gamma\subsetneqq\Gamma_1}\left[e^{-cld(j,\gamma)}\left(\sum_{o\in L(\gamma_\sigma)} e^{-cl|o|}\right)\right] \\ &\leqq \prod_{\sigma\in B} e^{-cld(\sigma)} \sum_{o\in\mathscr{L}(\Gamma_1)} (e^{-cld(j,o)} e^{-cl|o|}) \\ &\leqq \prod_{\sigma\in B} O(1) e^{-cld(\sigma)} \leqq \exp\left[\sum_{\sigma\in B}(O(1)-cld(\sigma))\right] \\ &\leqq \exp\left[\sum_{\sigma\in B} O(1) e^{-O(1)ld(\sigma)}\right] \leqq e^{O(1)|Z|}.\end{aligned} \tag{6.17}$$

In the last step $\sum\limits_{\sigma\in B}$ has been estimated as in (6.5).

We combine Lemma 6.2 with (6.14) and (6.9) to bound (6.7) by

$$\sum_{\{j_\gamma\}_{\gamma \in \not{p}}} T(\theta)\|w\|_{L^p} e^{K_2 l(N(\theta) + \deg R_Z)} \lambda^{-N_\Sigma(\theta)} \lambda^{N_0(\theta)/2} \lambda^{-\deg R_Z/2} e^{-cl|\Gamma_s|}$$
$$\cdot \prod_{\sigma \in B} e^{-3cld(\sigma)} \prod_{\gamma \in \not{p}} e^{-2cld(j_\gamma, \gamma)} e^{-d_5 \lambda^{-1/2}(|\Sigma| + |\chi'|)} e^{O(1)|Z|}$$
$$\cdot \prod_{\Delta} \left[(p_1 N(\Delta))!^{1/p_1} e^{K_3 N(\Delta)} \prod_{\Delta^1 \subseteqq \Delta} n(\Delta^1)! \right]. \tag{6.18}$$

Here a supremum over $\{\gamma_\sigma\}$ and $\not{p} \in \mathscr{P}(\Gamma_2)$ is implicit, and $\lambda^{-\deg R_Z/2}$ is absent if $\Sigma \equiv +$.

Lemma 6.3. *Given positive constants d_6, c, and K_2, there exist $K_4, K > 0$ such that*

$$e^{K_2 l(N(\theta) + \deg R_Z)} \lambda^{-N_\Sigma(\theta)} \lambda^{N_0(\theta)/2} \prod_{\sigma \in B} e^{-cld(\sigma)} \prod_{\gamma \in \not{p}} e^{-cld(j_\gamma, \gamma)}$$
$$\cdot e^{-d_6 \lambda^{-1/2}(|\Sigma| + |\chi'|)} \leqq e^{-K_4 l(|B| + |\not{p}|)} \lambda^{\delta(R_Z)/2} e^{Kl \deg R_Z} \tag{6.19}$$

for λ sufficiently small. For a bound uniform in $\deg R_Z$, $\lambda^{\delta(R_Z)/2}$ *is omitted. Here* $|B|$, $|\not{p}|$ *are the number of elements in* B, $\not{p}$, *respectively.*

Proof. We pin derivatives of e^{-Q} within $L/2$ of Σ and derivatives of χ_Σ to the convergence $e^{-d_6 \lambda^{-1/2}(|\Sigma| + |\chi'|)}$. Let $\mathscr{S}_\Delta$ denote the set of such derivatives which are localized in l-lattice square Δ. We shall prove that

$$\left(\prod_{\mathscr{S}_\Delta} \lambda^{-2} e^{-cl\bar{d}/2} \right) e^{-d_6 \lambda^{-1/2} l^2/L^2} \leqq 1 \tag{6.20}$$

for fixed $\deg R_Z$. Here $\bar{d}$ is either $d(\sigma)$ or $d(j_\gamma, \gamma)$. Taking logarithms, we need to show

$$\sum_{\mathscr{S}_\Delta} (2l^4 - cl\bar{d}/2) - d_6 \lambda^{-1/2} l^2/L^2 \leqq 0. \tag{6.21}$$

Throwing out terms of $\sum_{\mathscr{S}_\Delta}$ with $2l^4 - cl\bar{d}/2 < 0$ we bound the left-hand side of (6.21) by

$$2l^4(O(1) l^{12}) - d_6 \lambda^{-1/2} l^2/L^2 \leqq 0 \tag{6.22}$$

for $\deg R_Z$ fixed and λ small. Here we have used the fact that given j_1 there are no more than $O(1) l^6$ choices for j_2, α, or γ such that $d(\sigma)$ or $d(j_\gamma, \gamma)$ is less than $4l^3/c$. For $\deg R_Z$ arbitrarily large (6.20) holds provided the derivatives in R_Z are not included in $\mathscr{S}_\Delta$.

