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On Bounded Solutions of a Classical Yang-Mills Equation

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Abstract. We discuss bounded solutions of the equation

$$r^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial t^2}\right)=u^3-u$$

in the halfspace $r>0$. All solutions depending only on t/r are characterized topologically. Then we prove the existence of infinite dimensional manifolds of t-periodic as well as nonperiodic solutions which are small in a suitable norm.

0. Introduction

It was shown recently by Glimm and Jaffe [1] that multimeron solutions to the classical SU(2) Yang-Mills field equations in Euclidean space are characterized by the following singular elliptic boundary value problem:

$$\begin{aligned}&r^2\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial t^2}\right)=u^3-u \qquad t\in R, \qquad r>0, \\ &\lim_{r,t\to\infty} u(r,t)=1, \qquad u(0,t)=(-1)^i \quad \text{for} \quad t_i<t<t_{i+1}(i=0,1,\dots,2n),\end{aligned} \tag{0.1}$$

where $-\infty=t_0<t_1<\dots<t_{2n-1}<t_{2n}<t_{2n+1}=\infty$.

Jonsson et al. proved in [2] that this boundary value problem has at least one solution for every choice of the t_i. In this paper we investigate some kinds of *bounded solutions to (0.1), which satisfy different boundary conditions.*

We first prove (Sect. 1) that a bounded solution of (0.1) which has a continuous extension to the t-axis except for a countable number of points must satisfy $|u|\leqq 1$ in the whole half-plane and cannot be positive everywhere, unless it is constant.

The special solutions which we discuss then are of two different types. In Sect. 2 we are concerned with solutions depending only the independent variable $\frac{t}{r}$, for which (0.1) is reduced to an ordinary differential equation; in Sects. 3 and 4 we discuss solutions which are "small" in a suitable norm.

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Solutions depending only on r or only on $r/(r^2+t^2)$ have been considered by Protogenov [5]. A second class of solutions for which (0.1) is reduced to an ODE are those depending only on $t/r =: \tau$. We prove in Sect. 2 that there exists a two dimensional continuum of such solutions satisfying $\lim_{\tau\to\pm\infty} u(\tau)=0$. By a limit procedure we find that there exists one (unique) solution assuming the boundary values $\lim_{\tau\to-\infty} u(\tau)=-1$ and $\lim_{\tau\to\infty} u(\tau)=1$. In addition we find a one dimensional continuum of solutions approaching ± 1 only on one side and 0 on the other side.

Small solutions are discussed in Sect. 3 for the case of solutions that are periodic in t with a given period and in Sect. 4 for a certain class of nonperiodic solutions. In both cases we find a one-to-one correspondence between small solutions and the nullspace of the linearization. This proves the existence of an infinite-dimensional manifold of bounded solutions. All these solutions approach 0 as $r\to 0$.

1. A Priori Estimates for Bounded Solutions

Theorem 1.1. *Let u be a bounded C^2-solution of* (0.1) *in $r>0$ which can be continuously extended to the axis $r=0$ except at a countable number of points. Then $|u|\leqq 1$ in the whole halfplane $r>0$.*

Remark. The condition that u is C^2 is not really a restriction. As proved in [2], Theorem 3.1, every weak solution to (0.1) which is in L^∞ is real analytic in $r>0$.

Proof. (i) Let $(0,t_0)$ be a point on the t-axis where u has a continuous limit. We are going to prove that $u(0,t_0)$ must take one of the values 0, ± 1. According to Green's formula we have

$$u(r,t) = -\int_{\partial G} u(r_1,t_1)\frac{\partial \Gamma(r,t,r_1,t_1)}{\partial \nu}\,d\sigma - \int_G \Delta u(r_1,t_1)\Gamma(r,t,r_1,t_1)\,dr_1\,dt_1$$
$$= -\int_{\partial G}\dots - \int_G r_1^{-2}(u^3(r_1,t_1)-u(r_1,t_1))\Gamma(r,t,r_1,t_1)\,dr_1\,dt_1\,.$$

Here Γ is Green's function for the bounded domain $G\subset R^2_+$.

