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The Ginibre Inequality

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Abstract. In the Ising-type models of statistical mechanics and the related quantum field theories, an inequality of Ginibre implies useful positivity and monotonicity properties: the Griffiths correlation inequalities. Essentially, the Ginibre inequality states that certain functions on the cycle group of a graph are positive definite. This has been proved for arbitrary graphs when the spin dimension is 1 or 2 (classical Ising or plane rotator models). We give a counterexample to show that these spin dimensions are the only ones for which the Ginibre inequality is generally true: there are graphs for which it never holds when the spin dimension is at least 3. On the other hand, we show that for any graph the inequality holds for the apparent leading term in the large-spin-dimension limit. (The leading term vanishes in the graph of the counterexample.) Based on these results, one expects the Ginibre inequality to be true in most instances, with infrequent exceptions. A numerical survey supports this. The surprising failure of the Ginibre inequality in higher dimensions need not necessarily mean the Griffiths inequalities fail as well, but a different approach to them is required.

1. Introduction

This paper is an analysis of the Ginibre inequality. In the present introduction we state the inequality and its main application in statistical mechanics (Griffiths correlation inequalities), and summarize our new results.

The Ginibre inequality for a graph G is [2]:

$$I(G,\chi,D) \equiv \int \prod_{(i,j)\in G} [\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j + \chi_{(i,j)} \boldsymbol{\sigma}'_i \cdot \boldsymbol{\sigma}'_j] \prod_{k\in\Lambda} d\omega(\boldsymbol{\sigma}_k) d\omega(\boldsymbol{\sigma}'_k) \geqq 0 \qquad \forall \chi_{(i,j)} = \pm 1 . \tag{1.1}$$

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(There are more general formulations, but we shall not consider them here.) Note that this is actually a collection of inequalities, since nonnegativity is required for any choice of signs $\chi_{(i,j)}$. In (1.1), Λ is the vertex set of G. A typical edge is written as an unordered vertex pair (i,j). This notation is abusive, because we permit multiple edges between the same vertex pair, possibly with different signs χ. The measure ω is the natural normalized rotation-invariant measure on the sphere S^{D-1}, and the region of integration is the full product $\prod_{k\in\Lambda}(S^{D-1}\times S^{D-1})$.

We use the customary notation $\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j$ for the standard scalar product. The spin dimension is the dimension D of the Euclidean space surrounding S^{D-1}. We may assume without loss of generality that G is an even graph (all vertices of even degree), for if it is not the invariance of ω under the transformation $\boldsymbol{\sigma}\to-\boldsymbol{\sigma}$ causes $I(G,\chi,D)$ to vanish.

Inequality (1.1) was introduced by Ginibre as a device to prove the Griffiths correlation inequalities of ferromagnetic spin systems [3, 5]. These inequalities, which generalize to Euclidean quantum field theories, are:

$$\left\langle\prod_{(i,j)\in G}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\right\rangle\geqq 0, \tag{1.2a}$$

$$\left\langle\left[\prod_{(i,j)\in G_1}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\right]\left[\prod_{(k,l)\in G_2}\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_l\right]\right\rangle - \left\langle\prod_{(i,j)\in G_1}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\right\rangle\left\langle\prod_{(k,l)\in G_2}\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_l\right\rangle\geqq 0. \tag{1.2b}$$

Here the subscripts i,j,k,l range over a finite set Λ interpreted as the lattice sites. G, G_1, and G_2 are arbitrary graphs with vertices in Λ. The average $\langle\cdot\rangle$ is with respect to the normalized Gibbs measure

$$d\mu=Z^{-1}\exp\left(\sum_{a,b\in\Lambda}J_{ab}\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b+\sum_{c\in\Lambda}h_c\mathbf{e}\cdot\boldsymbol{\sigma}_c\right)\prod_{d\in\Lambda}d\omega(\boldsymbol{\sigma}_d). \tag{1.3}$$

We assume the ferromagnetism conditions J_{ab}, $h_c\geqq 0$ hold in (1.3). The vector $\mathbf{e}$ is an arbitrary constant.

