On a Characterization of the State Space of Quantum Mechanics

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Abstract. A characterization of state spaces of Jordan algebras by Alfsen and Shultz is improved to a form with more physical appeal (proposed by Wittstock) in the simplified case of a finite dimension.

1. Introduction

In recent years there have been a number of works on the characterization of state spaces of W^* and C^* algebras [1–3]. We present here another version, which seems to have a somewhat better physical appeal, though applicable only to a special situation (finite type I cases in the sequel).

Our axioms are very close to those of Alfsen and Shultz ([4], Sect. 6) for type I Jordan algebras except that we replace their P-projections by a weaker notion of filtering projections, which has been suggested by Wittstock [5].

The state space is assumed to be a compact convex set K as usual, where the convex combination of points in K represents a mixture of physical states, extremal points of K corresponding to pure states. We make the simplifying assumption that *the dimension of K is finite*.

The first axiom is in terms of *filtering projections*, which have a physical interpretation of the measuring process of the first kind for questions (observables with yes or no answers), as will be described in Definition 2.2. Axiom 0 in Sect. 3 requires the existence of sufficiently many filtering projections (i.e. one for each face of K) satisfying a certain consistency condition with an obvious physical interpretation.

The filtering projection P_φ associated with a pure state φ defines a number $\langle \varphi, \psi \rangle$ for another pure state ψ through the relation

$$P_\varphi \psi = \langle \varphi, \psi \rangle \varphi, \tag{1.1}$$

with an interpretation of the transition probability. Our second Axiom is

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Axiom $\mathscr{H}$. For any pair of pure states φ and ψ,

$$\langle \varphi, \psi \rangle = \langle \psi, \varphi \rangle. \tag{1.2}$$

The importance of this symmetry property of the transition probability in the axiomatization of quantum mechanics has been emphasized to the author by Haag since 1960 [6]. It has been treated also by other authors (see [7–9]).

The Axioms $\mathcal{O}$ and $\mathscr{H}$ combined are much stronger than each of them separately, as can be seen from simple examples of Sect. 11. In particular, the Axioms $\mathcal{O}$ and $\mathscr{H}$ together imply that the filtering projection of the Axiom $\mathcal{O}$ is a P-projection in the sense of Alfsen and Shultz [10]. The main mathematical content of this sequel is the proof of this statement.

Axioms $\mathcal{O}$ and $\mathscr{H}$ completely determine the case of rank 2 (i.e. when K has at most two pure states which are mutually orthogonal in the sense defined later). The set K in this case is a ball (of radius $1/\sqrt{2}$), which is the state space of the so-called spin factors [11, 12].

We have not been able to decide whether the Axioms $\mathcal{O}$ and $\mathscr{H}$ already lead to our conclusion. Therefore we tentatively introduce an additional Axiom:

Axiom $\mathscr{P}$. The filtering projections in Axiom $\mathcal{O}$ map pure states to multiples of pure states.

We feel that this Axiom is stronger than we need and are using only a small part of it to conclude that a finite dimensional K satisfies Axioms $\mathcal{O}$, $\mathscr{H}$, and $\mathscr{P}$ if and only if it is a direct sum of state spaces of the full matrix algebras (quantum mechanics) over real, complex and/or quaternion, the spin factors and/or the exceptional Jordan algebra M_3^8 [13]. This result follows from Axioms $\mathcal{O}$, $\mathscr{H}$, and $\mathscr{P}$ by [4] (especially Theorem 6.16) once we prove that the filtering projections of Axiom $\mathcal{O}$ are P-projections.

The case of complex field (the ordinary quantum mechanics) can be distinguished from other cases by its good behavior under composition of independent systems, as is discussed in Sect. 12.

2. Filtering Measurement and Filtering Projections

The following is an idealization of the quantum mechanical measurement (of the first kind) for questions (observables with yes and no answers alone):

Definition 2.1. A *filtering* $\hat{p}$ is a mapping from the state space K into $(K \times (0, 1]) \cup (0 \times 0)$, sending a state $\varphi \in K$ to a pair of a state $\hat{p} \cdot \varphi \in K \cup 0$ (a state coming out of the measurement when the measured value is "yes") and a number $\hat{p}(\varphi) \in [0, 1]$ (the probability of obtaining the measured value "yes" on the state φ, the special point 0 corresponding to the situation where the probability is 0 for the measured value "yes" and hence nothing comes out almost surely), which satisfies the following conditions:

(1) The repetition yields the same state with probability 1 unless $\hat{p} \cdot \varphi = 0$:

$$\hat{p} \cdot (\hat{p} \cdot \varphi) = \hat{p} \cdot \varphi, \qquad \hat{p}(\hat{p} \cdot \varphi) = 1. \tag{2.1}$$

(2) The mixture $\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2$ $(0 \leq \lambda \leq 1,\ \varphi_1 \in K,\ \varphi_2 \in K)$ behaves under a filtering $\hat{p}$ as if it is the state φ_1 with probability λ and the state φ_2 with probability $1-\lambda$:

$$\hat{p}(\varphi) = \lambda\hat{p}(\varphi_1) + (1-\lambda)\hat{p}(\varphi_2), \tag{2.2}$$

$\hat{p} \cdot \varphi = 0$ if $\hat{p}(\varphi) = 0$ and

$$\hat{p} \cdot \varphi = \{\hat{p}(\varphi)^{-1} \lambda \hat{p}(\varphi_1)\} \hat{p} \cdot \varphi_1 + \{\hat{p}(\varphi)^{-1}(1-\lambda)\hat{p}(\varphi_2)\} \hat{p} \cdot \varphi_2 \tag{2.3}$$

if $\hat{p}(\varphi) \neq 0$ where the term with $\hat{p} \cdot \varphi_i$ should be omitted whenever $\hat{p} \cdot \varphi_i = 0$.

(3) If the state comes out of the filtering with probability 1 for "yes" answer, then the state is not altered by the filtering:

$$\hat{p} \cdot \varphi = \varphi \quad \text{if} \quad \hat{p}(\varphi) = 1. \tag{2.4}$$

(4) There exists another map $\hat{p}'$ from K into $(K \times (0,1]) \cup (0 \times 0)$ (corresponding to the filtering measurement of the "no" answer) satisfying the same properties (1)–(3), such that it is complementary to $\hat{p}$ (i.e. the total probability is 1) in the following sense:

$$\hat{p}(\varphi) + \hat{p}'(\varphi) = 1 \quad \text{for any} \quad \varphi \in K. \tag{2.5}$$

To linearize this definition, we imbed K in a vector space V of one dimension higher such that the affine span of K in V does not contain the origin of V. Then the cone V_+ with the base K $\left(V_+ \equiv \bigcup_{\lambda \geq 0} \lambda K\right)$ defines an order $(\varphi_1 \leq \varphi_2$ iff $\varphi_2 - \varphi_1 \in V_+)$ and a norm

$$\|\varphi\| = \inf\{|\alpha| + |\beta|;\ \varphi = \alpha\varphi_1 + \beta\varphi_2,\ \varphi_1 \in K,\ \varphi_2 \in K\} \tag{2.6}$$

in V.

Definition 2.2. A *filtering projection* p is a linear mapping of V into V with the following properties:

(a) p is a projection: $p^2 = p$.
(b) p is positive: $pV_+ \subset V_+$.
(c) p is contractive: $\|p\| \leq 1$. (Equivalently, $\varphi \in K$ implies $\|p\varphi\| \leq 1$.)
(d) p is neutral: $\|p\varphi\| = \|\varphi\|$ implies $p\varphi = \varphi$.
(e) p is complemented: there exists another positive, contractive, neutral projection p' satisfying for every $\varphi \in K$

$$\|p\varphi\| + \|p'\varphi\| = 1. \tag{2.7}$$

The two definitions are related by the following:

Proposition 2.3. *If p is a filtering projection, then $\hat{p}$ defined for $\varphi \in K$ by*

$$\hat{p} \cdot \varphi = 0 \quad \text{if} \quad p\varphi = 0, \tag{2.8}$$

$$\hat{p} \cdot \varphi = \|p\varphi\|^{-1} p\varphi \quad \text{if} \quad p\varphi \neq 0, \tag{2.9}$$

$$\hat{p}(\varphi) = \|p\varphi\| \tag{2.10}$$

is a filtering. Conversely, if $\hat{p}$ is a filtering, there exists a unique filtering projection p such that (2.8), (2.9), and (2.10) holds for the given $\hat{p}$ and any $\varphi \in K$.

Remark 2.4. [In Definition 2.2, the contractive property (c) for p and p' actually follows from the properties (b) and (2.7).]

Remark 2.5. The dual of V can be identified with the set $A(K)$ of all affine functions on K (by restriction and linear extension) and the polar of V_+ is given by

$$A^+(K) = \{x \in A(K);\ x(\varphi) \geq 0\ (\varphi \in K)\}.$$

In particular the norms of $\varphi \in V_+$ and $x \in A(K)$ can be expressed in terms of $e \in A(K)$ satisfying $e(\psi) = 1$ for all $\psi \in K$ by

$$\|\varphi\| = e(\varphi), \tag{2.11}$$

$$\|x\| = \inf\{\lambda \geq 0;\ -\lambda e \leq x \leq \lambda e\}. \tag{2.12}$$

[The unit balls of V and $A(K)$ are the convex hull of K and $-K$, and the double cone with vertices e and $-e$.]

Proof of Remark 2.4. By (b) and (e), pK is in the convex hull of K and 0. Hence p maps the unit ball of B into itself.

Proof of Proposition 2.3. If p is a filtering projection, then $\hat{p}$ defined by (2.8), (2.9), and (2.10) maps K into $(K \times (0,1]) \cup (0 \times 0)$ due to (b) and (c) and clearly satisfies (2.1) due to (a), (2.2) due to (2.10) and (2.11), (2.3) due to the definition (2.9) and (2.10), (2.4) due to the neutrality (d) and the condition (4) due to (e).