Assume $u(0,t_0)\neq 0, \pm 1$ and let U be a neighbourhood of $(0,t_0)$, in which u never takes the values 0, ± 1. Let now G be a square as shown in the next diagram.

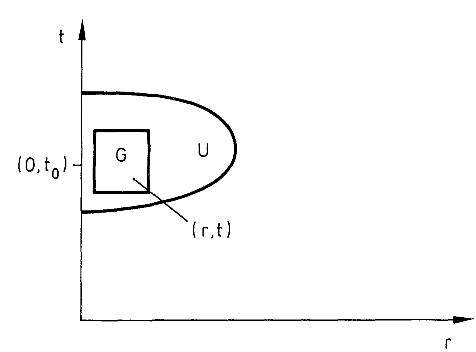

Fig. 1.1

The second integral in the formula above diverges as the left boundary of G is shifted to the t-axis (the integrand contains a factor r_1^{-2}, and only one r_1 is compensated by the Green's function, since the normal derivative of Γ on a smooth portion of the boundary does not vanish [6]). This contradicts the boundedness of u.

(ii) The following argument modifies an idea of Glimm and Jaffe [1]. Let F be a C^∞-function $R \to R$ with the following properties: $F(u)=u$ for $u \leqq 1$ and $F(u)<u$ for $u>1$, $F' \leqq 1$, $F'' \leqq 0$. Then

$$\Delta F(u) = F'\Delta u + F''(\nabla u)^2 \leqq F'\Delta u \leqq \Delta u .$$

Hence $u-F(u)$ is non-negative, subharmonic, bounded, and is equal to 0 on the t-axis except at a countable number of points. By inversion with respect to a circle we can map the half-plane onto the interior of a circle. From $u-F(u)$ we obtain then a function, which is non-negative, subharmonic and bounded in the interior of the cricle and vanishes on the boundary with at most countably many exceptional points. From [3, p. 204] we conclude that $u-F(u)$ is non-positive. This proves $u \leqq 1$, and replacing u by $-u$ we find $u \geqq -1$.

Next we prove the non-existence of non-trivial positive solutions.

Theorem 1.2. *Let* $u \geqq 0$ *be a bounded solution to* (0.1) *which can be continuously extended to the t-axis with at most countably many exceptional points. Then* $u=1$ *or* $u=0$.

Proof. Since $u \geqq 0$ we have $\Delta u \leqq 0$, and if the boundary value is 1 everywhere on the t-axis, then $u=1$. So we may assume there exists a point $(0, t_0)$ such that $\lim_{(r,t)\to(0,t_0)} u(r,t)=0$. Let $\zeta(t)$ be a cut-off function supported by a sufficiently small interval containing t_0.

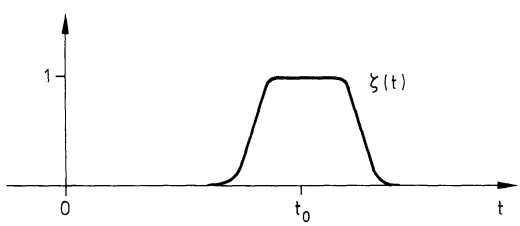

Fig. 1.2

We define $v(r) = \int_{-\infty}^{\infty} \zeta(t) u(r,t)\, dt$. Then Eq. (0.1) implies

$$\begin{aligned} r^2 \frac{d^2 v}{dr^2} &= -v + \int_{-\infty}^{\infty} \zeta(t) u^3(r,t)\, dt - r^2 \int_{-\infty}^{\infty} \zeta(t) u_{tt}(r,t)\, dt \\ &= -v + \int_{-\infty}^{\infty} \zeta(t) u^3(r,t)\, dt - r^2 \int_{-\infty}^{\infty} \zeta_{tt} u(r,t)\, dt . \end{aligned}$$

If ζ is as shown in the diagram above, then ζ_{tt} is either positive or its modulus is less than some constant times ζ. Hence we can arrange that

$$\int_{-\infty}^{\infty} \zeta(t)u^3(r,t)dt - r^2 \int_{-\infty}^{\infty} \zeta_{tt}u(r,t)dt \leqq \varepsilon v$$

for $r<\delta(\varepsilon)$. Hence for $r<\delta(\varepsilon)$ we find $r^2\dfrac{d^2v}{dr^2} < -(1-\varepsilon)v$.