While the first Griffiths inequality (1.2a) is straightforward to check directly for any spin dimension D, the second Griffiths inequality (1.2b) is presently known only for the values $D=1,2$ for which Ginibre proved (1.1). It is natural to conjecture that this restriction to low spin dimension is an artifact of the method of proof: that the Ginibre inequality in fact holds for all D, with the second Griffiths inequality the following as a corollary.

We show that this conjecture is false – the Ginibre inequality (1.1) breaks down if the spin dimension is three or more. For example, if one takes G to be the complete graph on five vertices, then for an essentially unique choice of signs $\chi_{(i,j)}$ the integral $I(G,\chi,D)$ in (1.1) is strictly negative for all $D\geqq 3$. Since the Griffiths inequalities follow from the Ginibre inequality but do not seem to imply it, failure of (1.1) need not necessarily imply failure of (1.2b).

We also investigate the large-spin-dimension limit $D\to\infty$. Here we prove that the coefficient $c(\chi,G)$ of the apparent leading term in the $1/D$ expansion of $I(G,\chi,D)$ is nonnegative. In particular, if $c(\chi,G)$ is strictly positive for some choice of signs

$\chi_{(i,j)}$ on a certain graph G, then $I(G,\chi,D)>0$ for all sufficiently large D. On the other hand, $c(\chi,G)$ vanishes in the counterexample, permitting normally subdominant terms to make $I(G,\chi,D)$ negative.

Our final results concern a survey of examples. As a preliminary step, we associate the choice of signs $\chi_{(i,j)}$ with a character χ of the cycle group $Z(G)$ of G, and then identify $I(G,\chi,D)$ with the Fourier transform $\hat{F}_D(\chi)$ of a related function F_D on $Z(G)$. It follows that (1.1) holds (for all choices of $\chi_{(i,j)}$) if and only if F_D is positive definite. This reformulation is desirable both because of the familiarity of positive definite functions and because it eliminates redundancy in the choice of signs $\chi_{(i,j)}$. Also, it yields an easy proof of (1.1) when $D=1$, since F_1 is always the constant function 1. We calculated the numerical values of the Fourier transform $\hat{F}_D$ for a selection of twenty-five graphs by means of a digital computer routine. (Exponential computational complexity makes extensive hand calculations impractical). The machine results indicate that F_D is not positive definite in just four of the graphs examined, and in these four graphs $\hat{F}_D(\chi)$ is negative for only a few of the many characters.

To subsume, our theoretical results suggest that the Ginibre inequality (1.1) is true in most instances, with definite but infrequent exceptions. Our numerical results support this hypothesis.

The organization of the remainder of this paper is as follows. Section 2 analyzes the large-spin-dimension limit. Section 3 reformulates (1.1) in terms of positive definiteness and presents a discussion of examples, including counterexamples to (1.1). A brief concluding section comments on the relationship between the Ginibre inequality and the Griffiths inequalities. The methods and results of the numerical survey are outlined in an Appendix.

2. The Large-Spin-Dimension Limit

This section considers the Ginibre inequality in the $D\to\infty$ limit. The technique is to approximate the natural measure ω on S^{D-1} by a Gaussian measure ν on $\mathbb{R}^D$ with the same mean and covariance. This approximation is asymptotically correct to leading order as $D\to\infty$. We then verify (1.1) with ω replaced by ν.

