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Renormalized G-Convolution of N-Point Functions in Quantum Field Theory: Convergence in the Euclidean Case II

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Abstract. The notion of Feynman amplitude associated with a graph G in perturbative quantum field theory admits a generalized version in which each vertex v of G is associated with a *general* (non-perturbative) n_v-point function H^{n_v}, n_v denoting the number of lines which are incident to v in G. In the case where no ultraviolet divergence occurs, this has been performed directly in complex momentum space through Bros–Lassalle's G-convolution procedure.

Note for the Reader

The general introduction of this work and the necessary mathematical material have been published in Commun. Math. Phys. Vol. 72, pp. 175–205. We now present our result on the convergence of renormalized G-convolution. In Sect. 2, a definition of our generalized renormalized integrand R_G is given: this definition closely follows Zimmermann's algorithm [7] and involves a sum of counterterms which are associated with all the G-forests: a G-forest is a subset of "non-overlapping" subgraphs of G.

In Sect. 3, we introduce the notion of "complete forest with respect to a nested set of subspaces of $E^{rm}_{(k)}$" (this is also an extension of a notion defined in [7]). This notion allows to write new expressions of R_G which are used in the following Sect. 4. The latter contains the proof of our main theorem: R_G satisfies Weinberg's convergence criterion, and thus the renormalized integral $H_G^{(\mathrm{ren})}(K)$ is a well-defined function in the Euclidean region.

2. A Generalization of Zimmermann's Renormalized Integrand

2.1. The Unrenormalized Integrand I_G

Let us consider a general connected graph G with n external lines and m independent loops. Let $\mathscr{L}$ denote the set of internal lines of G, $\mathscr{N}$ the set of its vertices, X the set of its external lines: $|X| = n$. Each internal line is considered as oriented;

it means that a sign ε_{iv} is prescribed for each couple $(i\in\mathscr{L}, v\in\mathscr{N})$, which is $+1$ or -1 according to whether the line i points towards the vertex v or not.

With each external line of G, we associate an "Euclidean" r-vector $K_j\in E^r$, which is also represented[1] as:

$$K_j = \vec{P}_j + iQ_j^0, \quad \vec{P}_j\in\mathbb{R}^{r-1}, \quad Q_j^0\in\mathbb{R}, (j\in X) \tag{2.1}$$

and whose norm is $\|K_j\| = [\langle\vec{P}_j,\vec{P}_j\rangle + (Q_j^0)^2]^{1/2}$: each vector K_j represents the external momentum which is carried by the line j, and these vectors satisfy the total energy-momentum conservation relation: $\sum_{j\in X} K_j = 0$

We denote by $K\in\mathscr{E}_{(K)}^{r(n-1)}$ the set:

$$K = \{K_j\in\mathbb{R}^{r-1} + i\mathbb{R},\ j\in X;\ \sum_{j\in X} K_j = 0\} \tag{2.2}$$

Definition 2a

i) For every vertex v of G, we call X_v the set of the lines of G which are incident to v, *and call their number* $n_v = |X_v|$.

If $\alpha\in X_v$, it coincides either with an internal line $i\in\mathscr{L}$ or with an external line $j\in X$; we shall then write: $\alpha = \alpha(i)$ or $\alpha(j)$.

With each vertex v, we associate a set of r-momenta:

$$K^v = \{K_\alpha^v;\ \alpha\in X_v;\ \sum_{\alpha\in X_v} K_\alpha^v = 0\} \tag{2.3}$$

which varies in the Euclidean space $\mathscr{E}_v$ with dimension $r(n_v - 1)$.

ii) With each internal line $i\in\mathscr{L}$, we associate an r-momentum l_i which varies in the (r-dimensional) euclidean space $\mathscr{E}_i$.

iii) A set of independent "internal" (or "integration") Euclidean r-momenta of G is defined as a set of r-vectors

$$k = \{k_1,\dots,k_m\}\in E_{(k)}^{rm}$$

which satisfies the following property.

For every vertex $v\in\mathscr{N}$ (resp. internal line $i\in\mathscr{L}$) there exists a *linear* mapping λ_v (resp. λ_i) from the space $\underline{\mathscr{E}}_{(K,k)}^{rN} = \mathscr{E}_{(K)}^{r(n-1)}\times E_{(k)}^{rm}$ onto $\mathscr{E}_v$ (resp. $\mathscr{E}_i$) such that the corresponding substitutions:

$$(K,k)\overset{\lambda_v}{\leftrightarrow} K^v(K,k) \tag{2.4}$$

$$(K,k)\overset{\lambda_i}{\leftrightarrow} l_i(K,k) \tag{2.5}$$

be solutions of the following system:

$$-\ \forall\alpha\in X_v : K_{\alpha(j)}^v(K,k) = K_j, \quad \text{if } \alpha = \alpha(j) \tag{2.6}$$

is an external line, i.e. $j\in X$;

1 This representation is inessential in the present work, but refers to the usual complex four-momentum Minkowskian space $\mathbb{C}^r$ in which E^r is imbedded: analytic continuation problems in $\mathbb{C}^r$ will be considered in a further work.