If we substitute $r=e^\tau$, this yields

$$\ddot{v}-\dot{v} < -(1-\varepsilon)v\,. \tag{1.1}$$

If u does not vanish identically, we may find a τ_0 such that $v(\tau_0)>0$, $\dot{v}(\tau_0)>0$. The following diagram shows the solution w of $\ddot{w}-\dot{w}=-(1-\varepsilon)w$ with the initial condition $w(\tau_0)=v(\tau_0)$, $\dot{w}(\tau_0)=\dot{v}(\tau_0)$.

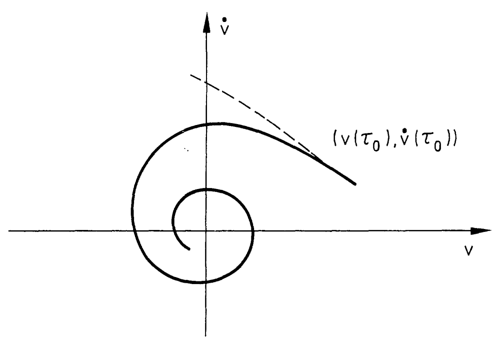

Fig. 1.3

According to (1.1) for $\tau \lesssim \tau_0$ the phase plane curve for v is above the curve for w and the two curves do not intersect again as long as $v>0$, $\dot{v}>0$. This is indicated by the dashed line in the diagram. In particular, we conclude from this that for $\tau<\tau_0$ $\dot{v}$ remains strictly positive as long as $v>0$, and hence v must change its sign for a certain finite $\tau_1<\tau_0$. (It is a priori clear from the definition of v that $\dot{v}$ cannot diverge to ∞ for finite τ.) This contradicts the positivity of v. Hence $v=0$, i.e. $u=0$ in a whole subdomain of R^2_+. Since u is real analytic, this implies that u vanishes identically.

2. Solutions which Depend only on t/r

Substituting $t/r=:\sinh\tau$, we obtain the ordinary differential equation

$$\ddot{u}+\tanh\tau\dot{u}=u^3-u \tag{2.1}$$

from which we conclude (after multiplication by $\dot{u}$)

$$\frac{d}{d\tau}\left(\frac{1}{2}\dot{u}^2+\frac{1}{2}u^2-\frac{1}{4}u^4\right)=-\tanh\tau\dot{u}^2\,.$$

The following diagram shows the lines of constant level for the function $\frac{1}{2}\dot{u}^2+\frac{1}{2}u^2-\frac{1}{4}u^4$.

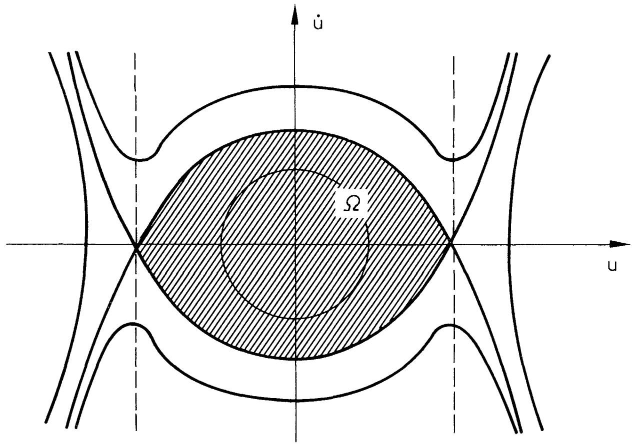

Fig. 2.1

We see immediately that there exists a two dimensional continuum of bounded solutions.

