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An Inequality for Trace Ideals

J. L. van Hemmen[1] and T. Ando[2]

[1] Universität Heidelberg, Sonderforschungsbereich 123, Im Neuenheimer Feld 294, D-6900 Heidelberg 1, Federal Republic of Germany

[2] Division of Applied Mathematics, Research Institute of Applied Electricity, Hokkaido University, Sapporo, Japan

Abstract. We prove an inequality for trace ideals which relates the difference of two positive operators to the difference of their square roots. Inequalities involving operator-monotone functions more general than the square root, are considered as well.

1. Introduction

In practical work involving, for instance, the approach to an equilibrium in a harmonic chain [1], or the elementary excitation spectrum of a random ferromagnet [2], the following problem shows up. One is given two positive bounded operators whose difference is finite-dimensional, and one would like to prove that the difference of the square roots is trace class [3]. So we are led to the question: If A and B are positive bounded operators such that $(A-B)$ is trace class, when is it true that the difference of their square roots is trace class as well? We cannot expect that this is true in general. Simply take $B=0$ and A trace class; $A^{1/2}$ is Hilbert-Schmidt, but not necessarily trace class. So we need a supplementary condition. In Sect. 2 (Proposition 2.1) we will meet such a condition and get an estimate of the trace norm $\|A^{1/2}-B^{1/2}\|_1$ in terms of $\|A-B\|_1$. In connection with this estimate we then may look for meaningful generalizations.

They are two natural ways of generalizing the above problem. We consider them in turn. First we notice that the function $f(\lambda)=\lambda^{1/2}$ is of a very special nature. It has the property that for any positive A and B such that $A \leqq B$ we have $f(A) \leqq f(B)$. Such a function is called operator-monotone [4]. Is it true that Proposition 2.1, when appropriately modified, also holds for operator-monotone functions?

Next we observe that up to now we have singled out the trace norm. But what can be said if we replace the trace norm, which is in fact an l^1-norm, by an l^p-norm or, even more generally, by a symmetric norm [5, 6]? This question is considered in Sect. 3. Operator-monotone functions are included in the analysis in Sect. 4.

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2. A Trace Class Inequality

It is worthwhile to consider the problem first in the simplest possible context. The identity is denoted by $\mathbb{1}$.

Proposition 2.1. *If A and B are positive bounded operators on a Hilbert space $\mathfrak{H}$, one has for any μ such that*

$$A^{1/2}+B^{1/2}\geqq \mu\mathbb{1}, \tag{2.1}$$

the following inequality:

$$\mu\|A^{1/2}-B^{1/2}\|_1\leqq \|A-B\|_1, \tag{2.2}$$

where $\|A\|_1=\operatorname{tr}|A|=\operatorname{tr}(A^*A)^{1/2}$ is the trace norm.

Proof. We partly follow Powers and Størmer [7]. Denote the trace class by $\mathscr{I}_1$, i.e. A is in $\mathscr{I}_1$ if and only if $\|A\|_1<\infty$. There is nothing to prove if $(A-B)$ is not in $\mathscr{I}_1$, so let us suppose that $(A-B)$ is trace class.

The compact operators form a closed two-sided ideal Comp$(\mathfrak{H})$ in $B(\mathfrak{H})$, the bounded operators on $\mathfrak{H}$. Let π be the mapping from $B(\mathfrak{H})$ onto the quotient space $B(\mathfrak{H})/\text{Comp}(\mathfrak{H})$, a C^*-algebra too [8]. $\mathscr{I}_1$ is contained in Comp$(\mathfrak{H})$, so $(A-B)$ being in $\mathscr{I}_1$ implies

$$\pi(A)=\pi(B)\Rightarrow\pi(A)^{1/2}=\pi(B)^{1/2}\Rightarrow\pi(A^{1/2})=\pi(B^{1/2}), \tag{2.3}$$

because $\pi(A^{1/2})$ satisfies the requirement that $\pi(A^{1/2})^2=\pi(A)$ and the square root is unique in a C^*-algebra. Ergo, $(A^{1/2}-B^{1/2})$ is compact.