Every φ-derivative counted in N_Σ is contracted with a $\partial^\alpha_{rt} C_j$ with $d(\sigma) \geqq L/2l$ because the bonds and lines in α are farther than L from any minus spin (Condition A). Using $\lambda^{-1} e^{-cL/2} \leqq \lambda^{1/2}$, the factor $\lambda^{N_0(\theta)/2}$, and (6.20) applied to each Δ, we obtain at least a factor $\lambda^{1/2}$ for every derivative applied to χ_Σ, to e^{-Q}, or to e^{-V}. Hence the factor $\lambda^{\delta(R_Z)/2}$ in (6.19) for fixed $\deg R_Z$. For arbitrary $\deg R_Z$ we must omit this factor.

Since $\deg Q = \deg V = 4$, we must have

$$N(\theta) - \delta(R_Z) - \delta_\chi(R_Z) - N_R(\theta) \leqq 4(N_0(\theta) + N_\chi(\theta) + N_\Sigma(\theta) - \delta(R_Z)). \tag{6.23}$$

Hence

$$|B|+|\not{h}|=\tfrac{1}{2}(N(\theta)-\delta(R_Z)-\delta_\chi(R_Z))\leqq\tfrac{1}{2}\deg R_Z+2(N_0(\theta)+N_\chi(\theta)+N_\Sigma(\theta)-\delta(R_Z)) \tag{6.24}$$

and

$$N(\theta)+\deg R_Z\leqq 2\deg R_Z+4(N_0(\theta)+N_\chi(\theta)+N_\Sigma(\theta)-\delta(R_Z)). \tag{6.25}$$

Therefore,

$$\lambda^{(N_0(\theta)+N_\chi(\theta)+N_\Sigma(\theta)-\delta(R_Z))/2}\leqq e^{-K_4 l(|B|+|\not{h}|)}e^{-K_2 l(N(\theta)+\deg R_Z)}e^{Kl\deg R_Z}. \tag{6.26}$$

Note that derivatives in R_Z are always counted in N_0 or N_χ. We have established an overall factor of $\lambda^{(N_0(\theta)+N_\chi(\theta)+N_\Sigma(\theta)-\delta(R_Z))/2}$, so (6.26) completes the proof of (6.19).

The next two lemmas follow closely the analogous lemmas in [15, 20].

Lemma 6.4. *Let $M(\Delta)$ denote the number of derivatives not in R_Z that are localized in l-lattice square Δ. Given K_3, p_1, there exist constants $M_1(\deg R_Z)$, K_6 such that*

$$\begin{aligned}T(\theta)\prod_\Delta&\left[(p_1 N(\Delta))!^{1/p_1}e^{K_3 N(\Delta)}\prod_{\Delta^1\subseteq\Delta}n(\Delta^1)!\right]\\&\leqq M_1(\deg R_Z)e^{K_6(|B|+|\not{h}|)}\Big(\prod_\Delta M(\Delta)!\Big)^6.\end{aligned} \tag{6.27}$$

Proof. Let $m(\Delta)=\deg R_\Delta$, $m=\deg R_Z$, and $\bar{M}=\sum_\Delta M(\Delta)=2|B|+2|\not{h}|$. The number of terms resulting from as many as $M(\Delta)+m(\Delta)$ 4 differentiations in Δ is at most

$$2^{(m(\Delta)+4M(\Delta))}(m(\Delta)+11)(m(\Delta)+15)\dots(m(\Delta)+7+4M(\Delta)+m(\Delta)). \tag{6.28}$$

The first factor comes from expanding W_θ in terms of ψ and the 11 comes from a possible $4+3+2+1$ terms resulting from differentiating Q or V plus one term from differentiating χ. We apply the inequalities

$$\begin{aligned}(a+b)!&\leqq(a+b)^a b!\\(a+b)^a&\leqq a^a e^a e^b\\(ab)!&\leqq a^{ab}(b!)^a\end{aligned} \tag{6.29}$$

to bound (6.28) by

$$\begin{aligned}&2^{m(\Delta)}[(2m(\Delta)+10)+4M(\Delta)]!^{1/4}2^{4M(\Delta)}\\&\leqq M_2(m(\Delta))e^{M(\Delta)}(4M(\Delta))!^{1/4}2^{4M(\Delta)}\\&\leqq M_2(m(\Delta))e^{O(1)M(\Delta)}M(\Delta)!.\end{aligned} \tag{6.30}$$