Lemma 2.1. *Whenever* $(u(0),\dot{u}(0))$ *is in* $\bar{\Omega}\backslash\{(1,0),(-1,0)\}$, *the solution of* (2.1) *with these initial conditions for* $\tau=0$ *is bounded and approaches the origin as* $\tau\to\pm\infty$.

We shall next prove

Theorem 2.2. *Given any* λ *between* -1 *and* 0, *there exists at least one solution of* (2.1) *satisfying an initial condition* $\dot{u}(0)>0$, $u(0)=\lambda$ *which approaches* -1 *as* $\tau\to-\infty$ *and* 0 *as* $\tau\to\infty$. *For* $\lambda=0$ *there is a solution approaching* 1 *as* $\tau\to\infty$ *and* -1 *as* $\tau\to-\infty$, *and for each* $0<\lambda<1$ *there is a solution approaching* 1 *as* $\tau\to\infty$ *and* 0 *as* $\tau\to-\infty$.

Proof. We confine ourselves to the case $\lambda<0$, the other cases are discussed in the same way.

If $u(0)=\lambda$ and $\dot{u}(0)$ is sufficiently small, then $(u,\dot{u})$ stays in Ω for every $\tau\in R$.

On the other hand

$$\frac{d}{d\tau}\left(\frac{1}{2}\dot{u}^2+\frac{1}{2}u^2-\frac{1}{4}u^4\right)=-\tanh\tau\left(\dot{u}^2+u^2-\frac{1}{2}u^4\right)+\tanh\tau\left(u^2-\frac{1}{2}u^4\right)$$

$$\geqq-\left(\dot{u}^2+u^2-\frac{1}{2}u^4\right)\quad\text{if}\quad\tau>0\quad\text{and}\quad-1\leqq u\leqq 1.$$

Hence if $u(0)=\lambda$, we find

$$\dot{u}^2(\tau)\geqq e^{-2\tau}(\dot{u}^2(0)+\lambda^2-\tfrac{1}{2}\lambda^4)+\tfrac{1}{2}u^4-u^2\geqq e^{-2\tau}(\dot{u}^2(0)+\lambda^2-\tfrac{1}{2}\lambda^4)-\tfrac{1}{2}$$

as long as $-1\leqq u(\tau)\leqq 1$.

We see from this that the solution leaves the strip $-1\leqq u\leqq 1$ for $\tau>0$, provided $\dot{u}(0)$ is large enough. Similarly, the solution leaves the strip for $\tau<0$, if $\dot{u}(0)$ is large enough. A continuity argument now shows that there exists $C>0$, such that the solution u_0 of (2.1) satisfying $u_0(0)=\lambda$, $\dot{u}_0(0)=C$ remains in the strip and approaches 1 as $\tau\to\infty$ or -1 as $\tau\to-\infty$. We now derive estimates that make sure that actually $\lim_{\tau\to-\infty} u_0(\tau)=-1$ and $\lim_{\tau\to\infty} u_0(\tau)=0$.

Denoting $u_0(-\tau)$ by $v_0(\tau)$, we obtain from (2.1)

$$\begin{aligned}(u_0+v_0)^{\cdot\cdot}+\tanh\tau(u_0+v_0)^{\cdot}&=u_0^3-u_0+v_0^3-v_0\\ u_0(0)=v_0(0)<0,\qquad \dot{u}_0(0)&=-\dot{v}_0(0)>0.\end{aligned} \tag{2.3}$$

For any $\tau>0$ we have $-1<u_0, v_0<1$, and as long as $(u_0,\dot{u}_0)$, $(v_0,-\dot{v}_0)\notin\Omega$ (which is true for small τ) $\dot{u}_0>0$ and $\dot{v}_0<0$.

The right side of (2.3) is positive for sufficiently small $\tau>0$, i.e.

$$\frac{d}{d\tau}(u_0+v_0)^{\cdot}+\tanh\tau(u_0+v_0)^{\cdot}>0.$$

Since we have $(u_0+v_0)^{\cdot}=0$ [whence $(u_0+v_0)^{\cdot\cdot}>0$] at $\tau=0$, we may conclude from (2.3) that $(u_0+v_0)^{\cdot}\geqq 0$ for small enough τ, let us say for $\tau\leqq\tau_0$.