Define

$$S=(A^{1/2}-B^{1/2})\quad\text{and}\quad T=(A^{1/2}+B^{1/2}). \tag{2.4}$$

S is a compact self-adjoint operator having a complete orthonormal set of eigenvectors $\{e_i\}$ in $\mathfrak{H}$; $Se_i=\lambda_i e_i$. Notice moreover $|S|e_i=|\lambda_i|e_i$.

Evidently $\frac{1}{2}(ST+TS)=(A-B)$, and $T\geqq\mu\mathbb{1}$ by (2.1). Thus we get

$$\begin{aligned}\operatorname{tr}|A-B|&=\sum_i \tfrac{1}{2}(e_i,|ST+TS|e_i)\\ &\geqq\sum_i \tfrac{1}{2}|(e_i,[ST+TS]e_i)|\\ &=\sum_i |\lambda_i|(e_i,Te_i)\\ &\geqq\sum_i \mu|\lambda_i|=\mu\sum_i(e_i,|S|e_i)\\ &=\mu\operatorname{tr}|A^{1/2}-B^{1/2}|,\end{aligned} \tag{2.5}$$

i.e. the desired inequality (2.2). □

In particular, if A and B are strictly positive, with $(A-B)$ trace class, we get the result we alluded to above: $(A^{1/2}-B^{1/2})$ is trace class too. Phrased differently: taking the square root is continuous on the interior of the positive cone of $\mathscr{I}_1$.

3. Symmetric Norms

In this section the proof is given that Proposition 2.1 also holds if we replace the trace norm by a more general norm. Let Φ be a *symmetric norm* on sequences (see [6]). Given a compact operator A, let $\{\mu_n(A)\}$ be the singular values of A and

$$\|A\|_\Phi := \Phi(\{\mu_n(A)\}). \tag{3.1}$$

For non-compact A we put $\|A\|_\Phi = \infty$.

For $\Phi_p(\{x_n\}) := (\sum |x_n|^p)^{1/p} (1 \leqq p < \infty)$ we use $\|A\|_p$ instead of $\|A\|_{\Phi_p}$.

Lemma 3.1. *If S is compact and $T \geqq \mu \mathbb{1} \geqq 0$, then for any symmetric norm Φ*

$$\tfrac{1}{2}\|TS + ST\|_\Phi \geqq \mu \cdot \|S\|_\Phi. \tag{3.2}$$

Proof. Let $S = \sum \mu_n(S)(\phi_n, \cdot)\psi_n$ be the canonical decomposition of S, with orthonormal $\{\phi_n\}$ and $\{\psi_n\}$. Since $S\phi_n = \mu_n(S)\psi_n$ and $S^*\psi_n = \mu_n(S)\phi_n$, we have (by [6], Proposition 2.6; cf. also [5], Lemma 2.3.4)

$$\begin{aligned}
\|\tfrac{1}{2}(ST + TS)\|_\Phi &\geqq \Phi(\{(\psi_n, \tfrac{1}{2}(ST + TS)\phi_n)\}) \\
&= \Phi\left(\left\{\frac{\mu_n(S)}{2}(T\phi_n, \phi_n) + \frac{\mu_n(S)}{2}(T\psi_n, \psi_n)\right\}\right) \\
&\geqq \mu \cdot \Phi(\{\mu_n(S)\}) \\
&= \mu \cdot \|S\|_\Phi. \quad \square
\end{aligned} \tag{3.3}$$

Proposition 3.2. *If $A, B \geqq 0$ and $A^{1/2} + B^{1/2} \geqq \mu \mathbb{1} \geqq 0$, then for any symmetric norm Φ*

$$\|A - B\|_\Phi \geqq \mu \cdot \|A^{1/2} - B^{1/2}\|_\Phi. \tag{3.4}$$

In particular, for $1 \leqq p < \infty$,

$$\|A - B\|_p \geqq \mu \cdot \|A^{1/2} - B^{1/2}\|_p. \tag{3.5}$$

Proof. Define S and T according to (2.4). If $(A - B)$ is compact, so is S. Then apply Lemma 3.1. $\square$