Conversely, if $\hat{p}$ is a filtering, then define $p\varphi = 0$ if $\hat{p} \cdot \varphi = 0$ and

$$p\varphi = \hat{p}(\varphi)\hat{p} \cdot \varphi \tag{2.13}$$

if $\hat{p} \cdot \varphi \neq 0$ for $\varphi \in K$ and

$$p\varphi = \alpha p\varphi_1 + \beta p\varphi_2 \tag{2.14}$$

whenever $\varphi = \alpha\varphi_1 + \beta\varphi_2$ with $\varphi_1 \in K$ and $\varphi_2 \in K$. Then p defined on K by (2.13) is affine due to (2.3) and hence $p\varphi$ given by (2.14) does not depend on the decomposition $\varphi = \alpha\varphi_1 + \beta\varphi_2$, and p is a linear map of V into V. The property (a) follows from (2.1), (b) from (2.13) and $\hat{p}(\varphi) \geq 0$, (d) from (2.4) and (2.13) which implies $\|p\varphi\| = \hat{p}(\varphi)$ for $\varphi \in K$, (c) by Remark 2.4 and (e) from (4).

3. Faces of K

Definition 3.1. A *face* f of K is a convex subset of K such that

$$\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2 \quad \text{for} \quad \varphi \in f, \quad \varphi_1 \in K, \quad \varphi_2 \in K$$

and

$$0 < \lambda < 1 \quad \text{imply} \quad \varphi_1 \in f \quad \text{and} \quad \varphi_2 \in f.$$

A face is a subset of K stable under mixing $(\varphi_1, \varphi_2 \to \varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2)$ and purification $(\varphi \to \varphi_1, \varphi_2)$. A face consisting of one point is an *extremal point* (a *pure state*).

We denote the set of all faces of K by $F(K)$ and the set of all extremal points by $F_0(K)$. Likewise, we denote the set of all non-empty faces of V_+ and $A^+(K)$ by

$F'(V_+)$ and $F'(A^+)$, respectively, and the set of all one-dimensional faces (extremal rays) of V_+ and $A^+(K)$ by $F_1(V_+)$ and $F_1(A^+)$.

Lemma 3.2. *The following gives* $1-1$ *relation between* $F(K)$ *and* $F'(V_+)$.

$$\hat{f} = \bigcup_{\lambda \geqq 0} \lambda f \in F'(V_+), \qquad f = \hat{f} \cap K \in F(K), \tag{3.1}$$

where $F_0(K)$ *is in* $1-1$ *relation with* $F_1(K)$ *and* $\hat{f} \equiv 0$ *if* $f = \emptyset$.

The proof is immediate as V is a convex cone with a compact base K.

Lemma 3.3. *If* p *is a filtering projection,*

$$\mathrm{Im}^+\, p \equiv pV \cap V_+ = pV_+ \in F'(V_+), \tag{3.2}$$

$$pV \cap K = pV_+ \cap K = pK \cap K \in F(K). \tag{3.3}$$

The two faces are related by the $1-1$ *correspondence of Lemma 3.2.*

By this lemma, a face of K is associated to each filtering projection. It is the totality of states coming out of the corresponding filtering measurement. In general, a face can be associated to many different filtering projections (in contrast to a P-projection of Alfsen and Shultz) as in an example of Sect. 11.

We are now ready to state one of our Axioms.

Axiom $\mathcal{O}$. For each $f \in F(K)$, there corresponds a filtering projection p_f such that

   (i) $p_f V \cap K = f$ (cf. Lemma 3.3),
   (ii) if $f_1 \subseteqq f_2$, then $p_{f_1} p_{f_2} = p_{f_1}$,
   (iii) for each $f \in F(K)$, there is another $f' \in F(K)$ for which $p_{f'}$ is complementary to p_f in the sense of (2.7).

Remark 3.4. The condition (ii) expresses a coherence of filtering to a bigger face and a smaller face. The following relation, however, holds without an assumption:

$$p_{f_2} p_{f_1} = p_{f_1} \quad \text{if} \quad f_1 \subseteqq f_2, \tag{3.4}$$

because of the following general property of a filtering projection:

Lemma 3.5. *The image of a filtering projection p (denoted by* $\mathrm{Im}\, p = pV$ *is the linear span of the associated face $pK \cap K$).*

Because of (3.4), the condition (ii) can also be formulated as the compatibility $\hat{p}_{f_1} \hat{p}_{f_2} = \hat{p}_{f_2} \hat{p}_{f_1}$ of the associated filtering measurements. The condition (i) simply says $\hat{p}_f \cdot K = f \cup 0$ (or K if $f = K$).

Proposition 3.6. *If K satisfies the Axiom $\mathcal{O}$, any of its faces also satisfies the Axiom $\mathcal{O}$. The same holds for the Axiom $\mathcal{H}$.*

Main reasons for this are Lemma 3.5 and the following immediate consequence of Definition 3.1:

Lemma 3.7. *For two faces f_1 and f_2 of K, $F(f_1) \subseteqq F(f_2)$ and $F_0(f_1) \subseteqq F_0(f_2)$ if $f_1 \subseteqq f_2$.*

Proof of Lemma 3.3. Since $V \supset V_+ \supset K$, we have

$$pV \cap K \supset pV_+ \cap K \supset pK \cap K,$$

while $\varphi \in pV$ satisfies $p\varphi = \varphi$ due to $p^2 = p$ and hence $\varphi \in pV \cap K$ implies $\varphi = p\varphi \in pK \cap K$. Therefore equalities in (3.3) hold.

Next $pV \cap K$ is convex, being an intersection of convex sets. To prove that it is a face of K, let

$$\varphi \in pV \cap K, \quad \varphi = \lambda \varphi_1 + (1 - \lambda)\varphi_2, \quad \varphi_1 \in K, \quad \varphi_2 \in K, \quad 0 < \lambda < 1.$$

Since $p\varphi = \varphi$, we have $\lambda p\varphi_1 + (1 - \lambda)p\varphi_2 = \varphi$ and hence

$$\lambda e(p\varphi_1) + (1 - \lambda)e(p\varphi_2) = e(\varphi) = 1.$$

By (2.11) and (c), we have $e(p\varphi_j) \leq 1$ $(j = 1, 2)$ and hence $e(p\varphi_1) = e(p\varphi_2) = 1$. By the neutrality (d), we conclude that $p\varphi_1 = \varphi_1$, $p\varphi_2 = \varphi_2$ and hence $\varphi_1 \in pK \cap K$ and $\varphi_2 \in pK \cap K$. This shows that $pV \cap K$ is a face of K.

Since any subcone V_1 of V_+ is determined by $V_1 \cap K$ and is a face if and only if $V_1 \cap K \in F(K)$ due to Lemma 3.2. Hence (3.3) implies the rest of the conclusion of the lemma.

Proof of Lemma 3.5. Since $V = V_+ - V_+$, $pV = pV_+ - pV_+$. Since pV_+ is in the linear span of $pV_+ \cap K$ (due to $pV_+ \subset V_+$), pV is in the linear span of $pK \cap K$, by (3.3).

Proof of Proposition 3.6 will be given in the next section.

4. Lattice of Faces

Faces of a convex set form a lattice relative to the order by inclusion with the whole set and empty set as the largest element and the smallest element, respectively, due to the following Lemma.

Lemma 4.1. (i) *An intersection of an arbitrary number of faces is a face.*
   (ii) *There exists the smallest face $f_a(L)$ containing a given subset L of K.*
   (iii) $f_a(L) = \{\varphi ; \exists \psi_j \in L, \varphi' \in K, \lambda \in (0, 1]$,

$$\lambda_j \geq 0, \sum \lambda_j = 1, \sum \lambda_j \psi_j = \lambda \varphi + (1 - \lambda)\varphi'\}. \tag{4.1}$$

Our aim of this section is to prove the following:

Theorem 4.2. *Under Axiom $\mathcal{O}$, $F(K)$ is an orthocomplemented, orthomodular lattice.*

Here the *orthocomplementation* is $f \to f'$ given by Axiom $\mathcal{O}$(iii). If $f_1 \subseteq f_2'$, then we say that f_1 and f_2 are *(lattice) orthogonal*. The *orthomodularity* means that a sublattice generated by such f_1 and f_2 is modular, which is equivalent to the validity of the following equality for all such f_1, f_2:

$$f_2' = f_1 \vee (f_1' \wedge f_2'). \tag{4.2}$$

(These properties have been treated, for example, in [14, 8].)

As a tool for the proof, we denote the transpose of p_f acting on the dual $A(K)$ of V by p_f^* and

$$e_f \equiv p_f^* e. \tag{4.3}$$

The Eq. (2.7) for p_f and $p_{(f')}$ is equivalent to

$$e_f + e_{(f')} = e \tag{4.4}$$

because K is total in V.

Lemma 4.3. *Under Axiom $\mathcal{O}$, the following holds for $f \in F(K)$.*
   (i) $f = \{\varphi \in K; e_f(\varphi) = 1\}$.
   (ii) $f = \{\varphi \in K; e_{f'}(\varphi) = 0\}$.
   (iii) $f = K \cap \ker p_{f'}$ (ker *denotes "the kernel of"*).
   (iv) $f \wedge f' = \emptyset$.

Proof of Lemma 4.1. (i) follows from Definition 3.1 and (ii) from (i) (take the intersection of all faces containing L). The right hand side of (4.1) is convex, contains L (set $\psi_j = \varphi$, $\lambda = 1$) and is contained in any face containing L. Hence it must be equal to $f_a(L)$.

Proof of Lemma 4.3. (i) $\|p_f \varphi\| = e(p_f \varphi) = e_f(\varphi) = 1$ for $\varphi \in K$ implies $p_f \varphi = \varphi$ by the neutrality of p_f and hence $\varphi \in f$ by Axiom $\mathcal{O}$(i). Conversely, $\varphi \in f$ implies $e_f(\varphi) = e(p_f \varphi) = 1$.
   (ii) follows from (i) and (4.4) due to $e(\varphi) = 1$ for $\varphi \in K$.
   (iii) follows from (ii) since $(\|p_{f'} \varphi\| =) e_{f'}(\varphi) = 0$ is equivalent to $p_{f'} \varphi = 0$.
   (iv) follows from (i) for f' and (ii) for f because $e_{f'}(\varphi) = 1$ and $e_{f'}(\varphi) = 0$ are incompatible.