As long as $(u_0,\dot{u}_0)$, $(v_0,-\dot{v}_0)\notin\Omega$, the inequality $(u_0+v_0)^{\cdot}>0$ implies $\dot{u}_0>-\dot{v}_0\Rightarrow\dot{u}_0^2>\dot{v}_0^2$, which, together with (2.2), yields

$$\begin{aligned}\tfrac{1}{2}\dot{u}_0^2+\tfrac{1}{2}u_0^2-\tfrac{1}{4}u_0^4<\tfrac{1}{2}\dot{v}_0^2+\tfrac{1}{2}v_0^2-\tfrac{1}{4}v_0^4&\Rightarrow\tfrac{1}{2}u_0^2-\tfrac{1}{4}u_0^4<\tfrac{1}{2}v_0^2-\tfrac{1}{4}v_0^4\\ &\Rightarrow|u_0|<|v_0|.\end{aligned} \tag{2.4}$$

This proves that $(u_0(\tau_0),\dot{u}_0(\tau_0))\in\Omega$ or else $|u_0(\tau_0)|<|v_0(\tau_0)|$. If $\tau_0=\infty$ or $(u_0(\tau_0),\dot{u}_0(\tau_0))\in\Omega$, this implies the statement of the theorem. So we assume $\tau_0<\infty$ and $(u_0(\tau_0),\dot{u}_0(\tau_0))\notin\Omega$. Then $|u_0(\tau_0)|<|v_0(\tau_0)|$ and

$$u_0^3(\tau_0)-u_0(\tau_0)+v_0^3(\tau_0)-v_0(\tau_0)\leqq 0.$$

This is only possible if $3v_0^2-1>0$.

Therefore, $\frac{d}{d\tau}(u_0^3-u_0+v_0^3-v_0)=(3u_0^2-1)\dot{u}_0+(3v_0^2-1)\dot{v}_0<0$ as long as $|u_0|<|v_0|$ and $\dot{v}_0\leqq-\dot{u}_0<0$.

Therefore the right side of (2.3) is negative for $\tau\gtrsim\tau_0$, and it remains negative as long as $|u_0|<|v_0|$ and $\dot{v}_0\leqq-\dot{u}_0<0$. Since $(u_0+v_0)^{\cdot}=0$ at $\tau=\tau_0$, this and (2.3) imply that $(u_0+v_0)^{\cdot}$ is negative for $\tau>\tau_0$, as long as $\dot{v}_0$ and $\dot{u}_0$ do not change their sign. But this implies that $\dot{v}_0$ cannot change its sign unless $\dot{u}_0$ does. Therefore either $\dot{u}_0$ has a change of sign for some $\tau_1>\tau_0$, which implies $(u_0(\tau_1),\dot{u}_0(\tau_1))\in\Omega$, or $\dot{u}_0+\dot{v}_0$ is negative for every $\tau>\tau_0$, which implies that $|v_0|-|u_0|$ increases monotonically for $\tau>\tau_0$. This concludes the proof.

Remark. In the special case $\lambda=0$, the solution given by the last theorem (which is later proved to be unique) is the well known single meron solution given explicitly by $u=\tanh\tau$ (cf. [4]).

Theorem 2.3. *In Theorem 2.2 "at least one" may be replaced by "one and only one".*

Proof. (i) If the solutions with the initial conditions $u(0)=\lambda\geqq 0$, $\dot{u}(0)=a, b$ resp. approach 1 as $\tau\to\infty$, then all the solutions with an initial condition $u(0)=\lambda$, $a<\dot{u}(0)<b$ have the same behaviour.