4. Operator-Monotone Functions

In what follows $f(\lambda)$ is always a function which is well-defined and real-valued for all $\lambda \geqq 0$. The function $f(\lambda)$ is said to be *operator-monotone* if, for any bounded (positive) self-adjoint A and B, the relation $A \leqq B$ implies $f(A) \leqq f(B)$. It is a celebrated theorem of Löwner (see [4]) that a function is operator-monotone if and only if it belongs to the so-called Pick class. Then f has the unique integral representation [4]

$$f(\lambda) = \alpha + \beta\lambda - \int_0^\infty \left[\frac{1}{t + \lambda} - \frac{t}{t^2 + 1}\right] d\nu(t), \tag{4.1}$$

where α is real, $\beta \geqq 0$, and, in the present case, ν is a positive Borel measure on $(0, \infty)$ such that $\int_0^\infty (t^2 + 1)^{-1} d\nu(t)$ is finite. In Eq. (4.1) λ ranges through the complex plane cut along the negative real axis. For instance,

$$\sqrt{\lambda}=\frac{1}{\sqrt{2}}-\int_0^\infty\left[\frac{1}{t+\lambda}-\frac{t}{t^2+1}\right]\frac{\sqrt{t}}{\pi}dt \tag{4.2}$$

is a Pick function, equipped with the afore mentioned properties: well-defined and real-valued for all $\lambda \geqq 0$. Moreover we notice that $\sqrt{\lambda}$ is monotonically increasing when λ goes from 0 to ∞. This is generally true for any Pick function $f(\lambda)$ that is considered here.

Proposition 4.1. *Let A and B be positive. We assume $A+B\geqq\mu\mathbb{1}$ with $\mu>0$. Then for any operator-monotone function $f(\lambda)$ and any symmetric norm Φ,*

$$\|f(A)-f(B)\|_\Phi \leqq \left(\frac{f\left(\frac{\mu}{2}\right)-f(0)}{\frac{\mu}{2}}\right)\cdot\|A-B\|_\Phi. \tag{4.3}$$

Proof. Let $C=(A+B)/2$. Then $C\geqq\frac{\mu}{2}\mathbb{1}$ and $(C-A)=(B-A)/2$. The reason to introduce the intermediate C will become apparent later on. The integral representation (4.1) implies

$$f(A)-f(C)=\beta(A-C)-\int_0^\infty[(t\mathbb{1}+A)^{-1}-(t\mathbb{1}+C)^{-1}]d\nu(t). \tag{4.4}$$

Because

$$(t\mathbb{1}+A)^{-1}-(t\mathbb{1}+C)^{-1}=(t\mathbb{1}+A)^{-1}(C-A)(t\mathbb{1}+C)^{-1}, \tag{4.5}$$

we get

$$\|f(A)-f(C)\|_\Phi\leqq\beta\|A-C\|_\Phi+\int_0^\infty\|(t\mathbb{1}+A)^{-1}(A-C)(t\mathbb{1}+C)^{-1}\|_\Phi d\nu(t). \tag{4.6}$$

We now have to estimate the integrand in (4.6). Thereto we notice that, since $(t\mathbb{1}+A)^{-1}$ is invertible, $(A-C)(t\mathbb{1}+C)^{-1}(t\mathbb{1}+A)^{-1}$ has precisely the same eigenvalues as $(t\mathbb{1}+A)^{-1}(A-C)(t\mathbb{1}+C)^{-1}$; the latter is self-adjoint by (4.5). Thus ([6], Theorem 1.19; cf. also [5], pp. 86, 89, and 97)

$$\begin{aligned}&\|(t\mathbb{1}+A)^{-1}(A-C)(t\mathbb{1}+C)^{-1}\|_\Phi\\&\leqq\|A-C\|_\Phi\|(t\mathbb{1}+C)^{-1}(t\mathbb{1}+A)^{-1}\|\leqq t^{-1}\left(t+\frac{\mu}{2}\right)^{-1}\cdot\|A-C\|_\Phi.\end{aligned} \tag{4.7}$$