Therefore,

$$\begin{aligned}T(0)&\leqq e^{O(1)\bar{M}}\prod_{\Delta:M(\Delta)+m(\Delta)>0}[M_2(m(\Delta))M(\Delta)!]\\&\leqq M_3(m)e^{O(1)\bar{M}}\prod_\Delta M(\Delta)!.\end{aligned} \tag{6.31}$$

Similarly, using $N(\Delta)\leqq m(\Delta)+4(M(\Delta)+m(\Delta))$ we have

$$\prod_{\Delta}[(p_1N(\Delta))!^{1/p_1}e^{K_3N(\Delta)}]\leqq\prod_{\Delta}[p_1^{N(\Delta)}N(\Delta)!e^{K_3N(\Delta)}]$$
$$\leqq M_4(m)e^{O(1)\bar{M}}\left(\prod_{\Delta}M(\Delta)!\right)^4e^{(K_3+\log p_1)N(\Delta)}$$
$$\leqq M_5(m)e^{O(1)\bar{M}}\left(\prod_{\Delta}M(\Delta)!\right)^4. \tag{6.32}$$

Finally,

$$\prod_{\Delta}\prod_{\Delta^1\subseteq\Delta}n(\Delta^1)!\leqq\prod_{\Delta}(m(\Delta)+M(\Delta))!\leqq M_6(m)e^{O(1)\bar{M}}\prod_{\Delta}M(\Delta)!. \tag{6.33}$$

This establishes the lemma.

Lemmas 6.3 and 6.4 yield the following bound on (6.18):

$$\sum_{\{j_\gamma\}_{\gamma\in\not{p}}}\|w\|_{L^p}M_1(\deg R_Z)\lambda^{\delta(R_Z)/2}e^{Kl\deg R_Z}e^{-cl|\Gamma_s|}e^{-d_4\lambda^{-1/2}|\Sigma|_+}$$
$$\cdot\prod_{\sigma\in B}e^{-2cld(\sigma)}\prod_{\gamma\in\not{p}}e^{-cld(j_\gamma,\gamma)}e^{O(1)|Z|}\left(\prod_{\Delta}M(\Delta)!\right)^6 \tag{6.34}$$

with $\lambda^{\delta(R_Z)/2}$ replaced by $\lambda^{-\deg R_Z/2}$ if Σ is not identically $+$ or if (6.34) is to be uniform in $\deg R_Z$. The final lemma proves (6.3) from (6.34) and hence completes the proof of Lemma 5.2.

Lemma 6.5. *The following bound holds independently of* $\not{p}$, B:

$$\sum_{\{j_\gamma\}_{\gamma\in\not{p}}}\prod_{\sigma\in B}e^{-cld(\sigma)}\prod_{\gamma\in\not{p}}e^{-cld(j_\gamma,\gamma)}\left(\prod_{\Delta}M(\Delta)!\right)^6\leqq e^{O(1)|Z|}. \tag{6.35}$$

Proof. The sum over $\{j_\gamma\}$ is handled as follows:

$$\sum_{\{j_\gamma\}_{\gamma\in\not{p}}}\prod_{\gamma\in\not{p}}e^{-cld(j_\gamma,\gamma)/2}\leqq\prod_{\gamma\in\not{p}}\sum_{j_\gamma}e^{-cld(j_\gamma,\gamma)/2}$$
$$\leqq\prod_{\gamma\in\not{p}}O(1)\leqq e^{O(1)|Z|}. \tag{6.36}$$

Fixing $\Delta=\Delta_{j_1}$ or Δ_{j_2}, there are at most $O(1)(a^2+1)$ choices of $\gamma\in\not{p}$ with $d(j_\gamma,\gamma)\leqq a$, or of j_2, j_1, or α, with $d(\sigma)\leqq a$. Altogether there are $O(1)(a^4+1)$ choices with both $d(j_\gamma,\gamma)$ and $d(\sigma)$ less than a. Hence there are less than $M(\Delta)/2$ choices of σ, γ such that

$$d(\sigma)+d(j_\gamma,\gamma)\leqq(O(1)M(\Delta)-1)^{1/4}. \tag{6.37}$$

Thus there are $M(\Delta)/2$ choices with a convergence factor $e^{-\frac{1}{4}cl(O(1)M(\Delta)-1)^{1/4}}$ in (6.35). Since

$$(M(\Delta)!)^6e^{-\frac{1}{8}cl(O(1)M(\Delta)-1)^{1/4}M(\Delta)}\leqq O(1) \tag{6.38}$$

for large $M(\Delta)$, we have

$$\prod_{\sigma\in B}e^{-cld(\sigma)}\prod_{\gamma\in\not{p}}e^{-cld(j_\gamma,\gamma)/2}\left(\prod_{\Delta}M(\Delta)!\right)^6\leqq e^{O(1)|Z|} \tag{6.39}$$

which establishes the lemma.