Our primary tool is an explicit formula for the moments of ω [4]. In vector notation this formula is

$$\int_{S^{D-1}} \left(\prod_{i=1}^{2e} \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}_i\right) d\omega(\boldsymbol{\sigma}) = \frac{1}{D^e \prod_{\alpha=0}^{e-1}\left(1+\frac{2\alpha}{D}\right)} \sum_{(\pi,\pi')\in\mathscr{P}_{2e}} \prod_{\beta=1}^{e} \boldsymbol{\sigma}_{\pi(\beta)}\cdot\boldsymbol{\sigma}_{\pi'(\beta)}, \tag{2.1a}$$

where $\mathscr{P}_{2e}$ is the set of all pair partitions of $\{1,2,\ldots,2e\}$; that is, the set of all pairs of injective maps $\pi,\pi':\{1,2,\ldots,e\}\to\{1,2,\ldots,2e\}$ with $\pi<\pi'$ and π increasing. All the vectors $\boldsymbol{\sigma}_i\in\mathbb{R}^D$ are arbitrary. (We take an even number of $\boldsymbol{\sigma}_i$, since an odd number yields 0 by reflection symmetry). In tensor notation, using superscripts to denote components of the vector $\boldsymbol{\sigma}\in S^{D-1}\subset\mathbb{R}^D$, (2.1a) becomes

$$\int_{S^{D-1}} \sigma^{i_1}\sigma^{i_2}\ldots\sigma^{i_{2e}} d\omega(\boldsymbol{\sigma}) = \frac{1}{D^e \prod_{\alpha=0}^{e-1}\left(1+\frac{2\alpha}{D}\right)} \cdot \Delta^{i_1 i_2\ldots i_{2e}}. \tag{2.1b}$$

Here

$$\Delta^{i_1 i_2 \dots i_{2e}} \equiv \sum_{(\pi,\pi')\in\mathscr{P}_{2e}} \prod_{\beta=1}^{e} \delta^{i_{\pi(\beta)} i_{\pi'(\beta)}},$$

δ being the usual Kronecker symbol. By way of comparison, the corresponding formulas for the Gaussian measure v with the same mean (zero) and covariance matrix $\left(\frac{1}{D}\cdot\mathbb{1}\right)$ are:

$$\int_{\mathbb{R}^D} \left(\prod_{i=1}^{2e} \boldsymbol{\sigma}\cdot\boldsymbol{\sigma}_i\right) dv(\boldsymbol{\sigma}) = \frac{1}{D^e} \sum_{(\pi,\pi')\in\mathscr{P}_{2e}} \prod_{\beta=1}^{e} \boldsymbol{\sigma}_{\pi(\beta)}\cdot\boldsymbol{\sigma}_{\pi'(\beta)}, \tag{2.2a}$$

$$\int_{\mathbb{R}^D} \sigma^{i_1}\sigma^{i_2}\dots\sigma^{i_{2e}} dv(\boldsymbol{\sigma}) = \frac{1}{D^e} \Delta^{i_1 i_2 \dots i_{2e}}. \tag{2.2b}$$

Note that (2.2) differs from (2.1) solely by the omission of the factor $\prod_{\alpha=0}^{e-1} (1+2\alpha/D)$, which approaches 1 as $D\to\infty$.

The integral $I(G,\chi,D)$ in (1.1) expands to a linear combination of products of terms

$$\int \left(\prod_{(i,j)\in G_1} \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\right) \prod_{k\in\Lambda_1} d\omega(\boldsymbol{\sigma}_k), \tag{2.3}$$

where G_1 is an even subgraph of G having vertex set Λ_1. Evaluation of (2.3) by repeated applications of (2.1b) yields a complicated expression of the form

$$\left\{\prod_{k\in\Lambda_1} \left[\prod_{\alpha=0}^{|k|/2-1} (1+2\alpha/D)\right]\right\}^{-1} \times (D^{-|G_1|}) \\ \cdot(\text{Contractions of products of the tensors } \Delta). \tag{2.4}$$