Proof of Theorem 4.2 is divided into several steps:
   (I) $f \rightarrow f'$ is involutive: By (4.4), we have $e_f = e_{(f')'}$ and by Lemma 4.3 (i), we have $f = (f')'$.
   (II) $f_1 \subseteq f_2$ if and only if $f_1' \supseteq f_2'$: If $f_1' \supseteq f_2'$, Axiom $\mathcal{O}$(ii) implies $\ker p_{f_1'} \subseteq \ker p_{f_2'}$. This implies $f_1 \subseteq f_2$ due to Lemma 4.3(iii). The converse follows from this conclusion due to the step (I).
   (III) $f \vee f' = K$ and $f \wedge f' = \emptyset$: The latter due to Lemma 4.3(iv) and the former due to the latter and the step I. Note that $p_\emptyset$ must be 0, p_K must be an identity, and $K' = \emptyset$.
   The above three steps proves that $F(K)$ is an orthocomplemented lattice. We now derive a few identities leading to the proof of orthomodularity.
   (IV) If $f_1 \leq f_2$, then $p_{f_2}^* e_{(f_1')} = p_{f_1}^* e_{f_2}$: By applying $p_{f_2}^*$ on (4.4) for f_1 and using Axiom $\mathcal{O}$(ii) in the form $p_{f_2}^* p_{f_1}^* = p_{f_1}^*$, we obtain [cf. (4.3)]

$$e_{f_1} + p_{f_2}^* e_{(f_1')} = e_{f_2}. \tag{4.5}$$

By adding $e_{(f_2')}$ and using (4.4), we obtain

$$e_{f_1} + p_{f_2}^* e_{(f_1')} + e_{(f_2')} = e. \tag{4.6}$$

Likewise by applying $p_{f_1'}^*$ on (4.4) for f_2 and using Axiom $\mathcal{O}$(ii) for $f_1' \geq f_2'$ in the form $p_{f_1'}^* p_{f_2'}^* = p_{f_2'}^*$, we obtain

$$p_{f_1'}^* e_{f_2} + e_{f_2'} = e_{f_1'} \tag{4.7}$$

and hence

$$e_{f_1} + p_{f_1'}^* e_{f_2} + e_{f_2'} = e. \tag{4.8}$$

Comparing (4.6) and (4.8), we obtain the desired relation.

(V) If $f_1 \leq f_2$, then $p_{f_1'}^* e_{f_2} = e_{(f_1' \wedge f_2)}$: Consider

$$f \equiv K \cap \ker(p_{f_2} p_{f_1'}). \tag{4.9}$$

It is convex as an intersection of convex sets. If $\varphi \in f$, $0 < \lambda < 1$, $\varphi_1 \in K$, $\varphi_2 \in K$ and $\varphi = \lambda \varphi_1 + (1 - \lambda)\varphi_2$, then

$$\lambda p_{f_2} p_{f_1'} \varphi_1 + (1 - \lambda) p_{f_2} p_{f_1'} \varphi_2 = 0. \tag{4.10}$$

Hence $p_{f_2} p_{f_1'} \varphi_1$ and $p_{f_2} p_{f_1'} \varphi_2$ being both in V_+ due to definition (2.2) (b), must be 0. $[0 = e^\perp \cap V_+$ is in $F_0(V_+)$.] This implies that both φ_1 and φ_2 are in f. Hence $f \in F(K)$.

By (4.9), $f \supseteq K \cap \ker p_{f_1'} = f_1$ [Lemma 4.3(iii)]. By the step (IV) and the same argument, we also have $f \supseteq f_2'$. Hence

$$f \geq f_1 \vee f_2'. \tag{4.11}$$

Since the image of $p_{(f_1 \vee f_2')}$ is the linear span of $f_1 \vee f_2'$ by Lemma 3.5, (4.11) and the definition (4.9) imply

$$p_{f_2} p_{f_1'} p_{(f_1 \vee f_2')} = 0 \tag{4.12}$$

and hence

$$p_{(f_1 \vee f_2')}^* p_{f_1'}^* e_{f_2} = 0. \tag{4.13}$$

By multiplying $p_{(f_1 \vee f_2')}^*$ on (4.8) and using Axiom $\mathcal{O}$(ii) for $f_1 \vee f_2' \geq f_1$ and $f_1 \vee f_2' \geq f_2'$, we obtain

$$e_{f_1} + e_{f_2'} = e_{(f_1 \vee f_2')}. \tag{4.14}$$

Using (4.4) for $(f_1 \vee f_2')$ and (4.8), we obtain from (4.14)

$$e_{(f_1 \vee f_2')'} = e - e_{f_1} - e_{(f_2')} = p_{f_1'}^* e_{f_2}. \tag{4.15}$$

In view of the relation $(f_1 \vee f_2')' = f_1' \wedge f_2$, this is the desired relation.

(VI) If $f_1 \leq f_2$, then $f_2 = f_1 \vee (f_1' \wedge f_2)$: since $f_1' \geq (f_1' \wedge f_2)$, the equality (4.14) for the pair $(f_1' \wedge f_2)$ and f_1' instead of f_1 and f_2 implies

$$e_{f_1 \vee (f_1' \wedge f_2)} = e_{(f_1' \wedge f_2)} + e_{f_1} = e_{f_2}, \tag{4.16}$$

where the last equality is due to (4.5) and the steps (IV), (V). By Lemma 4.3(i), this implies the desired equality and hence the orthomodularity. [Replace f_2 by f_2' to get (4.2).]

Proof of Proposition 3.6. Given $g \in F(K)$. For each $f \in F(g)$, we associate the restriction p_f^g of p_f to the linear span V_g of g. By Lemma 3.7, f is a face of K and hence p_f is provided by Axiom $\mathcal{O}$ for K. Since the image of p_f is the linear span of $f (\subseteq g)$, p_f^g is a linear map of V_g into V_g and is a projection. By Axiom $\mathcal{O}$(i) for K, and (3.2), $V_+ \cap V_g \equiv V_+^g$ is the cone with base g. Hence p_f^g is positive and neutral. We use

$$f^{(g)} = f' \cap g$$

to be the complementation in Axiom $\mathcal{O}$(iii) for g. Due to (4.2), we have $g = f \vee f^{(g)}$ [in $F(K)$] and hence

$$e_f + e_{f^{(g)}} = e_g$$

due to (4.14) for the pair f and $(f^{(g)})' = f \vee g' \geq f$ instead of f_1 and f_2. Since $e_g(\varphi) = 1$ for all $\varphi \in g$, we have

$$e_g(p_f \varphi) + e_g(p_{f^{(g)}} \varphi) = 1$$

for $\varphi \in g$. Hence Axiom $\mathcal{O}$(iii) is satisfied with the complementation $f \to f^{(g)}$. The contractivity of p_f then follows due to Remark 2.4. Thus Axiom $\mathcal{O}$ is satisfied for g.

Since $F_0(g) \subset F_0(K)$ (Lemma 3.7), Axiom $\mathcal{H}$ holds automatically for g for this choice of p_f if it is true for K.

5. Spectral Theory

As a technical tool, we derive decomposition of an element in $A(K)$. The role of spectral projections is played by e_{f_i} with mutually lattice-orthogonal f_i.

Lemma 5.1. If f_i $(i = 1, \ldots, n)$ are mutually lattice-orthogonal faces of K and $f = \bigvee_{i=1}^{n} f_i$, then Axiom $\mathcal{O}$ implies

$$e_f = \sum_{i=1}^{n} e_{f_i}. \tag{5.1}$$

This has already been obtained in (4.14) for $n = 2$ and the general case follows by repetition.

For each face f of K, we can associate the following faces of $A(K)$:

Definition 5.2. For $f \in F(K)$, we denote

$$f^{\perp} \equiv \{a \in A^+(K); \, a(\varphi) = 0 \text{ for all } \varphi \in f\}, \tag{5.2}$$

$$f^* \equiv \text{the face of } A^+(K) \text{ generated by } e_f. \tag{5.3}$$

A similar notation $F^{\perp}$ is used for $F \in F(A^+)$, where $F^{\perp} \subset K$.

It is straightforward to prove that $f^{\perp}$ is a face of $A^+(K)$ and $F^{\perp}$ is a face of K. [This is true for any subsets f of K and F of $A^+(K)$.]

Proposition 5.3. *Under Axiom $\mathcal{O}$, the following holds:*
  (i) $f \to (f')^{\perp}$ *is an order isomorphism of $F(K)$ into $F'(A^+)$.*
  (ii) $(f^*)^{\perp} = f', \, f^* \subseteq (f')^{\perp}$.

In general, the map of Proposition 5.3(i) is neither surjective nor a lattice isomorphism. If it is surjective, then the spectral analysis can be performed as follows:

Proposition 5.4. *If $(f')^{\perp}$ with $f \in F(K)$ exhausts non-empty faces of $A^+(K)$, the following holds under Axiom $\mathcal{O}$:*
  (i) *For any face f of K and any non-empty face F of $A^+(K)$,*

$$f^* = (f')^{\perp}, \qquad (F^{\perp})^{\perp} = F. \tag{5.4}$$

(ii) *Any $x \in A(K)$ has the following unique decomposition:*

$$x = \sum_{i=1}^{n} \mu_i e_{f_i}, \tag{5.5}$$

where μ's are a strictly increasing sequence of non-zero reals, and f's are mutually lattice-orthogonal non-empty faces of K.

Proof of Proposition 5.3. (i) For any $f \in F(K)$, the definition (5.2) implies $(f^\perp)^\perp \supset f$ and therefore

$$(f^\perp)^\perp = f \tag{5.6}$$

because Lemma 4.3(ii) implies $e_{f'} \in f^\perp$ and $(f^\perp)^\perp \subset (e_{f'})^\perp = f$. Therefore $f_1^\perp = f_2^\perp$ implies $f_1 = f_2$. Since $f_1 \supseteq f_2$ implies $f_1^\perp \subseteq f_2^\perp$, $f \to (f')^\perp$ is an order isomorphism.

(ii) By usual argument, $(x = \lambda x_1 + (1 - \lambda)x_2$ and $x(\varphi) = 0$ implies $x_i(\varphi) = 0$ if $x_i(\varphi) \geq 0$ and $0 < \lambda < 1)$, (5.3) implies $(f^*)^\perp = (e_f)^\perp$ which coincides with f' due to Lemma 4.3(ii). This implies

$$f^* \subseteq \{(f^*)^\perp\}^\perp = (f')^\perp. \tag{5.7}$$

Proof of Proposition 5.4(i). $(F^\perp)^\perp = F$ follows from (5.6) for example (or from a general argument). Then $f^* = (f')^\perp$ follows from Proposition 5.3(ii). Proof of Proposition 5.4(ii) is in Appendix.