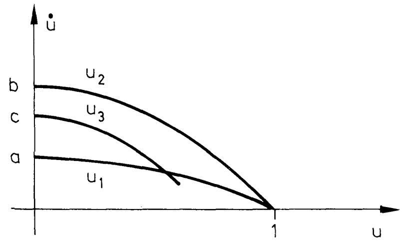

Fig. 2.2

It is obviously sufficient to prove that the solution u_3 starting from (λ, c) never crosses the two other solutions. Assume the contrary as indicated in the diagram, i.e. $\exists \tau_1, \tau_2 > 0$ such that $u_3(\tau_1) = u_1(\tau_2)$, $\dot{u}_3(\tau_1) = \dot{u}_1(\tau_2)$. Since

$$\tau_1 = \int_{\lambda}^{u_3(\tau_1)} (\dot{u}_3)^{-1} du_3, \qquad \tau_2 = \int_{\lambda}^{u_1(\tau_2)} (\dot{u}_1)^{-1} du_1,$$

we see that $\tau_1 < \tau_2$. Hence (3.1) implies that

$$\ddot{u}_3(\tau_1) = -\tanh \tau_1 \cdot \dot{u}_3(\tau_1) + u_3^3(\tau_1) - u_3(\tau_1) > \ddot{u}_1(\tau_2),$$

which is an apparent contradiction. The same argument shows that u_3 and u_2 cannot cross.

(ii) Given $T > 0$ large enough, there exists a neighbourhood U of (1.0) and an analytic curve C in U passing through the point (1, 0), such that for $(u(T), \dot{u}(T))$ in U the solution u of (2.1) with this initial condition approaches 1 as $\tau \to \infty$ iff $(u(T), \dot{u}(T)) \in C$.

From (ii) the theorem follows easily, since the mapping $(u(T), \dot{u}(T)) \to (u(0), \dot{u}(0))$ is (locally) analytic and hence takes C to an analytic curve $\tilde{C}$. But an analytic curve cannot contain an interval on the line $u = \lambda$.

So it remains to prove (ii). After the substitution $u - 1 = v$, (2.1) reads

$$\ddot{v} + \dot{v} - 2v = v^3 + 3v^2 + (1 - \tanh \tau)\dot{v}.$$

We rewrite this equation as an operator equation

$$Lv = M(v) \qquad L, M : C_b^2([T, \infty)) \to C_b([T, \infty)). \tag{2.5}$$

L has a one dimensional nullspace and full range, i.e. there exists a right inverse $\hat{L}$ and we may rewrite (2.5) in the form

$$v = \hat{L}M(v) + f, \tag{2.6}$$

where f is in the nullspace of L. If T is large enough, we conclude from the implicit function theorem that locally (2.6) has a unique analytic resolution $v = v(f)$. Since the projection $v \to (v(T), \dot{v}(T))$ is a continuous linear operator from $C_b^2([T, \infty))$ into R^2, this implies (ii).

3. Solutions which are Periodic in *t*

Substituting $r = e^x$ and denoting differerentiation w.r.t. x by $'$, we obtain from (0.1)

$$u'' - u' + e^{2x}u_{tt} + u = u^3. \tag{3.1}$$

Using the Fourier expansion $u=\sum_{k\in Z} u_k(x)e^{ik\omega t}$ we find

$$u_k''-u_k'-k^2\omega^2e^{2x}u_k+u_k=(u^3)_k=\sum_{l,m\in Z} u_{k-l}(x)u_{l-m}(x)u_m(x). \tag{3.2}$$

We are looking for small solutions of (3.2) in the space

$$l^1(C_b(R)):=\left\{u|u_k \text{ continuous}, \sum_{k\in Z}\sup_{x\in R}|u_k(x)|<\infty\right\}.$$

We see that (3.2) is of the form $Qu=u^3$, and we shall prove that there exists an operator $M\in\mathscr{L}(l^1(C_b(R))$ such that $QM=1$. (M has regularizing properties so that Q is defined on the range of M.) The equation

$$u-Mu^3=f \tag{3.3}$$

has a unique solution $u=u(f)$ in a neighbourhood of 0, and this solution gives a solution of (3.2) iff f is in the kernel of Q. Hence we obtain a one-to-one correspondence between small solutions of (3.2) and members of the kernel N of the linearization Q. We now construct M. Consider the problem