Here $|k|$ is the number of edges incident on vertex k, while $|G_1|$ is the number of edges in G_1. Detailed consideration of the factor involving the contractions shows that it is a polynomial in D of degree $\gamma^*(G_1)$, the largest number of edge-disjoint cycles into which G_1 can be decomposed. Thus, the integral (2.3) can be written as

$$\left\{\prod_{k\in\Lambda_1} \left[\prod_{\alpha=0}^{|k|/2-1} (1+2\alpha/D)\right]\right\}^{-1} \times R(D),$$

where $R(D)$ is a rational function of D with degree $\gamma^*(G_1)-|G_1|$. Replacement of ω by v in (2.3) merely eliminates the factor in curly brackets. Of course, this factor is 1 to lowest order in $1/D$. Combining the terms from the expansion of (1.1) shows that $I(G,\chi,D)$ is a linear combination of rational functions of D having degree $\gamma^*(G_1)+\gamma^*(G_2)-|G|$, $G=G_1\oplus G_2$, with poles in the closed left half-plane. The apparent degree of this linear combination is the maximum degree $\gamma^*(G)-|G|$ of the functions involved. (The actual degree may be reduced by cancellations.) Moreover, as we have seen, the coefficient of the apparent leading term in the $1/D$ expansion of (1.1) (Laurent expansion about $D=0$) is unchanged when ω is replaced by the Gaussian measure v. We collect these conclusions in

Proposition 2.1. *The integrals*

$$I(G,\chi,D)=\int \prod_{(i,j)\in G} (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+\chi_{(i,j)}\boldsymbol{\sigma}'_i\cdot\boldsymbol{\sigma}'_j)\prod_{k\in\Lambda} d\omega(\boldsymbol{\sigma}_k)d\omega(\boldsymbol{\sigma}'_k), \tag{2.5a}$$

$$I_0(G,\chi,D)=\int \prod_{(i,j)\in G} (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+\chi_{(i,j)}\boldsymbol{\sigma}'_i\cdot\boldsymbol{\sigma}'_j)\prod_{k\in\Lambda} dv(\boldsymbol{\sigma}_k)dv(\boldsymbol{\sigma}'_k) \tag{2.5b}$$

are rational functions of D of degree $-(|G|-\gamma^*(G))$ *or less. Moreover, they have the same coefficient of* $D^{-(|G|-\gamma^*(G))}$ *in their Laurent expansion about* 0:

$$\lim_{D\to\infty} D^{(|G|-\gamma^*(G))}|I(G,\chi,D)-I_0(G,\chi,D)|=0.$$

Proof. The proof is immediate from the preceding discussion. QED

We next verify the Ginibre inequality with the spherical measure ω replaced by the Gaussian measure v.

Proposition 2.2. *Let* v *be the Gaussian measure on* $\mathbb{R}^D$ *with mean* 0 *and covariance matrix* $(1/D)\cdot\mathbb{1}$. *Then*

$$I_0(G,\chi,D)\equiv\int \prod_{(i,j)\in G} (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+\chi_{(i,j)}\boldsymbol{\sigma}'_i\cdot\boldsymbol{\sigma}'_j)\prod_{k\in\Lambda} dv(\boldsymbol{\sigma}_k)dv(\boldsymbol{\sigma}'_k)\geqq 0$$

$$\forall\chi_{(i,j)}=\pm 1, \forall G, \forall D.$$

Proof. Define $\mathbf{s}_i=2^{-1/2}(\boldsymbol{\sigma}_i+\boldsymbol{\sigma}'_i)$ and $\mathbf{d}_i=2^{-1/2}(\boldsymbol{\sigma}_i-\boldsymbol{\sigma}'_i)$. The integrand

$$\prod_G (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+\chi_{(i,j)}\boldsymbol{\sigma}'_i\cdot\boldsymbol{\sigma}'_j)$$

is a polynomial with nonnegative coefficients in the scalar products $\mathbf{s}_i\cdot\mathbf{s}_j$, $\mathbf{s}_i\cdot\mathbf{d}_j$, $\mathbf{d}_i\cdot\mathbf{s}_j$, $\mathbf{d}_i\cdot\mathbf{d}_j$. Moreover, the product measures are invariant under the change of variables:

$$dv(\boldsymbol{\sigma}_k)dv(\boldsymbol{\sigma}'_k)=dv(\mathbf{s}_k)dv(\mathbf{d}_k).$$