6. The Inner Product

We investigate the consequence of Axiom $\mathcal{H}$ alone. By this we mean that the existence of a filtering projection p_f is assumed for each pure states but not for any other face f, in addition to (1.2).

Proposition 6.1. *Under Axiom $\mathcal{H}$,*

$$\hat{e}\psi \equiv \sum \lambda_j e_{\varphi_j} \in A(K) \tag{6.1}$$

for

$$\psi = \sum \lambda_j \varphi_j \quad \varphi_j \in F_0(K) \tag{6.2}$$

is independent of the decomposition (6.2) of $\psi \in V$ and linearly depends on $\psi \in V$.

$$\langle \psi_1, \psi_2 \rangle \equiv \hat{e}\psi_1(\psi_2) \tag{6.3}$$

defines a symmetric bilinear form on V and induces a non-degenerate symmetric bilinear form

$$\langle \hat{e}\psi, \hat{e}\varphi \rangle \equiv \langle \psi, \varphi \rangle \tag{6.4}$$

on the image $A^e \equiv \hat{e}V$ of the map $\hat{e}$. The kernel V_0 of this map is the intersection of $\ker p_\varphi$ for all $\varphi \in F_0(K)$ and the annihilator of V_0 in $A(K)$ is A^e.

The target of the next two sections is the proof of $V_0 = 0$ and a stronger property that the inner product on V is positive definite. The following technical lemma, which shows some positive definiteness, can be proved under Axiom $\mathcal{H}$ alone.

Lemma 6.2. *For $\psi_1 \in K$ and $\psi_2 \in K$,*

$$1 \geqq \langle \psi_1, \psi_2 \rangle \geqq 0. \tag{6.5}$$

The equality $\langle \psi_1, \psi_2 \rangle = 1$ holds if and only if $\psi_1 = \psi_2$ and it is extremal. For any $\psi \in K$,

$$\langle \psi, \psi \rangle > 0. \tag{6.6}$$

If ψ_1 and ψ_2 are distinct extremal points of K,

$$\langle \psi_1 - \psi_2, \psi_1 - \psi_2 \rangle > 0. \tag{6.7}$$

Proof of Proposition 6.1. For any $\varphi \in F_0(K)$,

$$\sum \lambda_j e_{\varphi_j}(\varphi) = \sum \lambda_j e_\varphi(\varphi_j) = e_\varphi(\psi), \tag{6.8}$$

which depends only on ψ and is independent of the decomposition (6.2). Since K is the convex hull of φ in $F_0(K)$ and is total in V, we obtain the independence of (6.1) on the decomposition (6.2). Likewise, the linear dependence of $\hat{e}\psi$ on ψ follows from the linear dependence of (6.8) on ψ. By (6.8),

$$\hat{e}\psi(\varphi) = \hat{e}\varphi(\psi) \tag{6.9}$$

holds for $\psi \in V$ and $\varphi \in F_0(K)$ and hence for any $\varphi \in V$ due to the linear dependence of both sides of the equation on φ. Therefore (6.3) defines a symmetric bilinear form on V.

The kernel V_0 of the map $\hat{e}$ is the set of all $\varphi \in V$ which is orthogonal to every ψ in V relative to $\langle \varphi, \psi \rangle$. Hence (6.4) depends only on $\hat{e}\psi$ and $\hat{e}\varphi$ (and not ψ and φ directly) and is non-degenerate.

By the symmetry of (6.3), V_0 is also the set of all $\psi \in V$ which are annihilated by all $\hat{e}\psi$, $\psi \in V$ and hence is the annihilator of A^e in V. By bipolar theorem, A^e is the annihilator of V_0 in $A(K)$.

By (1.1), which is consistent with our definition (6.3) for a pure state ψ and hence for any $\psi \in V$ for any fixed $\varphi \in F_0(K)$, $\psi \in V_0$ is in $\ker p_\varphi$ for all $\varphi \in F_0(K)$. Conversely, if $p_\varphi \psi = 0$ for all $\varphi \in F_0(K)$, then $\hat{e}\varphi(\psi) = 0$ for all $\varphi \in F_0(K)$ and hence for all $\varphi \in V$. Hence $\psi \in V_0$.

Proof of Lemma 6.2. If ψ_1 and ψ_2 are extremal points of K, then the positivity and the contractivity of the filtering projection p_φ implies (6.5). In addition, $\langle \psi_1, \psi_2 \rangle = 1$ implies $p_{\psi_1} \psi_2 = \psi_1 \in K$ and hence $p_{\psi_1} \psi_2 = \psi_2$ due to neutrality. On the other hand, $\langle \psi, \psi \rangle = 1$ for a pure state ψ due to $p_\psi \psi = \psi$. Thus $\langle \psi_1, \psi_2 \rangle = 1$ for pure ψ_1 and ψ_2 holds if and only if $\psi_1 = \psi_2$.

K is the convex hull of its extremal points. Let $\psi_i = \sum_k \lambda_{ik} \psi_{ik}$, $\lambda_{ik} \geqq 0$, $\psi_{ik} \in F_0(K)$ $(i = 1, 2)$. Then $\psi_i \in K$ implies $\sum_k \lambda_{ik} = 1$. We have

$$\langle \psi_i, \psi_j \rangle = \sum_{k,l} \lambda_{ik} \lambda_{jl} \langle \psi_{ik}, \psi_{jl} \rangle. \tag{6.10}$$

Therefore $\langle \psi_{1j}, \psi_{2k} \rangle \geqq 0$, which has been already established, implies $\langle \psi_i, \psi_j \rangle \geqq 0$ and, since $\langle \psi_{ik}, \psi_{ik} \rangle = 1$, we also have $\langle \psi_i, \psi_i \rangle \geqq \sum_j (\lambda_{ij})^2 > 0$, i.e. (6.6).

If we use $\langle \psi_{ik}, \psi_{jl} \rangle \leqq 1$, which has been already established, in (6.10), we obtain

$$\langle \psi_i, \psi_j \rangle \leqq \sum_{k,l} \lambda_{ik} \lambda_{jl} = 1 \,.$$

The equality holds if and only if $\langle \psi_{ik}, \psi_{jl} \rangle = 1$, i.e. if and only if $\psi_{ik} = \psi_{jl}$ for all k and l, which is equivalent to $\psi_i = \psi_{ik} = \psi_{jl} = \psi_j$. Thus $\langle \psi_1, \psi_2 \rangle = 1$ for $\psi_1, \psi_2 \in K$ if and only if $\psi_1 = \psi_2$ and it is extremal.

(6.7) follows from $(\psi_1, \psi_1) = (\psi_2, \psi_2) = 1$ and $(\psi_1, \psi_2) < 1$ for distinct extremal ψ_1 and ψ_2.

7. Lattice and Metric Orthogonalities

This section treats easy consequences of Axioms $\mathcal{O}$ and $\mathcal{H}$ together. Axiom $\mathcal{O}$ brings in the notion of lattice-orthogonality of faces and Axiom $\mathcal{H}$ brings in the metric orthogonality relative to the inner product (6.3). The two coincides due to the following:

Proposition 7.1. *Under Axioms $\mathcal{O}$ and $\mathcal{H}$,*

$$f' = \{ \varphi \in K \, ; \langle \varphi, \psi \rangle = 0 \text{ for all } \psi \in f \} \,. \tag{7.1}$$

Two faces f_1 and f_2 of K are lattice-orthogonal if and only if they are metrically orthogonal, which is also equivalent to $\langle e_{f_1}, e_{f_2} \rangle = 0$.

The notion of the rank of a face can be introduced in terms of mutually orthogonal extremal points of the face.

Proposition 7.2. *Under Axioms $\mathcal{O}$ and $\mathcal{H}$, $e_f \in A^e$ and*

$$r_f = \langle e_f, e_f \rangle \tag{7.2}$$

is a positive integer for each non-empty face f of K. (It is 0 for the empty set $f = \emptyset$.) The number k of mutually orthogonal non-empty faces $\varphi_1, \ldots, \varphi_k$ of f is at most r_f. $k = r_f$ can be achieved. Then each φ_i is necessarily extremal and

$$f = \bigvee_{i=1}^{r_f} \varphi_i, e_f = \sum_{i=1}^{r_f} e_{\varphi_i} \,. \tag{7.3}$$

Definition 7.3. r_f is called the *rank* of f.

Corollary 7.4. (i) *If $f_1 \gneqq f_2$, then $\operatorname{rank} f_1 > \operatorname{rank} f_2$.*

(ii) *If $f = \bigvee_{i=1}^{k} f_i$ and all f_i are mutually orthogonal, then*

$$\operatorname{rank} f = \sum_{i=1}^{k} \operatorname{rank} f_i \,. \tag{7.4}$$

Proof of Proposition 7.1. If φ and ψ are extremal points of K satisfying $\varphi' \geqq \psi$ (lattice-orthogonality), then ψ is in $\ker p_\varphi$ due to Lemma 4.3(iii) (with f replaced by φ') and hence $\langle \varphi, \psi \rangle = 0$ (metric orthogonality). Since any face of K is a convex hull of its extremal points, this result implies that two faces are metrically orthogonal if they are lattice-orthogonal.

If $\varphi \in K$ satisfies $\langle \varphi, \psi \rangle = 0$ for all $\psi \in f$ for some face f of K, and if $\varphi = \lambda \varphi_1 + (1 - \lambda) \varphi_2$ with $0 < \lambda < 1$, $\varphi_1 \in K$ and $\varphi_2 \in K$, then

$$0 = \langle \varphi, \psi \rangle = \lambda \langle \varphi_1, \psi \rangle + (1 - \lambda) \langle \varphi_2, \psi \rangle. \tag{7.5}$$

Due to Lemma 6.2, $\langle \varphi_j, \psi \rangle$ ($j = 1, 2$) are positive and hence must vanish. Therefore the right hand side of (7.1) is a face of K. Let us call this face g. By the earlier argument, we have $g \geq f'$.