$$u_k''-u_k'+u_k-k^2\omega^2e^{2x}u_k=v_k(x) \qquad (v_k)\in l^1(C_b(R)).$$

For $k=0$ this has a unique solution $u_0=M_0v_0\in C_b(R)$. So let now be $k\neq 0$. Substituting $\zeta=x-x_k'$, where x_k' is defined by $\exp(2x_k')=(k\omega)^{-2}$, we reduce the problem to the equation

$$u''-u'+u-e^{2\zeta}u=v. \tag{3.4}$$

So we have to construct a linear operator $\tilde{M}$ in $\mathscr{L}(C_b(R))$, such that $u=\tilde{M}v$ solves (3.4).

With $r=e^\zeta$ (3.4) reads

$$u_{rr}-u+r^{-2}u=r^{-2}v. \tag{3.5}$$

If r_0 is sufficiently large, the term $r^{-2}u$ can be treated as a perturbation, and from the characteristic exponents of $u_{rr}-u=0$ we see that (3.5) has a unique bounded solution for $r\geqq r_0$ which obeys the initial condition $u_r(r_0)=0$. This solution depends continuously on v, i.e. there exists a constant C such that

$$\sup_{r\geqq r_0}|u(\mathrm{r})|\leqq C\sup_{r\geqq r_0}|v(r)|.$$

Now continue this solution to the left side according to Eq. (3.4). This gives a solution u in any interval $-\zeta_0\leqq\zeta<\infty$, which depends continuously on v:

$$\sup_{\zeta\geqq-\zeta_0}|u(\zeta)|\leqq C(\zeta_0)\sup_{\zeta\geqq\zeta_0}|v(\zeta)|.$$

If we choose ζ_0 large enough, $e^{2\zeta}$ becomes arbitrarily small for $\zeta<-\zeta_0$, so that now the term $e^{2\zeta}u$ can be treated as a perturbation, and from the characteristic exponents of $u''-u'+u=0$ we see that on $(-\infty,-\zeta_0)$ the Eq. (3.4) defines u as a continuous function of v, $u(-\zeta_0)$ and $u'(-\zeta_0)$. So taking everything together we have found a solution u of (3.4) depending continuously on v:

$$\sup_{\zeta\in R}|u(\zeta)|\leqq C'\sup_{\zeta\in R}|v(\zeta)|.$$

The solution u constructed in this manner is defined as $\tilde{M}v$, which completes the construction of M. The elements of N are obtained by the same construction as above, if we take $v=0$ and replace the condition $u_r(r_0)=0$ by $u_r(r_0)=c$, $c\neq 0$. This shows that for every $k\neq 0$ we find a one-dimensional nullspace. We thus have an infinite dimensional nullspace and hence an infinite dimensional manifold of nontrivial solutions of (3.1).

The condition $\lim\limits_{x\to-\infty} u_k(x)=0$, $k\in Z$ defines a closed subspace Y of $l^1(C_b(R))$, which is mapped into itself by M. Since the members of the kernel of Q belong to Y, we see from (3.3) that our solutions are in Y as well.

We have thus proved

Theorem 3.1. *For any given period T there exists an infinite dimensional manifold of solutions to* (0.1) *which are T-periodic w.r.t. t and obey the boundary condition* $\lim\limits_{r\to 0} u(r,t)=0$ *uniformly in t.*

4. Nonperiodic Solutions

In the argument above we replace Fourier expansion by the Fourier integral, i.e.

$$u(x,k)=\int_{-\infty}^{\infty} u(x,t)e^{-ikt}dt$$

and we replace the space $l^1(C_b(R))$ by

$$L^1(C_b(R))=\left\{u(x,k)|u \text{ measurable, } u(\cdot,k)\in C_b(R) \text{ for almost every } k, \int \sup_x |u(x,k)|dk<\infty\right\}.$$

The operator M is constructed exactly as above, however, we have to discuss some technical details. First we make sure that the definition of $L^1(C_b(R))$ makes sense:

Lemma 4.1. *If u is measurable and $u(\cdot,k)\in C_b(R)$ for almost every k, then* $\sup\limits_x u(x,k)$ *is measurable.*

Proof. $\left\{k|\sup\limits_x u(x,k)>K\right\}=\bigcup\limits_{x\in Q}\{k|u(x,k)>K\}$ modulo a null set.