Here we used the inequality for C. And therefore

$$\|f(A)-f(C)\|_\Phi\leqq\mu^{-1}\|A-B\|_\Phi\cdot\left\{\beta\frac{\mu}{2}-\int_0^\infty\left[\frac{1}{t+\frac{\mu}{2}}-\frac{1}{t}\right]d\nu(t)\right\}, \tag{4.8}$$

i.e.

$$\|f(A)-f(C)\|_{\Phi} \leqq \mu^{-1}\left\{f\left(\frac{\mu}{2}\right)-f(0)\right\} \cdot \|A-B\|_{\Phi}. \tag{4.9}$$

Analogously we find

$$\|f(C)-f(B)\|_{\Phi} \leqq \mu^{-1}\left\{f\left(\frac{\mu}{2}\right)-f(0)\right\} \cdot \|A-B\|_{\Phi}. \tag{4.10}$$

Then the triangle inequality,

$$\|f(A)-f(B)\|_{\Phi} \leqq \|f(A)-f(C)\|_{\Phi} + \|f(C)-f(B)\|_{\Phi}, \tag{4.11}$$

implies the assertion (4.3). □

Plainly, there exists a ξ such that $0<\xi<\frac{1}{2}\mu$ and $[f(\mu/2)-f(0)]/(\frac{1}{2}\mu)=f'(\xi)$. This suggests that, after all, the requirement $\mu>0$ is superfluous and we can take the limit $\mu \downarrow 0$ so as to obtain the constant $f'(0^+)$ in (4.3). Typical operator-monotone functions are $f(\lambda)=\lambda^m$ with $0<m\leqq 1$. If $m=1$, $f'(0^+)$ does make sense in (4.3). However, for the other values of m we mentioned, $f'(0^+)=+\infty$. So we have to keep $\mu>0$ to get nontrivial results.

Corollary 4.2. *If $A, B \geqq 0$ and $A+B \geqq \mu \mathbb{1}$ with $\mu>0$, then for any symmetric norm Φ, and $0<m\leqq 1$,*

$$\|A^m - B^m\|_{\Phi} \leqq (2/\mu)^{1-m}\|A-B\|_{\Phi}. \tag{4.12}$$

Finally we return to our original problem where $m=\frac{1}{2}$ and $A^{1/2}+B^{1/2} \geqq \mu \mathbb{1}$. If $A^{1/2}+B^{1/2} \geqq \mu \mathbb{1}$, then

$$(A+B)^{1/2} \geqq (A^{1/2}+B^{1/2})/\sqrt{2} \geqq \mu/\sqrt{2}\mathbb{1}, \tag{4.13}$$

and hence

$$A+B \geqq \tfrac{1}{2}\mu^2 \mathbb{1}. \tag{4.14}$$

By the previous corollary $(\mu := \frac{1}{2}\mu^2)$

$$\|A^{1/2}-B^{1/2}\|_{\Phi} \leqq 2\mu^{-1} \cdot \|A-B\|_{\Phi}, \tag{4.15}$$

while Proposition 3.2 gives

$$\|A^{1/2}-B^{1/2}\|_{\Phi} \leqq \mu^{-1} \cdot \|A-B\|_{\Phi}. \tag{4.16}$$

In fact, μ^{-1} is the "best" constant. To see this, take $A^{1/2}=\frac{1}{2}\mu P_0+\mu P_1$ and $B^{1/2}=\frac{1}{2}\mu P_0$ with P_0 and P_1 orthogonal projection operators $(P_0P_1=0)$ such that $P_0+P_1=\mathbb{1}$ and P_1 is one-dimensional.

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