Let $\varphi \in g \wedge (f')' = g \wedge f$. Since it is in g and f, $\langle \varphi, \varphi \rangle = 0$ which is impossible due to Lemma 6.2. Therefore $g \wedge f = \emptyset$. The orthomodularity implies

$$g = (g \wedge (f')') \vee f' = \emptyset \vee f' = f'. \tag{7.6}$$

This proves (7.1) and the statement that two faces are lattice-orthogonal if they are metrically orthogonal.

The equivalence of orthogonality to $\langle e_{f_1}, e_{f_2} \rangle = 0$ will be proved at the end of this section.

Proof of Proposition 7.2. Let $\varphi_1, \ldots, \varphi_k$ be mutually orthogonal extremal points of f. If $f_1 = \bigvee_{i=1}^{k} \varphi_i$ is not f, then there exists an extremal point φ of $f_1' \wedge f$ ($\neq \emptyset$), which is automatically an extremal point of f orthogonal to all φ_i, $i = 1, \ldots, k$. By repeating this process, which terminates with $f = \bigvee_{i=1}^{r} \varphi_i$, we obtain a family of mutually orthogonal extremal points of f which contains the original family and span f. By Lemma 5.1, $e_f = \sum e_{\varphi_i} \in A^e$ and hence

$$\langle e_f, e_f \rangle = \sum_{i,j} \langle e_{\varphi_i}, e_{\varphi_j} \rangle = \sum_{i=1}^{r} \langle e_{\varphi_i}, e_{\varphi_i} \rangle = r, \tag{7.7}$$

where we used $\langle \varphi_i, \varphi_j \rangle = 0$ for $i \neq j$ and $= 1$ for $i = j$. This shows also $k \leq r$. If $\{f_i\}$ is a family of mutually orthogonal subfaces of f ($i = 1, \ldots, k$), then orthomodularity implies

$$f = \bigvee_{i=0}^{k} f_i, \quad f_0 \equiv f \wedge \left(\bigvee_{i=1}^{k} f_i \right)'. \tag{7.8}$$

Let φ_{ij} ($j = 1, \ldots, \operatorname{rank} f_i$) be a family of mutually orthogonal extremal points of f_i, Lemma 5.1 implies

$$e_f = \sum_{i=0}^{k} e_{f_i} = \sum_{i,j} e_{\varphi_{ij}}. \tag{7.9}$$

Hence, due to mutual orthogonality of φ_{ij},

$$\operatorname{rank} f = \langle e_f, e_f \rangle = \sum_{i,j} \langle \varphi_{ij}, \varphi_{ij} \rangle = \sum_{i=0}^{k} \operatorname{rank} f_i,$$

which proves $k \leq \operatorname{rank} f$ (the case with non-empty f_i) as well as (7.4) (the case with $f_0 = \emptyset$, $\operatorname{rank} f_0 = 0$).

Proof of Corollary 7.4. (ii) is just proved. If $f_1 \gtrless f_2$, then $f_1 = f_2 \vee (f_1 \wedge f_2')$ with $\operatorname{rank}(f_1 \wedge f_2') \neq 0$ and hence (i) follows from (ii).

Proof of Proposition 7.1 (continued). Let f_1 and f_2 be two faces and $f_i = \bigvee_j \varphi_{ij}$ with mutually orthogonal φ_{ij} for each $i = 1, 2$. If f_1 and f_2 are orthogonal, then all φ_{ij} are mutually orthogonal and hence $\langle e_{\varphi_{ij}}, e_{\varphi_{kl}} \rangle = 0$ if $(i,j) \neq (k,l)$. Therefore $e_{f_i} = \sum_j e_{\varphi_{ij}}$ for $i = 1$ and 2 also satisfies $\langle e_{f_1}, e_{f_2} \rangle = 0$.

Conversely, assume $\langle e_{f_i}, e_{f_j} \rangle = 0$. Since

$$\langle e_{\varphi_{1j}}, e_{\varphi_{2k}} \rangle = \langle \varphi_{1j}, \varphi_{2k} \rangle \geqq 0$$

by (6.5), $e_{f_i} = \sum_j e_{\varphi_{ij}}$ implies the vanishing of the above quantity for all pair j, k. This means that φ_{1j} is lattice orthogonal to φ_{2k} by earlier proof and hence $f_1 = \bigvee_j \varphi_{1j}$ is lattice orthogonal to $f_2 = \bigvee_k \varphi_{2k}$.

8. Positive Definiteness

Axioms $\mathcal{O}$ and $\mathcal{H}$ together imply the following nice consequence:

Theorem 8.1. *Under Axioms $\mathcal{O}$ and $\mathcal{H}$, the inner product (6.3) on V is positive definite.*

The first step is to prove that the inner product is non-degenerate:

Lemma 8.2. *Under Axioms $\mathcal{O}$ and $\mathcal{H}$, $V_0 = 0$, i.e. the inner product (6.3) on V is non-degenerate and the kernel of $\psi \in V \to e_\psi$ is 0.*

Once this lemma is established, $\hat{e}$ is a linear bijection of V onto $A(K)$ (the surjectivity due to dimensionality reason).

Proposition 8.3. *Under Axioms $\mathcal{O}$ and $\mathcal{H}$,*

$$\hat{e}V_+ = A^+(K), \tag{8.1}$$

$$\hat{e}\hat{f} = f^* = (f')^\perp, \tag{8.2}$$

where $\hat{f}$ is given by (3.1), f^* by (5.3), f' by Axiom $\mathcal{O}$ (iii) and $f^\perp$ by (5.2).

The last property (8.2) enables us to use the spectral theory of Sect. 5 and yields a simple proof of Theorem 8.1. At the same time we obtain the following:

Theorem 8.4. *Under Axioms $\mathcal{O}$ and $\mathcal{H}$, p_f for every face f is an orthogonal projection relative to the inner product (6.3) and is a P-projection.*

Proof of Lemma 8.2. This is divided into a few steps.

(I) We investigate the quotient map q from V to V/V_0. The first observation is that qK is a compact convex set as the image of a compact convex set under a continuous linear map q.

(II) If ψ is on the relative boundary of K, then the face g of qK generated by $q\psi$ is not qK. The reason is as follows: the face f generated by ψ is not K due to Lemma A.1 in the Appendix. Hence f' is not empty. Any $\varphi \in f'$ is (lattice and hence) metrically orthogonal to ψ. Let

$$q\psi = \lambda q\psi_1 + (1 - \lambda)q\psi_2, \quad \psi_1 \in K, \quad \psi_2 \in K, \quad 0 < \lambda < 1. \tag{8.3}$$

Due to the definition of V_0, we have

$$0 = \langle \varphi, \psi \rangle = \lambda \langle \varphi, \psi_1 \rangle + (1 - \lambda) \langle \varphi, \psi_2 \rangle. \tag{8.4}$$

By (6.5), we must have $\langle \varphi, \psi_1 \rangle = \langle \varphi, \psi_2 \rangle = 0$. Therefore gf' is metrically orthogonal to g. By (6.6), any $\varphi \in K$ cannot be orthogonal to itself and hence qf' has no intersection with g. Hence $g \neq qK$.

(III) If ψ is in the relative interior of K, then the face of qK generated by $q\psi$ is qK itself. The reason: $\psi_1 \equiv (1 + \varrho)\psi - \varrho\varphi$ is in K for sufficiently small ϱ for any given $\varphi \in K$, because a full (relative) neighborhood of ψ is in K. Hence $\psi = \lambda \psi_1 + (1 - \lambda)\varphi$ with $0 < \lambda = (1 + \varrho)^{-1} < 1$, which implies $q\psi = \lambda q\psi_1 + (1 - \lambda)q\varphi$ and hence $q\varphi$ is in the face of qK generated by $q\psi$.

(IV) Final arguments: Since $e = e_K \in A^e$ we have $e(\varphi_0) = 0$ for any $\varphi_0 \in V_0$. Therefore for any $\psi \in K$, $\psi + V_0$ is contained in the affine manifold $\{\varphi ; e(\varphi) = 1\}$ on which K lies with interior points. Take ψ to be an interior point of K. If $V_0 \neq 0$, V_0 is non-compact, being a linear set. Hence $\psi + V_0$ intersects with the boundary of K at least at some point ψ_1. Then $q\psi = q\psi_1$ which contradicts (II) and (III). Hence $V_0 = 0$.

Proof of Proposition 8.3. This is divided into a few steps.

(I) $\hat{e}V_+ \subset A^+(K)$: this follows from the positivity $e_{\psi_1}(\psi_2) \geq 0$ given by (6.5).

(II) If ψ is on the boundary of V_+; the face of $A^+(K)$ generated by $\hat{e}\psi$ does not contain an interior point of $A^+(K)$. The reason: The face of V_+ generated by ψ is of the form $\hat{f}$ ($f = \emptyset$ if $\psi = 0$) for some face f of K and $f \neq K$ if $\psi \in \partial V_+$. Hence ψ is orthogonal to $f' \neq \emptyset$. The same argument as (II) of the proof of Lemma 8.2 implies that $\hat{e}\psi$ generates a face of $A^+(K)$ annihilating f' and hence not containing $\hat{e}f'$. Therefore it is a proper face of $A^+(K)$ and hence is on the boundary of $A^+(K)$.

(III) *Proof of (8.1).* Let $\psi_1 \notin V_+$ and $\hat{e}\psi_1$ be in $A^+(K)$. Let ψ_2 be an interior point of V_+. Then there exists λ satisfying $0 < \lambda < 1$ and $\psi \equiv \lambda \psi_1 + (1 - \lambda)\psi_2 \in \partial V_+$. Since $\hat{e}\psi = \lambda \hat{e}\psi_1 + (1 - \lambda)\hat{e}\psi_2$ and $\hat{e}\psi_2 \in A^+(K)$ by (I), $\hat{e}\psi_2$ must be in the face generated by $\hat{e}\psi$, which is on the boundary of $A^+(K)$ by (II). This is impossible because $\hat{e}\psi_2$ is in the interior of $\hat{e}V_+ \subset A^+(K)$. Hence $\hat{e}\psi_1 \notin A^+(K)$ if $\psi_1 \notin V_+$. Together with (I), this proves (8.1).