Thus I_0 is a moment of a mean-zero positive-covariance Gaussian measure, and so is nonnegative. QED

Combining Propositions 2.1 and 2.2 yields

Proposition 2.3. *If the actual degree of the integral* $I(G,\chi,D)$, *regarded as a rational function of D, agrees with its apparent degree* $-(|G|-\gamma^*(G))$, *then*

$$I(G,\chi,D)\equiv\int \prod_{(i,j)\in G} (\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+\chi_{(i,j)}\boldsymbol{\sigma}'_i\cdot\boldsymbol{\sigma}'_j)\prod_{k\in\Lambda} d\omega(\boldsymbol{\sigma}_k)d\omega(\boldsymbol{\sigma}'_k)>0$$

for all sufficiently large D.

In particular, Proposition 2.3 suggests that counterexamples to (1.1) can occur when fortuitous cancellations reduce the degree of I by causing the leading term in the $1/D$ expansion of I to drop out.

3. Examples

We begin this section by restating the Ginibre inequality in terms of positive definiteness. We then present an example to prove that $I(G,\chi,D)$ can be negative. Finally, we discuss the results of a computer-aided analysis of $I(G,\chi,D)$ for twenty-five graphs G.

Consider the expression for $I(G,\chi,D)$:

$$I(G,\chi,D)=\int\prod_{(i,j)\in G}(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+\chi_{(i,j)}\boldsymbol{\sigma}'_i\cdot\boldsymbol{\sigma}'_j)\prod_{k\in\Lambda}d\omega(\boldsymbol{\sigma}_k)d\omega(\boldsymbol{\sigma}'_k).$$

Expanding the product in the integrand yields

$$I(G,\chi,D)=\sum_{G_1\subset G}\left\{\left(\int\prod_{(i,j)\in\tilde{G}_1}\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\prod d\omega\right)\left(\prod_{(k,l)\in G_1}\chi_{(k,l)}\right)\left(\int\prod_{(k,l)\in G_1}\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_l\prod d\omega\right)\right\}$$

$$=\sum_{G_1\subset G}f_D(\tilde{G}_1)\chi(G_1)f_D(G_1), \tag{3.1}$$

where f_D and χ are defined by

$$f_D(G_1)=\int_{\prod S^{D-1}}\left(\prod_{(k,l)\in G_1}\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_l\right)\prod d\omega, \tag{3.2}$$

$$\chi(G_1)=\prod_{(k,l)\in G_1}\chi_{(k,l)}. \tag{3.3}$$

Here $\tilde{G}_1$ denotes the edge complement in G of the subgraph G_1. The sum over subgraphs in (3.1) is effectively restricted to the set $Z(G)$ of even subgraphs (cycles) of G, since the invariance of ω under the symmetry $\boldsymbol{\sigma}\to-\boldsymbol{\sigma}$ causes all other terms to vanish. $Z(G)$ is a natural commutative group – the cycle group – with symmetric difference as the group operation. Moreover, the function χ defined in (3.3) is a character of $Z(G)$. It is easy to check that every character of $Z(G)$ arises from some (non-unique) choice of signs $\chi_{(i,j)}$ on the edges. Thus we obtain

Proposition 3. 1. *Define the function* $F_D:Z(G)\to\mathbb{R}$ *by* $F_D(G_1)=f_D(G_1)f_D(\tilde{G}_1)$, f_D *as in* (3.2), *and define the character* $\chi\in\hat{Z}(G)$ *by* (3.3). *Then*

$$I(G,\chi,D)=\hat{F}_D(\chi), \tag{3.4}$$

where $\hat{F}_D$ *is the Fourier transform* $\hat{F}_D(\chi)=\sum_{G_1\in Z(G)}F_D(G_1)\chi(G_1)$. *Moreover, the Ginibre inequality* $I(G,\chi,D)\geqq 0$ *holds for all choices of sign* $\chi_{(i,j)}$ *if and only if* F_D *is positive definite.*

Proof. The proof is immediate from the paragraph prior. QED

This restatement of (1.1) is an improvement from a computational viewpoint: there are $2^{|G|}$ possible choices of signs $\chi_{(i,j)}$ but only $2^{|G|-|\Lambda|+1}$ characters of $Z(G)$, all of which can be readily determined. Note also that the case D = 1 of the Ginibre inequality follows immediately from Proposition 3.1, as $F_1\equiv 1$.