Lemma 4.2. $L^1(C_b(R))$ *(with natural norm) is a Banach space.*

Proof. Let $\{u_m\}_{m\in N}$ be a Cauchy sequence in $L^1(C_b(R))$. Since after passing to a subsequence L^1-convergence implies convergence a.e.,

$$\lim_{m,n\to\infty}\ \sup_x |u_n(x,k)-u_m(x,k)|=0$$

for almost every k; hence u_m converges to a function u uniformly in x for a.e. k. Since $u=\lim\limits_{m\to\infty} u_m$ except for a null set, u is measurable, and clearly $u(\cdot,k)\in C_b(R)$ for a.e. k. It remains to be shown that

$$\int \sup_x |u(x,k)-u_m(x,k)|dk\to 0 \quad \text{as} \quad m\to\infty$$

$v_n(k) = \sup_x |u_n(x,k) - u_m(x,k)|$ is a Cauchy sequence in L^1 and hence convergent to $\sup_x |u(x,k) - u_m(x,k)|$. Therefore we find

$$\int \sup_x |u(x,k) - u_m(x,k)| dk \leqq \limsup_{n \to \infty} \int \sup_x |u_n(x,k) - u_m(x,k)| dk$$

which implies the lemma.

Lemma 4.3. *M (constructed as before) maps* $L^1(C_b(R))$ *into itself.*

Proof. Clearly the only difficult step is to show that Mu is measurable. Since the transformation $(x,k) \to (\zeta + x'_k, k)$ transforms measurable functions to measurable functions, it is sufficient to prove that $\tilde{M}u$ is measurable. But $L^1(C_b(R)) \subset C_b(L^1(R))$ [the space of all bounded continuous functions $R \to L^1(R)$], and one easily concludes from the construction of $\tilde{M}$ that $\tilde{M}$ maps $C_b(L^1(R))$ into itself. [To see this, we only have to reinterpret u and v in (4.4) as elements of $L^1(R)$.] Since all elements of $C_b(L^1(R))$ are measurable functions, the lemma is proved.

From Lemmas 4.1 to 4.3 we conclude that we may now perform the same construction as in Sect. 3 and obtain an infinite dimensional manifold of nonperiodic solutions.

The condition $\lim_{x \to -\infty} \int \sup_{\zeta \leqq x} |u(\zeta, k)| dk = 0$ defines a closed subspace of $L^1(C_b(R))$, which is mapped into itself by M, and as before we conclude from this fact that the solutions we have constructed vanish in the limit $x \to -\infty$.

Altogether we have proved:

Theorem 4.4. *There exists an infinite dimensional manifold of solutions to* (0.1), *which are nonperiodic in t and obey the boundary condition* $\lim_{r \to 0} u(r,t) = 0$ *uniformly with respect to t.*

5. Some Remarks on the Physical Significance of our Solutions

Equation (0.1) has been derived [1] from the SU(2) Yang-Mills equations by the special ansatz $A = d\theta$, $\phi = \Theta(0, u)$, where Θ is the matrix

$$\Theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

If θ has no singularities, this leads to a vanishing charge density. Therefore, all the solutions having continuous boundary values (particularly $u = 0$) on the t-axis may be interpreted as solutions of the Yang-Mills equations with zero charge. The physically more interesting solutions are those assuming the boundary values ± 1 on the t-axis. The solutions considered in [1] are of this type, and there the singularities on the t-axis are compensated by singularities of θ in such a way that ϕ is constant on the t-axis. These singularities in θ lead to point charges [1] located on the t-axis. We have seen in Sect. 2 that a special solution with boundary values ± 1 can be obtained as a limiting case of solutions with boundary value 0. We suspect that in a similar way solutions with boundary values ± 1 can be found on the boundary of the manifolds, the existence of which we have established in Sects. 3 and 4.

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