(IV) *Proof of (8.2).* By (7.3), and $\hat{e}V_+ \subset A^+(K)$, any extremal points ψ of f satisfies $\hat{e}\psi \in f^*$ and hence $\hat{e}\hat{f} \subset f^*$. On the other hand, (8.1) implies that $\hat{e}\hat{f}$ is a face of $A^+(K)$ because $\hat{f}$ is a face of V_+, and $\hat{e}\hat{f}$ contains e_f due to (7.3). Therefore $f^* = \hat{e}\hat{f}$. Proposition 7.1 shows that

$$\hat{e}\hat{f} = \{x \in \hat{e}V_+ ; x(\varphi) = 0 \text{ for all } \varphi \in (f')\} = (f')^{\perp}. \tag{8.5}$$

[Alternatively, we obtain $f^* = (f')^{\perp}$ from Proposition 5.4(i).]

Proof of Theorem 8.1. By (8.1), the hypothesis of Proposition 5.4 is satisfied and hence any non-zero $x \in A$ has a decomposition (5.5) with $\mu_i \neq 0$ and $f_i \neq \emptyset$. By Proposition 7.1 and (7.3), e_{f_i} are mutually orthogonal and hence

$$\langle x, x \rangle = \sum_{i=1}^{n} (\mu_i)^2 \operatorname{rank} f_i > 0. \tag{8.6}$$

Proof of Theorem 8.4. The proof of $p_f^* = p_f$ is divided into several steps.

(I) For $\varphi \in F_0(K)$, $p_\varphi^* = p_\varphi$ follows from the following computation for ψ_1 and ψ_2 in the total set $F_0(K)$ of V:

$$\langle \psi_1, p_\varphi \psi_2 \rangle = \langle \psi_1, \langle \varphi, \psi_2 \rangle \varphi \rangle$$
$$= \langle \psi_1, \varphi \rangle \langle \varphi, \psi_2 \rangle = \langle p_\varphi \psi_1, \psi_2 \rangle. \tag{8.7}$$

(II) By Axiom $\mathcal{O}$(ii), $\ker p_\varphi$ for any $\varphi \in F_0(f)$ contains $\ker p_f$ for $f \in F(K)$ and hence

$$\ker p_f \subset \cap \{\ker p_\varphi ; \varphi \in F_0(f)\}. \tag{8.8}$$

(III) Since f is the convex span of $F_0(f)$, Lemma 3.5 implies

$$\operatorname{Im} p_f = \vee \{\operatorname{Im} p_\varphi ; \varphi \in F_0(f)\} = (\cap \{\ker p_\varphi ; \varphi \in F_0(f)\})^\perp, \tag{8.9}$$

where the lattice notation is used for the lattice of subspaces and the second equality follows from (I).

(IV) (8.8) and (8.9) implies the orthogonality of $\ker p_f$ and $\operatorname{Im} p_f$ and hence p_f is an orthogonal projection.

To show that p_f is a P-projection, we note that p_f and $p_{f'}$ are quasicomplementary in the sense of [10]. Due to selfadjointness of p_f just proved, dual maps of p_f and $p_{f'}$ are the same as p_f and $p_{f'}$ (due to the selfduality given by Proposition 8.3) and hence are quasicomplementary. Therefore they are P-projections.

Remark 8.5. The above proof also shows

$$\ker p_f = \cap \{\ker p_\varphi ; \varphi \in F_0(f)\}. \tag{8.10}$$

9. Geometrical Characterization

Theorem 9.1. *A convex cone C of a finite dimensional vector space V is the V_+ associated with a finite dimensional compact convex set K satisfying Axioms $\mathcal{O}$ and $\mathscr{H}$ if and only if C has the following properties relative to a positive definite inner product.*

α) *Every non-empty face F of C (including C itself) is self-polar in its linear span $\operatorname{Lin} F$ in the following sense:*

$$F = \{\varphi \in \operatorname{Lin} F ; \langle \varphi, \psi \rangle \geq 0 \text{ for all } \psi \in F\}. \tag{9.1}$$

(β) *Every non-empty face F of C satisfies $(F^0)^0 = F$, where*

$$F^0 \equiv \{\varphi \in C ; \langle \varphi, \psi \rangle = 0 \text{ for all } \psi \in F\}. \tag{9.2}$$

(γ) *There exists a vector $e_0 \in C$ which has an orthogonal projection of length 1 on every extremal rays of C.*

(δ) *For any face F of C, e_0 is in the convex span of F and F^0.*

Proof. First assume Axioms $\mathcal{O}$ and $\mathscr{H}$. Set $C = V_+$.

(α) From the definition of $A^+(K)$ and (8.1), the self-polar property of V_+ follows.

By Proposition 3.6, each face f of K satisfies Axioms $\mathcal{O}$ and $\mathscr{H}$ with the linear span of f replacing V and $\hat{f}$ replacing V_+. By Lemma 3.2, a non-empty face F of V_+

is of the form $\hat{f}$ for some face f of K and hence the self-polar property of $F = \hat{f}$ follows from the previous argument.

(β) From (8.2) $\hat{e}\hat{f}' = f^{\perp}$ follows. Due to $\hat{e}(\hat{f}^0) = f^{\perp}$, we have $(\hat{f})^0 = \hat{f}'$ and hence $(F^0)^0 = F$ for $F = \hat{f}$.

(γ) Take e_0 to be such that $\hat{e}e_0 = e$. Then $\langle e_0, \psi \rangle = e(\psi) = 1$ and $\langle \psi, \psi \rangle = 1$ for any $\psi \in F_0(K)$ implies (γ) in view of Lemma 3.2.

(δ) This follows from $e = e_f + e_{f'}$, with $e_f \in \hat{e}\hat{f}$, $e_{f'} \in \hat{e}\hat{f}'$ and $\hat{f}' = \hat{f}^0$ in view of Lemma 3.2.

Now assume properties (α), (β), (γ), and (δ). The set K is defined as the set of all $\varphi \in C$ satisfying $\langle e_0, \varphi \rangle = 1$.

Due to (9.1) for C, C does not contain non-trivial linear subsets and hence is the convex span of its extremal rays. Every extremal ray of C has one point on K by (γ). Therefore K is a basis of C and is a compact convex set.

For every face f of K, let p_f be the orthogonal projection operator with its range = the linear span of f. Since any $\varphi \in C$ satisfies $\langle \varphi, \psi \rangle \geq 0$ with $\psi \in \hat{f}$ due to (9.1) for C and since $\langle p_f \varphi, \psi \rangle = \langle \varphi, \psi \rangle \geq 0$ for all $\psi \in \hat{f}$ implies $p_f \varphi \in \hat{f}$ due to (9.1) for the face $\hat{f}$ of C, we have the positivity of p_f.

We define f' by $\hat{f}' = \hat{f}^0$. Then ranges of p_f and $p_{f'}$ are orthogonal by definition. Due to (δ), $e_0 = p_f e_0 + p_{f'} e_0$ and hence the complementarity (2.7) is satisfied where $\|\psi\| = \langle e_0, \psi \rangle$ for any $\psi \in C$ due to (2.11) (and should not be confused with the Hilbert metric):

$$\|p_f \varphi\| + \|p_{f'} \varphi\| = \langle e_0, p_f \varphi \rangle + \langle e_0, p_{f'} \varphi \rangle$$
$$= \langle p_f e_0, \varphi \rangle + \langle p_{f'} e_0, \varphi \rangle = \langle e_0, \varphi \rangle = 1 \qquad (9.3)$$

for all $\varphi \in K$.

If $\varphi \in p_f K \cap K$, then $\langle e_0, p_f \varphi \rangle = 1$ and hence $\langle e_0, p_{f'} \varphi \rangle = 0$ by (9.3). Because K is a base of C and $\langle e_0, \psi \rangle = 1$ for $\psi \in K$, this implies $p_{f'} \varphi = 0$ due to the positivity $p_{f'} \varphi \in C$ (which is already proved). Since $p_{f'}$ is an orthogonal projection on the linear span of f', $p_{f'} \varphi = 0$ implies that $\varphi \in K$ is orthogonal to $\hat{f}'$ and hence $\varphi \in (\hat{f}')^0 = (f^0)^0 = f$ due to (β). Therefore $p_f \varphi = \varphi$ and p_f is neutral.

The contractive property of p_f follows from what we have proved due to Remark 2.4. Thus (i) and (iii) of Axiom $\mathcal{O}$ is now proved. (ii) is immediate because p_{f_j} ($j = 1, 2$) are orthogonal projections and $\mathrm{Im}\, p_{f_1} \subseteq \mathrm{Im}\, p_{f_2}$ if $f_1 \subseteq f_2$.

Because of (γ) and definition of K, the extremal point φ of K satisfies

$$\langle p_\varphi e_0, p_\varphi e_0 \rangle = 1 \quad [\text{due to } (\gamma)], \qquad (9.4)$$

$$\langle p_\varphi e_0, \varphi \rangle = \langle e_0, \varphi \rangle = 1, \qquad (9.5)$$

and $p_\varphi e_0$ is a multiple of φ (because it belongs to $\hat{\varphi}$). Hence

$$\langle \varphi, \varphi \rangle = 1. \qquad (9.6)$$

Therefore for any $\psi \in V$,

$$p_\varphi \psi = \langle \varphi, \psi \rangle \varphi. \qquad (9.7)$$

Axiom $\mathcal{H}$ follows from (9.7) and the symmetry of the inner product.

10. Decomposition into Irreducible Parts

The following are known notions and consequences.

Definition 10.1. A face f_1 of a compact convex set K is called a *split face* if there is another face f_2 called *complement* of f_1 such that any $\varphi \in K$ has the following unique decomposition:

$$\varphi = \lambda \varphi_1 + (1 - \lambda)\varphi_2 \tag{10.1}$$

with $0 \leq \lambda \leq 1$, $\varphi_1 \in f_1$ and $\varphi_2 \in f_2$. (λ is unique and if $0 < \lambda < 1$, φ_1 and φ_2 are unique.) We also say that f_1 and f_2 *split* K.

Proposition 10.2. *If f_1 and f_2 split K, then $f_1 \wedge f$ and $f_2 \wedge f$ split a face f of K and V is a direct sum of $\mathrm{Lin} f_1$ and $\mathrm{Lin} f_2$.*

Definition 10.3. A compact convex set K is said to be *irreducible* if it does not have a split face other than K and empty set.