We now turn to examples.

Example 3.1. *Let G be the complete graph on five points K_5, and let χ be the character resulting from the choice of signs $\chi_{(i,j)} = -1$ for all edges (i,j). Then*

$$I(G, \chi, D) = \hat{F}_D(\chi) = \frac{-96(D^2-1)(D-2)}{D^8(D+2)^4}, \tag{3.5}$$

which is strictly negative when $D \geqq 3$.

Proof. This is a calculation using (2.1). QED

Thus, the Ginibre inequality (1.1) is a low spin dimension phenomenon.

We comment on Example 3.1. The complete graph on five points is one of the two elementary nonplanar graphs. This may be coincidental, as several other nonplanar graphs tested did not violate (1.1). On the other hand, every counterexample to (1.1) so far found is nonplanar, and some recent results suggest a connection between inequalities and planarity [1]. Also, note that this example displays fortuitous reduction of degree: the apparent degree of $I(G, \chi, D)$ as a function of D (see Sect. 2) is -7, but cancellations occurring in the sum over $Z(G)$ reduce the true degree to -9.

We conclude this section by summarizing the results of a computer-aided investigation of $I(G, \chi, D)$ for twenty-five graphs, including that of Example 3.1. More detail is given in an Appendix. The first fifteen graphs considered were all the graphs of cyclomatic number $\gamma = |G| - |\Lambda| + 1$ at most six having no vertices of degree two or articulation vertices. (Reduction procedures permit one to make these two simplifying assumptions about the vertices with no loss in generality.) The remaining ten graphs were chosen with the hope of finding additional counterexamples to (1.1). $\hat{F}_D(\chi)$ was computed for all characters of all graphs, with D given the values 3, 4, 5, 10. Four graphs yielded counterexamples to (1.1), and for these graphs all values of $\hat{F}_D$ were also determined at the intervening values 6, 7, 8, 9 of D. The negative values of $\hat{F}_D(\chi)$ obtained in dimension $D=3$ turn positive for two graphs when D becomes sufficiently large, while the other two have characters such that $\hat{F}_D(\chi)$ apparently remains negative for all D. All graphs analyzed are tabulated in an Appendix, which gives more detail concerning the four exceptional graphs, and a brief outline of the computational methods. In sum, approximately 21,500 values of $\hat{F}_D(\chi)$ were calculated, and of these 45 were negative.

4. Concluding Remarks

This section explicates the connection between the Ginibre inequality (1.1) and the Griffiths inequalities (1.2a, 1.2b) it was designed to prove.

After multiplying through by Z^2, (1.2b) may be effectively rewritten as

$$\int \sigma_{G_1}(\sigma_{G_2} - \sigma'_{G_2}) \exp\left(\sum J_{ab}[\boldsymbol{\sigma}_a \cdot \boldsymbol{\sigma}_b + \boldsymbol{\sigma}'_a \cdot \boldsymbol{\sigma}'_b]\right) \prod d\omega(\boldsymbol{\sigma}_k) d\omega(\boldsymbol{\sigma}'_k) \geqq 0. \tag{4.1}$$

Here we use the abbreviated notation

$$\sigma_{G_i} = \prod_{(k,l) \in G_i} \boldsymbol{\sigma}_k \cdot \boldsymbol{\sigma}_l .$$

