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Lieb's Correlation Inequality for Plane Rotors

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Abstract. We prove a conjecture by E. Lieb, which leads to the Lieb inequality for plane rotors. As in the Ising model case, this inequality implies the existence of an algorithm to compute the transition temperature of this model.

Introduction

In [2], Simon proved and applied certain correlation inequalities. These inequalities are special cases of a class of inequalities proved earlier by Boel and Kasteleyn [3–6]. For a finite range pairwise interacting Ising ferromagnet on a lattice L, the inequalities in [2] are:

$$\langle s_a \cdot s_c \rangle < \sum_{b \in B} \langle s_a \cdot s_b \rangle \langle s_b \cdot s_c \rangle , \tag{1}$$

where B is any set of spins separating a from c.

Lieb has generalized this inequality. In [1], he showed the stronger assertion:

$$\langle s_a \cdot s_c \rangle \leqq \sum_{b \in B} \langle s_a \cdot s_b \rangle_A \langle s_b \cdot s_c \rangle , \tag{2}$$

where A is the union of B and the connected component of L–B containing a, and $\langle \cdot \rangle_A$ denotes expection values with respect to the A system only. He also reduced the proof of a similar inequality for plane rotors to a conjecture on directed graphs, which he proves in a special case, and which we prove generally in the next section. Hence we obtain:

Theorem. *Let us consider a plane rotors model with pairwise interaction between two spins* $\mathbf{s}_a$ *and* $\mathbf{s}_b$ *of type* $-J_{a,b}\mathbf{s}_a \cdot \mathbf{s}_b$, *with* $J_{a,b} \geqq 0$.
Then

$$\langle \mathbf{s}_a \cdot \mathbf{s}_c \rangle \leqq \sum_{b \in B} \langle \mathbf{s}_a \cdot \mathbf{s}_b \rangle_A \langle \mathbf{s}_b \cdot \mathbf{s}_c \rangle , \tag{3}$$

where B *separates* a *from* c, *and* A *has same meaning as previously.*

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Among the consequences of (3), is the existence of an algorithm to compute the transition temperature of this model (in the sense that above, but not below, there is exponential decay of the two point function), and the continuity of the mass gap as function of the interaction, for nearest neighbor interactions [1, 2].

Finally we remark that the method used to prove our lemma extends to prove other combinatorial results of the same kind. We intend to explore more completely in the future the consequences of these results for the plane rotors model.

I) We prove a slightly stronger result than the original conjecture of [1]:

Lemma. *Let G be a finite directed graph (possibly with several edges between two vertices) and let the valence at vertex M_i, i.e. the number of arrows in minus the number out of M_i, be m_i. Clearly, $\sum m_i = 0$. Suppose $m_1 > 0$, and m_2, $m_3, \ldots, m_k < 0, m_i \geqq 0$ otherwise. Let $N(G)$ be the set of subgraphs of G (subsets of edges), including the empty graph, having valence 0 at each vertex. Let $K(M_1, G)$ be the set of subgraphs of G with the following property: vertex M_1 has valence $+1$, some vertex M_i with $2 \leqq i \leqq k$ has valence -1, all other valences are 0.*
Then:

$$|N(G)| \leqq |K(M_1, G)|. \tag{4}$$

Proof. We prove the lemma by induction on the number of edges in G. The lemma is trivial for a graph with a unique edge. We suppose it is true up to l edges. Let G have $l+1$ edges, among which $l_1, \ldots, l_p$ are the arrows into M_1 and $l_{p+1}, \ldots, l_{p+q}$ are the arrows out of M_1.
We define:

$$\text{for} \quad 1 \leqq j \leqq p \qquad \begin{cases} N_j(G) = \{S \in N(G) : l_j \notin S\} \\ K_j(M_1, G) = \{S \in K(M_1, G) : l_j \in S\} \end{cases}$$

$$\text{for} \quad p+1 \leqq j \leqq p+q \begin{cases} N_j(G) = \{S \in N(G) : l_j \in S\} \\ K_j(M_1, G) = \{S \in K(M_1, G) : l_j \notin S\}. \end{cases}$$

We claim that for $1 \leqq j \leqq p+q$,

$$|N_j(G)| \leqq |K_j(M_1, G)|. \tag{5}$$

Assuming this, the lemma follows easily, since every subgraph in $N(G)$ belongs to exactly p subsets $N_j(G)$, and every subgraph in $K(M_1, G)$ belongs to exactly $q+1$ subsets $K_j(M_1, G)$. Hence:

$$p|N(G)| = \sum_j |N_j(G)| \leqq \sum_j |K_j(M_1, G)| = (q+1)|K(M_1, G)|$$

and $(q+1)/p \leqq 1$ is precisely the condition $m_1 > 0$.

To prove the claim, we consider first the case $1 \leqq j \leqq p$. Let l_j be the arrow $M_j M_1$. If $m_j < 0$, we have a natural injection of $N_j(G)$ into $K_j(M_1, G)$: to S we associate $S \cup \{l_j\}$. If $m_j \geqq 0$, we apply the induction hypothesis to the graph $G_j = G - \{l_j\}$: in G_j, $m_1 \geqq 0$, $m_j > 0$, hence $|N(G_j)| \leqq |K(M_j, G_j)|$. On the other hand, there are natural bijections of $N(G_j)$ into $N_j(G)$, and of $K(M_j, G_j)$ into $K_j(M_1, G)$ (respectively $S \to S$, and $S \to S \cup \{l_j\}$). In both cases, (5) is proved.

Finally, if $p+1 \leqq j \leqq p+q$, either $N_j(G)$ is empty, and (5) is trivial, or there exists a subgraph L_j containing l_j in $N_j(G)$. We then note the effect of reversing arrows in a given subgraph L belonging to $N(G)$. We call G^L the graph obtained from G by this transformation. The valences m_i do not change. Moreover, there are natural bijections between $N(G)$ and $N(G^L)$, and between $K(M_1, G)$ and $K(M_1, G^L)$, which to a subgraph S associate $S \Delta L$, where Δ is the symmetric difference of subsets. Returning to the proof of (5), we reverse the arrows in the subgraph L_j previously introduced, and transform the problem into one of the preceding kind for $G^L j$, hence already solved. Hence (5) holds in every case.

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