Proposition 10.4. *A finite dimensional compact convex set K is a direct convex sum of its minimal and hence irreducible split faces f_i in the sense that any $\varphi \in K$ has the following unique decomposition:*

$$\varphi = \sum \lambda_i \varphi_i \tag{10.2}$$

with $\lambda_i \geq 0$, $\sum \lambda_i = 1$, $\varphi_i \in f_i$. Any face f of K is a direct convex sum of $f \wedge f_i$ and V is a direct sum of $\mathrm{Lin} f_i$ (the linear hull of f_i in V).

Special situations under Axioms $\mathcal{O}$ and $\mathcal{H}$ are given by the following:

Theorem 10.5. *Under Axiom $\mathcal{O}$, $f_2 = (f_1)'$ if f_1 and f_2 split K. If the f_i are as in Proposition 10.4, then they are mutually lattice orthogonal and the lattice $F(K)$ is a direct sum of lattices $F(f_i)$, each of which is lattice irreducible (i.e. non-trivial further lattice direct sum decomposition).*

Theorem 10.6. *Under Axioms $\mathcal{O}$ and $\mathcal{H}$, the decompositions of K, $f \in F(K)$ and V in Proposition 10.4 is an orthogonal decomposition.*

Proof of Proposition 10.2. If $\varphi \in f$ for $f \in F(K)$, then φ_1 and φ_2 of (10.1) must be in f when $0 < \lambda < 1$. Therefore $f \wedge f_1$ and $f \wedge f_2$ split f. If $\mathrm{Lin} f_1$ and $\mathrm{Lin} f_2$ have the following common element:

$$\varphi = \sum_i \varrho_{1i}^+ \varphi_{1i}^+ - \sum_j \varrho_{1j}^- \varphi_{1j}^- = \sum_k \varrho_{2k}^+ \varphi_{2k}^+ - \sum_l \varrho_{2l}^- \varphi_{2l}^- \tag{10.3}$$

with $\varrho_{1i}^+ > 0$, $\varrho_{1j}^- > 0$, $\varrho_{2k}^+ > 0$, $\varrho_{2l}^- > 0$, $\varphi_{1i}^+ \in f_1$, $\varphi_{1j}^- \in f_1$, $\varphi_{2k}^+ \in f_2$, and $\varphi_{2l}^- \in f_2$, then

$$e(\varphi_{1j}^+) = e(\varphi_{1j}^-) = e(\varphi_{2k}^+) = e(\varphi_{2l}^-) = 1$$

implies

$$\varrho \equiv \sum \varrho_{1i}^+ + \sum \varrho_{2l}^- = \sum \varrho_{1j}^- + \sum \varrho_{2k}^+$$

and

$$\varrho^{-1}(\sum \varrho_{1i}^+ \varphi_{1i}^+ + \sum \varrho_{2i}^- \varphi_{2i}^-) = \varrho^{-1}(\sum \varrho_{1j}^- \varphi_{1j}^- + \sum \varrho_{2k}^+ \varphi_{2k}^+) \in K. \tag{10.4}$$

The uniqueness of the decomposition then implies $\varphi = 0$. Therefore V is the direct sum of $\mathrm{Lin} f_1$ and $\mathrm{Lin} f_2$.

Proof of Proposition 10.4. If f_1 and f_2 split K and f_3 and f_4 split f_1, then any $\varphi \in K$ has a unique decomposition

$$\varphi = \sum_{j=2}^{4} \lambda_j \varphi_j \tag{10.5}$$

with $\varphi_j \in f_j$, $\lambda_j \geq 0$ and $\sum \lambda_j = 1$. This immediately implies that the convex sum f_5 of f_2 and f_4 is a face of K and f_3 and f_5 split K. Therefore a split face of a split face of K is a split face of K.

If f and g are split faces of K, then $f \wedge g$ is a split face of g by Proposition 10.2 and hence is a split face of K by the previous argument.

By repeated use of the first argument and the finite dimensionality, any face contains a minimal split face. Repeated use of splitting into a minimal face and its complement then yields a decomposition of K into a direct convex sum of a finite number of minimal split faces. Such a decomposition splits any face f of K into a direct convex sum of $f \wedge f_i$ by the repeated application of Proposition 10.2. In particular, any minimal split face of K must coincides with one of the component of this decomposition and hence the decomposition uses all minimal split faces. Repeated use of Proposition 10.2 also shows that V is a direct sum of linear hulls of minimal split faces.

Proof of Theorem 10.5. If f_1 and f_2 split K, then f_1' must be a direct convex sum of $f_1' \wedge f_1$ and $f' \wedge f_2$ by Proposition 10.2. By Lemma 4.3(iv), $f_1' \wedge f_1$ is empty. Hence $f_1' \subset f_2$. Then $f_2 = f_1' \vee (f_1 \wedge f_2) = f_1'$ due to the orthomodularity and $f_1 \wedge f_2 = \emptyset$ [which follows from the uniqueness of the decomposition (10.1)]. In Proposition 10.4, f_i is in the complement of f_j for $j \neq i$ and hence they are lattice orthogonal.

By Proposition 10.4, any face f of K is a direct convex sum of faces $f \wedge f_i$ and hence is their lattice sum. Conversely, if $g_i \in F(f_i)$, then the (direct) convex sum g of g_i is a face of K because if

$$\varphi = \sum \lambda_i \varphi_i, \quad \lambda_i \geq 0, \quad \sum \lambda_i = 1, \quad \varphi_i \in g_i \tag{10.6}$$

has a decomposition

$$\varphi = \lambda \psi_1 + (1 - \lambda) \psi_2 \tag{10.7}$$

with $0 < \lambda < 1$, $\psi_1 \in K$ and $\psi_2 \in K$, then

$$\psi_k = \sum_i \mu_{ik} \psi_{ik} \quad (k = 1, 2) \tag{10.8}$$

with $\mu_{ik} \geq 0$, $\sum_i \mu_{ik} = 1$, $\psi_{ik} \in f_i$ and hence the uniqueness of decomposition of φ implies

$$\lambda_i \varphi_i = \lambda \mu_{i1} \psi_{i1} + \lambda \mu_{i2} \psi_{i2}, \tag{10.9}$$

which shows either $\mu_{i1} = \mu_{i2} = 0$ (for $\lambda_i = 0$) or $\psi_{i1} \in g_i$, $\psi_{i2} \in g_i$. Therefore $\vee g_i$ is the direct convex sum of g_i for any face g_i of f_i and the lattice $F(K)$ is a direct sum of lattices $F(f_i)$.

Finally suppose that $F(K)$ is a direct sum of two sublattices L_1 and L_2. This means that any $g_k \in F(K)$ has a unique decomposition $g_k = g_{k1} \vee g_{k2}$ with $g_{ki} \in L_i$ satisfying $\wedge g_k = (\wedge g_{k1}) \vee (\wedge g_{k2})$. Let $K = f_1 \vee f_2$ with $f_1 \in L_1$ and $f_2 \in L_2$. Then

$(g)_1 \vee (g')_1 = f_1$, $(g)_2 \vee (g')_2 = f_2$ due to $g \vee g' = K$ and hence $g = (g \wedge f_1) \vee (g \wedge f_2)$, for any $g \in F(K)$. In particular, $F_0(K) = F_0(f_1) \cup F_0(f_2)$ and hence K is a direct convex sum of f_1 and f_2. By the preceding argument, $f_2 = f_1'$ and hence f_1 and f_2 are affine independent. Therefore $F(K)$ has a non-trivial split face. This shows that $F(K)$ is lattice irreducible if and only if K is irreducible.

Proof of Theorem 10.6 follows from Theorem 10.5 and the equivalence of lattice and metric orthogonality (Proposition 7.1).

11. Cases of rank 2

If rank $K = 2$, all possible K satisfying Axioms $\mathcal{O}$ and $\mathcal{H}$ are determined and correspond to spin factors.

Theorem 11.1. *A finite dimensional compact convex K satisfies Axioms $\mathcal{O}$ and $\mathcal{H}$ and is of* rank 2 *if and only if it is affine isomorphic to a ball of radius* $1/\sqrt{2}$ *(given as an intersection of the unit ball in V with the affine manifold of codimension 1 at a distance* $1/\sqrt{2}$ *from the origin), where K is irreducible except for the case of* $\dim K = 1$ *($\dim V = 2$). (The filtering projection p_φ for $\varphi \in F_0(K)$ is the orthogonal projection with its range = the line passing through* 0 *and* φ.)

Proof. By $\langle \psi, \psi \rangle = 1$ for $\psi \in F_0(K)$ and $\langle e, e \rangle = \text{rank } K = 2$, $F_0(K)$ is a subset of the intersection of the unit sphere of V with the affine manifold $\{\varphi \in V; e(\varphi) = 1\}$ at a distance of $1/\sqrt{2}$ from the origin. Hence K is a subset of the intersection of the unit ball of V with the same affine manifold.

Since all faces of K other than K and $\emptyset$ are of rank 1, i.e. extremal points of K, $\partial K = F_0(K)$ (due to Lemma A1, for example). Therefore ∂K must be the whole intersection of the unit sphere of V with the above affine manifold.

The proof of the converse is straightforward.   Q.E.D.

The following examples show independence of Axioms $\mathcal{O}$ and $\mathcal{H}$, as well as the difference of a filtering projection and a P-projection.

Example 11.2. Consider an arbitrary strictly convex compact body K with a smooth surface ∂K in a finite dimensional space.

There are exactly two points of K with their tangent parallel to any given hyperplane in Lin K. These two points are taken to be φ and φ', both belonging to $F_0(K)$. The p_φ is defined by specifying its image to be the line joining the origin 0 of V and φ, and its kernel to be the hyperplane (linear subset of V with codimension 1) tangent to K at φ'. It is easy to check that Axiom $\mathcal{O}$ is satisfied. For any such K which is not an ellipsoid, Axiom $\mathcal{H}$ is violated (due to Theorem 11.1). In this example p_φ is a P-projection.

Example 11.3. In Example 11.2, we allow non-smooth surface in such a way that there are exactly two points of K with their tangent parallel to a given hyperplane. For example any strictly convex body symmetric relative to a point has such a property. We define φ' and p_φ as in the preceding example, except that we may have a choice of the tangent plane at a non-smooth point. If that happens at φ, p_φ is not a P-projection.