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References

1. Bałaban, T., Gawędzki, K.: A low temperature expansion for the pseudoscalar Yukawa model of quantum fields in two space-time dimensions. Harvard preprint (1980)
2. Brydges, D.: A rigorous approach to Debye screening in dilute classical coulomb systems. Commun. Math. Phys. **58**, 313–350 (1978)
3. Brydges, D., Federbush, P.: Debye screening. Commun. Math. Phys. **73**, 197–246 (1980)
4. Burnap, C.: Harvard University thesis (1976)
5. Burnap, C.: Isolated one particle states in boson quantum field theory models. Ann. Phys. **104**, 184–196 (1977)
6. Combescure, M., Dunlop, F.: n-particle-irreducible functions in Euclidean quantum field theory. Ann. Phys. **122**, 102–150 (1979)
7. Dimock, J.: Asymptotic perturbation expansion in the $P(\phi)_2$ quantum field theory. Commun. Math. Phys. **35**, 347–356 (1974)
8. Dimock, J., Eckmann, J.-P.: On the bound state in weakly coupled $\lambda(\phi^6-\phi^4)_2$. Commun. Math. Phys. **51**, 41–54 (1976)
9. Dimock, J., Eckmann, J.-P.: Spectral properties and bound-state scattering for weakly coupled $\lambda P(\varphi)_2$ models. Ann. Phys. **103**, 289–314 (1977)
10. Eckmann, J.-P., Epstein, H., Fröhlich, J.: Asymptotic perturbation expansion for the S-matrix and the definition of time ordered functions in relativistic quantum field models. Ann. Inst. Henri Poincaré **25**, 1–34 (1976)
11. Eckmann, J.-P., Magnen, J., Sénéor, R.: Decay properties and Borel summability for the Schwinger functions in $P(\Phi)_2$ theories. Commun. Math. Phys. **39**, 251–271 (1975)
12. Fröhlich, J.: Schwinger functions and their generating functionals. Helv. Phys. Acta **47**, 265–306 (1974)
13. Glimm, J., Jaffe, A.: The resummation of one particle lines. Commun. Math. Phys. **67**, 267–293 (1979)
14. Glimm, J., Jaffe, A., Spencer, T.: The Wightman axioms and particle structure in the $\lambda\mathscr{P}(\phi)_2$ quantum field model. Ann. Math. **100**, 585–632 (1974)
15. Glimm, J., Jaffe, A., Spencer, T.: The particle structure of the weakly coupled $\mathscr{P}(\varphi)_2$ model and other applications of high temperature expansions. In: Velo, G., Wightman, A. (eds.): Constructive quantum field theory. Lecture Notes in Physics, Vol. 25. Berlin, Heidelberg, New York: Springer 1973
16. Glimm, J., Jaffe, A., Spencer, T.: A convergent expansion about mean field theory. Ann. Phys. **101**, 610–669 (1976)
17. Koch, H.: Irreducible kernels and bound states in $\lambda\mathscr{P}(\varphi)_2$ models. Ann. Inst. Henri Poincaré **31**, 173–234 (1979)
18. Pirogov, S., Sinai, Y.: Phase diagrams of classical lattice systems. Theor. Math. Phys. **25**, 1185–1192 (1975) and **26**, 39–49 (1976)
19. Spencer, T.: The mass gap for the $P(\phi)_2$ quantum field model with a strong external field. Commun. Math. Phys. **39**, 63–76 (1974)
20. Spencer, T.: The decay of the Bethe-Salpeter kernel in $P(\varphi)_2$ quantum field models. Commun. Math. Phys. **44**, 143–164 (1975)
21. Spencer, T., Zirilli, F.: Scattering states and bound states in $\lambda\mathscr{P}(\phi)_2$. Commun. Math. Phys. **49**, 1–16 (1976)
22. Summers, S.: Harvard University thesis (1979)
23. Summers, S.: On the phase diagram of a $P(\phi)_2$ quantum field model. CNRS preprint (1979)
24. Koch, H.: Particles exist in the low temperature φ_2^4 model. University of Geneva preprint (1980)

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