Note that rotational invariance of (1.2b) as a function of **e** allows one to absorb the magnetic field term in the pair potential term – the "ghost spin" method of [3]. Upon decomposing (4.1) into a sum by expanding the exponential in its Maclaurin series, we see that (4.1) follows from the inequality

$$\int \sigma_{G_1}(\sigma_{G_2}-\sigma'_{G_2})\prod_{G_3}(\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b+\boldsymbol{\sigma}'_a\cdot\boldsymbol{\sigma}'_b)\prod d\omega(\boldsymbol{\sigma}_k)d\omega(\boldsymbol{\sigma}'_k)\geqq 0, \tag{4.2a}$$

or in equivalent symmetrized form

$$\int(\sigma_{G_1}-\sigma'_{G_1})(\sigma_{G_2}-\sigma'_{G_2})\prod_{G_3}(\boldsymbol{\sigma}_a\cdot\boldsymbol{\sigma}_b+\boldsymbol{\sigma}'_a\cdot\boldsymbol{\sigma}'_b)\prod d\omega(\boldsymbol{\sigma}_k)d\omega(\boldsymbol{\sigma}'_k)\geqq 0. \tag{4.2b}$$

It is apparent from (4.2b) that a counterexample to (1.1) requiring only two minus signs would cast significant doubt on (4.1), but all counterexamples presently in hand need four. On the other hand, if one could prove (1.1) with the added assumption that only two of the $\chi_{(i,j)}$ be negative, then the special case $\langle(\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)(\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_l)\rangle-\langle\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j\rangle\langle\boldsymbol{\sigma}_k\cdot\boldsymbol{\sigma}_l\rangle\geqq 0$ of (1.2b) would follow immediately from (4.2b).

A further decomposition of (4.2) permits one to derive it from (1.1). Define

$$\begin{Bmatrix} s_{ij}=\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j+\boldsymbol{\sigma}'_i\cdot\boldsymbol{\sigma}'_j \\ d_{ij}=\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j-\boldsymbol{\sigma}'_i\cdot\boldsymbol{\sigma}'_j \end{Bmatrix} \Leftrightarrow \begin{Bmatrix} \boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j=\frac{1}{2}(s_{ij}+d_{ij}) \\ \boldsymbol{\sigma}'_i\cdot\boldsymbol{\sigma}'_j=\frac{1}{2}(s_{ij}-d_{ij}) \end{Bmatrix}$$

and express the integrand of (4.2) in terms of these new variables. A sum with nonnegative coefficients of integrals of the type (1.1) results. One can cause the counterexamples to (1.1) to appear in this sum by suitable choice of G_1, G_2, G_3. However, these exceptional terms are generally of a lower order of magnitude than the other positive terms in the same sum – presumably because they are generated by fortuituous cancellations. Consequently their negative contribution is washed out in the summation of Ginibre inequalities needed for (4.2), which could yet prove true.

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Appendix

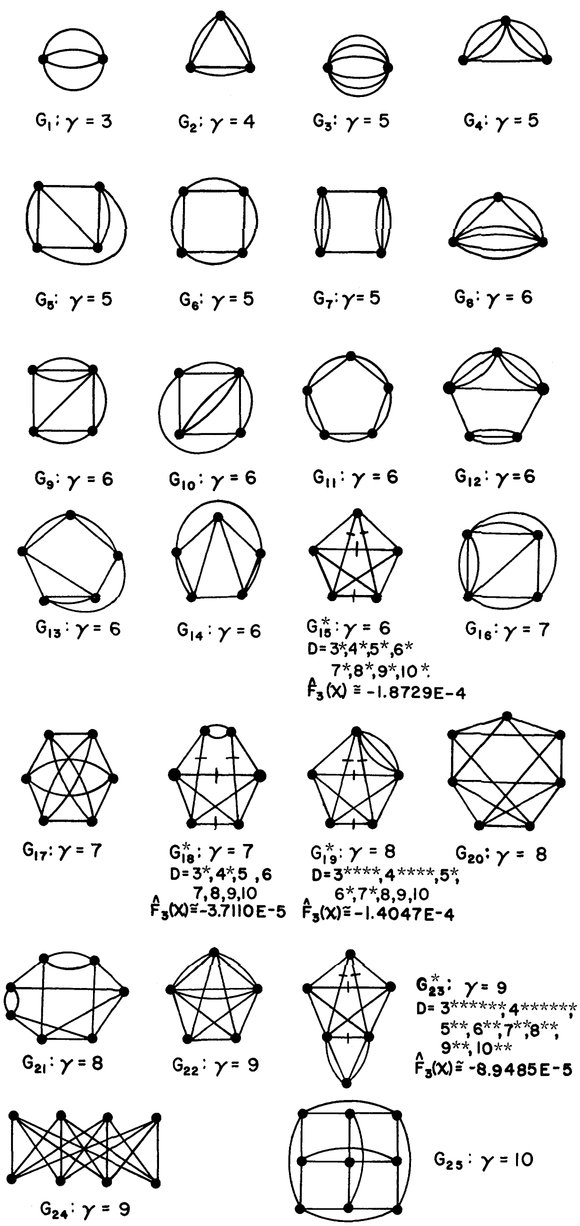
This Appendix sets forth the methods and results of a numerical investigation of (1.1).