Example 11.4. Consider the intersection S of the unit ball of a d-dimensional V and the affine manifold $H = \{\varphi \in V; \langle e_0, \varphi \rangle = 1\}$ with $e_0 \in V$, $\langle e_0, e_0 \rangle = 2$.

Cut off some part of S by two parallel $(d-2)$-dimensional affine submanifold of H at an equal non-zero distance of strictly less than $1/\sqrt{2}$ from $e_0/2 \in H$, and call the resulting compact convex set K.

If we take p_φ for $\varphi \in F_0(K)$ to be the orthogonal projection with its image = the line joining the origin 0 of V and φ, then Axiom $\mathscr{H}$ is satisfied.

The fact that it can not satisfy Axiom $\mathcal{O}$ for any choice of complementation is seen as follows: K has two faces, say f_1 and f_2 which are not K nor extremal. Let $\varphi \in F_0(f_1)$. Then in order Axiom $\mathcal{O}$ be satisfied, $f_1 = \varphi \vee (\varphi' \wedge f_1)$ due to orthomodularity and hence φ' contains an extremal point φ_1 of $\varphi' \wedge f_1 \subsetneqq f_1$. (Otherwise $\varphi' \wedge f_1 = \emptyset$ and hence $f_1 = \varphi$.) If φ' contains any other point, then φ' is either f_1 or K, both of which is impossible due to $\varphi \wedge \varphi' = \emptyset$. Therefore $\varphi' = \psi_1 \subsetneqq f_1$. However this implies $\varphi \vee \varphi' = f_1$ contradicting with $\varphi \vee \varphi' = H$.

12. Conclusion

We have described the consequences of Axioms $\mathcal{O}$ and $\mathscr{H}$ in detail. We are not sure how far the consequence of Axioms $\mathcal{O}$ and $\mathscr{H}$ is from the following final consequence, which we obtain by addition of Axiom $\mathscr{P}$ in Sect. 1 or Property $\mathscr{R}$ below, due to lack of suitable examples. It is enough to describe K when it is irreducible in view of results in Sect. 10.

Theorem 12.1. *An irreducible K satisfies Axioms $\mathcal{O}$, $\mathscr{H}$, and $\mathscr{P}$ if and only if it is a state space of a finite dimensional Jordan algebra factor.*

A finite dimensional Jordan algebra factor [13] is either $n \times n$ hermitian matrices over the real, complex or quaternion field or the exceptional Jordan algebra M_3^8 of 3×3 hermitian matrices over the Cayley numbers, which arises for the case of rank 3, or spin factors, which arise for the case of rank 2 and have already been treated in Sect. 11.

Remark 12.2. The conclusion of Theorem 12.1 holds if Axiom $\mathscr{P}$ is replaced by the following:

Property $\mathscr{R}$. The face generated by any two pure states has rank 2.

Remark 12.3. For rank $K > 3$, we have Jordan algebras of all self-adjoint operators on Hilbert spaces over real, complex or quaternion field. These three fields may be distinguished by the behavior of state spaces when we try to consider a combined system of two independent systems. The "independence" is expressed in the case of complex field by the tensor product of the underlying Hilbert spaces. This leads to the relation

$$\dim V = (\dim V_1)(\dim V_2) \tag{12.1}$$

for the linear span of state spaces for the combined and individual systems. [Similarly the Jordan algebra $A(K)$ is the tensor product of $A(K_1)$ and $A(K_2)$ as a linear space, but not as an algebra.] If we make a corresponding construction of the combined system for the case of the real field, then we obtain a strict inequality

> in (12.1) except for trivial cases essentially because the tensor product of two skew hermitian operators is hermitian.

In the case of the quaternion field, it is known that there is a difficulty in constructing something reasonable for the combined system due to the non-commutativity of the quaternion field. [15] However, even if we consider a Hilbert space H with $\dim H = (\dim H_1)(\dim H_2)$ from the analogy with the real and complex cases, the corresponding state space for the quaternion case will satisfy a strict inequality $<$ in (12.1).

Thus the complex field has the most pleasant feature that the linear span of state space of the combined syste (i.e. V's) is a tensor product of individual ones.

Proof of Theorem 12.1 and Remark 12.2. This can be obtained from Theorem 6.16 of [4], where the condition (i) is the irreducibility, (ii) follows from our Theorem 8.4, the pure state properties of (iii) is the combination of our Axiom $\mathscr{H}$ and $\mathscr{P}$. The Hilbert ball property (iii)' of [10] follows from Proposition 3.6, Property $\mathscr{R}$ and Theorem 11.1.

Conversely, let V be a finite dimensional irreducible real Jordan algebra, V_+ be its positive cone (consisting of all squares), $\langle a, b \rangle = \varphi(a \circ b)$ be the real inner product in V given by a trace state φ, $e \in V$ be the identity of the Jordan algebra giving $\varphi(a) = \langle e, a \rangle$ and $K = \{a \in V_+ ; \langle e, a \rangle = 1\}$. Then V_+ is self-polar ([3], Lemma 6.2) and hence K is the state space.

Any face F of V_+ is closed by the finite dimensionality and $F = (F^0)^0$ ([3], Lemma 2.2(iii) which implies $F = (F^\perp)^\perp$ in their notation due to finite dimensionality). Hence F are in one-to-one correspondence with idempotents e_F of V in such a way that $F = \{a \in V_+ ; \langle e - e_F, a \rangle = 0\}$. The orthogonal projection p_F with its image $= F - F$ satisfies $p_F V_+ = F$ ([3], Lemma 2.2(iii)) and hence the self-polar property of V_+ implies that of F in $\mathrm{Lin}\, F = F - F$. Furthermore $e_F + e_{(F^0)} = e$ with $e_F = p_F e$, $e_{(F^0)} = p_{(F^0)} e$ ([3], Theorem 6.3 and the definition of U, for example) and the projection of e to extremal rays has the same length 1 due to $\varphi(e^2) = \varphi(e) = 1$ for the minimal idempotent of V. Therefore Axioms $\mathcal{O}$ and $\mathscr{H}$ are satisfied due to Theorem 9.1. Since p_F is uniquely defined by the face F as a P-projection [Theorem 6.16(iii)], we have Axiom $\mathscr{P}$ by [2], Theorem 6.16(iii).

[Explicit form of p_F is given in terms of the Jordan product on V by $p_F a = 2 e_F \circ (a \circ e_F) - e_F \circ a$.]

Appendix: Proof of Proposition 5.4(ii)

We use the following general lemma:

Lemma A1. *The face generated by a point P on the boundary of a closed convex set C is a proper subset of C if C has a non-empty interior.*

In fact, there exists a supporting hyperplane H of C passing P due to the separation theorem and $H \cap C$ is a face of C containing P but not containing the interior of C.

We apply this Lemma for C with relative interior and P on the relative boundary within the affine span of C.

We first establish the existence and uniqueness of the decomposition (5.5) for the special case $x \in A^+(K)$.

Let $h_1 \in F(K)$ be such that h_1^* is the face generated by x. (If x is in the interior of $A^+(K)$, then $h_1^* = K$.) Then x is in the relative interior of h_1 due to Lemma A1. By (4.1) with $L = \{e_f\}$ and (5.3), f^* is the set of all $x \in A^+(K)$ such that $x \leq \lambda e_f$ for some $\lambda > 0$. Hence $x - \lambda e_{h_1}$ is in $-A^+(K)$ for some $\lambda > 0$, which means that it is in $-h_1^*$ for this λ due to $\lambda e_{h_1} \geq \lambda e_{h_1} - x \geq 0$ and consequently in the relative interior of $-h_1^*$, for sufficiently large λ. Therefore

$$\{x - \lambda e_{h_1}; \lambda \geq 0\} \cap h_1^* = \{x - \lambda e_{h_1}; \lambda_1 \geq \lambda \geq 0\} \tag{A.1}$$

for some $\lambda_1 > 0$ by the convexity of h_1^*. Then

$$x_1 = x - \lambda_1 e_{h_1} \tag{A.2}$$

is on the relative boundary of h_1^*. Let h_2^* be the face generated by x_1.

We can continue this process obtaining a strictly decreasing sequence of faces $h_1 > h_2 > \ldots$ of K, strictly positive numbers λ_i and points x_i on the relative boundary of h_i^* ($i = 1, 2, \ldots$) such that x_i generates h_{i+1}^* (and hence is in the relative interior of h_{i+1}^*) and

$$x_i = x_{i-1} - \lambda_i e_{h_i} \tag{A.3}$$

($i = 1, 2, \ldots; x_0 = x$) until h_{n+1}^* does not have a relative interior for some n, i.e. $h_{n+1}^* = 0$, $h_{n+1} = $ empty set, and $x_n = 0$. By setting

$$f_j = h_j \wedge h_{j+1}' \quad \text{(a non-empty face of } K), \tag{A.4}$$

$$\mu_j = \sum_{i=1}^{j} \lambda_i \tag{A.5}$$

for $j = 1, \ldots, n$, we obtain a family of mutually orthogonal f_j and a strictly increasing sequence of strictly positive numbers μ_j, which satisfy (5.5) due to Lemma 5.1 and the equality $h_j = \bigvee_{i=j}^{n} f_i$ following from (A.4) by orthomodularity.

Conversely, if we apply the above procedure for a given decomposition (5.5) where μ_j are strictly increasing, we find successively

$$h_j = \bigvee_{i=j}^{n} f_i, \quad \lambda_j = \mu_j - \mu_{j-1} \quad (\mu_0 \equiv 0), \tag{A.6}$$

whose unique solution is given by (A.4) and (A.5). Therefore the decomposition (5.5) exists and is unique for $x \in A^+(K)$.

For a general $x \in A$, we have $\|x\|e + x \in A^+(K)$ and the unique decomposition

$$\|x\|e + x = \sum_{j=1}^{n} \mu_j' e_{f_j} \tag{A.7}$$

implies the unique decomposition

$$x = \sum_{j=0}^{n} \mu_j e_{f_j}, \tag{A.8}$$

where $\mu_j = \mu'_j - \|x\|$ for $j = 1 \ldots n$, $\mu_0 = -\|x\|$, $f_0 = \bigwedge_{j=1}^{n} f'_j$, the term with $\mu_j = 0$ is to be omitted if it exists and the $j = 0$ term is to be omitted if $f_0 = \emptyset$.

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