The analysis goes by way of Proposition 3.1, using a computer routine to calculate the values of the Fourier transform $\hat{F}_D$. The routine has three main steps: determination of $f_D: Z(G)\to\mathbb{R}$, generation of the characters $\chi\in\hat{Z}(G)$, and summation over $Z(G)$ to obtain $\hat{F}_D(\chi)$. The last two steps, though of exponential complexity, are straightforward upon introducing a cycle basis. The first (also exponentially complex) was achieved by recursive implementation of the reduction formula

$$f_D(G_1)=\left[\prod_{\alpha=0}^{e-1}(D+2\alpha)\right]^{-1}\cdot\sum_{(\pi,\pi')\in\mathscr{P}_{2e}} f_D((\pi,\pi')\cdot G_1), \tag{A.1}$$

which follows from (2.1). Here we have selected arbitrarily a distinguished vertex in G_1, having $2e$ incident edges. The pair partition (π,π') of these incident edges acts on the graph G_1 to eliminate the distinguished vertex by merging together each pair of edges in the partition to yield a single edge in the reduced graph $(\pi,\pi')\cdot G_1$, and then omitting all trivial loops (edges incident on one vertex only). Some of the results of applying this routine to twenty-five graphs are tabulated below.

Table 1. This table enumerates twenty-five graphs analyzed numerically, listing them in order of increasing complexity. The cyclomatic number γ of each graph is displayed with it. All values of $\hat{F}_D(\chi)$ were determined for all graphs when $D=3,4,5,10$. Graphs 15, 18, 19, and 23 (marked by asterisks) had some negative values of $\hat{F}_D(\chi)$, and were also analyzed for $D=6,7,8,9$. (Graph 15 is Example 3.1.) The number of characters χ with $\hat{F}_D(\chi)<0$ in dimension D for these graphs is represented by the number of asterisks after D. The most negative value of $\hat{F}_3(\chi)$ is also shown, and a typical (minimal, nonunique) choice of minus signs yielding it is sketched on the graph across the edges in question. Exact calculations suggest that the relative numerical error is better than a few parts in 10^5. For example, the computed value of $\hat{F}_3(\chi)$ for graph 15 was -1.87287×10^{-4}, while the exact value is $-256/1366875 \cong 1.872885\times 10^{-4}$



References

1. Boel, R.J.: On certain relations between spin correlation of Ising models. Leiden Univ. Thesis (1978)
2. Ginibre, J.: General formulation of Griffiths' inequalities. Commun. Math. Phys. **16**, 310–328 (1970)
3. Griffiths, R.B.: Correlations in Ising ferromagnets I and II. J. Math. Phys. **8**, 478–483 and 484–489 (1967)
4. Lenard, A.: Private communication
5. Sylvester, G.S.: Inequalities for continuous-spin Ising ferromagnets. J. Stat. Phys. **15**, 327–341 (1976)

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