$$K^v_{\alpha(i)}(K,k) = \varepsilon_{iv} l_i(K,k), \quad \text{if } \alpha = \alpha(i) \text{ is an internal line, i.e. } i \in \mathscr{L}; \tag{2.7}$$

$$\sum_{\alpha \in X_v} K^v_\alpha(K,k) = 0 \ \text{(see (2.3))}.$$

Definition 2b. With G we shall associate a function $I_G(K,k)$ on $\underline{\mathscr{E}}^{rN}_{(K,k)}$ by the following procedure:

i) With each vertex v of G we associate a general n_v-point function defined on the primitive analyticity domain of Q.F.T. (see introduction) contained in $\mathbb{C}^{r(n_v-1)}$ and denoted by $H^{(n_v)}(K^v)$. With each line i of G we associate a general 2-point function $H^{(2)}_{(i)}(l_i)$ defined on the corresponding primitive analyticity domain contained in $\mathbb{C}^r$. Here we shall only consider the restrictions[2] of the above functions $H^{(n_v)}_{\mathscr{E}_v}(K^v)$ and $H^{(2)}_{i\mathscr{E}_i}(l_i)$ to the Euclidean regions $\mathscr{E}_v, \mathscr{E}_i$, the latter being respectively realized as the following subspaces of $\mathbb{C}^{r(n_v-1)}$ and $\mathbb{C}^r$.

$$\begin{aligned} \mathscr{E}_v &= \{K^v \in \mathbb{C}^{r(n_v-1)} : \operatorname{Im} \vec{K} = 0 \quad \operatorname{Re}(K^v)^0 = 0\} \\ \mathscr{E}_i &= \{\vec{l}_i \in \mathbb{C}^r : \operatorname{Im} \vec{l}_i = 0 \qquad \operatorname{Re} l^0_i = 0\} \end{aligned} \tag{2.8}$$

This is meaningful since, as it is known [2, 3], the Euclidean region $\mathscr{E}_v$ (resp. $\mathscr{E}_i$) is contained inside the primitive domain of analyticity of $H^{(n_v)}(K^v)$ (resp. $H^{(2)}_{(i)}(l_i)$). We shall put:

$$(\lambda^*_v H^{(n_v)})(K,k) = H^{(n_v)}(K^v(K,k)) \tag{2.9}$$

$$(\lambda^*_i H^{(2)}_i)(K,k) = H^{(2)}_i(l_i(K,k)) \tag{2.10}$$

We moreover define the *completely amputated* general n_v-point functions by:

$$\hat{H}^{(n_v)}(K^v) = H^{(n_v)}(K^v) \times \prod_{\alpha \in X_v} [H^{(2)}_\alpha(\varepsilon_\alpha K^v_\alpha)]^{-1} \tag{2.11}$$

where ε_α is equal to $+1$ if $\alpha = \alpha(j)$ with $j \in X$, and $\varepsilon_\alpha = \varepsilon_{iv}$ if $\alpha = \alpha(i)$ with $i \in \mathscr{L}$:

We then also put: $(\lambda^*_v \hat{H}^{(n_v)})(K,k) = \hat{H}^{(n_v)}(K^v(K,k))$ (2.12)

ii) We assume that for each vertex $v \in \mathscr{N}$ (resp. line $i \in \mathscr{L}$), there exists a class of functions $\sum^{\mu_v}_{(r(n_v-1))}$ $\left(\text{resp. } \sum^{\mu_i}_{r}\right)$ to which $\hat{H}^{(n_v)}$ (resp. $H^{(2)}$) belongs; in the following μ_v and μ_i will be always considered as integers.

From (2.12) (resp. from (2.10) and Lemma (1.3) it then follows that $\hat{H}^{(n_v)}(K^v(K,k)))$ (resp. $H^{(2)}_{(i)}(l_i(K,k))$ belongs to the class $A^{(\alpha_v)}_{rN}$ (resp. $A^{(\alpha_i)}_{rN}$) with corresponding asymptotic coefficients:

$$\alpha_v(S) = 0 \quad \text{if } S \subset \operatorname{Ker} \lambda_v \ (\text{resp. } \alpha_i(S) = 0 \ \text{if } S \subset \operatorname{Ker} \lambda_i) \tag{2.13}$$

$$\alpha_v(S) = \mu_v \quad \text{if } S \not\subset \operatorname{Ker} \lambda_v \ (\text{resp. } \alpha_i(S) = \mu_i \ \text{if } S \not\subset \operatorname{Ker} \lambda_i) \tag{2.14}$$

iii) We define the function I_G on $\underline{\mathscr{E}}^{rN}_{(K,k)}$ called the unrenormalized integrand by the following products:

$$I_G(K,k) = \prod_{v \in \mathscr{N}} \hat{H}^{n_v}(K^v(K,k)) \prod_{i \in \mathscr{L}} H^{(2)}_{(i)}(l_i(K,k)) \tag{2.15}$$

2 In the following we shall only consider these restrictions and omit for simplicity the subscript

We show the following properties:

Lemma 2.1

a) $\hat{H}^{n_v}(K^v(K,k))$ *(resp.* $H^{(2)}(l_i(K,k))$*) belongs to a class* $\mathscr{A}_{rN}^{(\alpha_v,\sigma_v,\omega_v)}$ *(resp.* $\mathscr{A}_{rN}^{(\alpha_i,\sigma_i,\omega_i)}$*) of admissible Weinberg functions. The set* σ_v *(resp.* σ_i*) is the set of subspaces* $S \subset E_{(k)}^{rm}$ *satisfying:* $S \not\subset \operatorname{Ker} \lambda_v$ *(resp.* $S \not\subset \operatorname{Ker} \lambda_i$*),* α_v, α_i *are defined through formulae* (2.13), (2.14) *and:*

$$\omega_v = \{S \subset \underline{\mathscr{E}}_{(K,k)}^{rN} : S \not\subset \operatorname{Ker} \lambda_v \, \pi(S) \in \sigma_v\}$$
$$(resp.\ \omega_i = \{S \subset \underline{\mathscr{E}}_{(K,k)} : S \not\subset \operatorname{Ker} \lambda_i \,;\, \pi(S) \in \sigma_i\})$$

b) $I_G(K,k)$ *belongs to a class* $\mathscr{A}_{rN}^{(\alpha_G,\sigma_G,\omega_G)}$ *of admissible Weinberg functions. The admissible couple* (ω_G, σ_G) *in* $\underline{\mathscr{E}}_{(K,k)}^{rN}$ *is given by:*

$$\sigma_G = (\bigcap_{i \in \mathscr{L}} \sigma_i) \tag{2.16a}$$

$$\omega_G = \{S \subset \mathscr{E}_{(K,k)}^{rN} : S \not\subset \operatorname{Ker} \lambda_i \forall i \in \mathscr{L},\, \pi(S) \in \sigma_G\} \tag{2.16b}$$

The asymptotic coefficient for each subspace $S \subset \mathscr{E}_{(K,k)}^{rN}$ *is:*

$$\alpha_G(S) = \sum_{\substack{v;\\ S \not\subset \operatorname{Ker} \lambda_v}} \mu_v + \sum_{\substack{i;\\ S \not\subset \operatorname{Ker} \lambda_i}} \mu_i \tag{2.17}$$

We remark that in (2.16a, b) the property $S \not\subset \operatorname{Ker} \lambda_i \forall i \in \mathscr{L}$ automatically implies (in view of (2.3) (2.7)) $S \not\subset \operatorname{Ker} \lambda_v \cdot \forall v \in \mathscr{N}$.

Proof

a) is a direct application of Lemma 1.7 to the case of the mappings λ_v and λ_i.

b) By taking the result of a) into account and applying proposition 1.3.b. and d. to the product (2.15), the result is obtained.

In perturbation theory, Zimmermann [7, 9] has proved that for a Feynman graph Bogoliubov-Parasiuk-Hepp's method of renormalization in configuration space can be worked out in momentum space independently of any ultraviolet regularization. Let us denote by R_G^F the renormalized Zimmermann integrand associated with a Feynman graph G, and H_G^F the corresponding finite part of the Feynman amplitude such that:

$$H_G^F(K) = \int R_G^F(K,k)\, dk_1 \ldots dk_m \tag{2.18}$$

The function R_G^F is defined by Zimmermann's method as a rational function of the internal and external momenta which is obtained by subtracting appropriate counterterms from the integrand I_G^F of the divergent Feynman graph.

In the present work we shall extend Zimmermann's prescription of renormalization to the case of a general integrand I_G defined by (2.15), and we shall show that in the Euclidean region $\mathscr{E}_{(K)}^{r(n-1)}$ a certain "renormalized convolution product" H_G can be defined through an integral of the form:

$$H_G(K) = \int_{E^{rm}} R_G(K,k)\, dk_1 \ldots dk_m, \tag{2.19}$$

where R_G is a functional of the $H^{(n_v)}$ and $H_i^{(2)}$ obtained by the following procedure.

2.2 Admissible Sets of Momenta for G and its Subgraphs

Let us recall some usual notions in the matter. A graph G is called one-particle irreducible or "proper" if it is connected and cannot be separated in two parts by cutting a single line. A graph γ is a *subgraph* of a graph G, written $\gamma \subset G$, if the vertices of γ form a subset $\mathcal{N}_\gamma$ of the set $\mathcal{N}$ of vertices of G and if all the lines of γ are lines of G ending to vertices of γ; moreover some of the internal lines of G may be cut to form external lines of γ. Two graphs γ_1, γ_2 are called nonoverlapping if[3] either $\gamma_1 \cap \gamma_2 = \emptyset$ or $\gamma_1 \subset \gamma_2$ or $\gamma_2 \subset \gamma_1$.

A G forest U is a set of non trivial nonoverlapping one-particle irreducible subgraphs of a graph G. U may also be the empty set. AG forest U is called full if $G \in U$; if $G \notin U$ then U is called a normal forest. If γ is any subgraph of U then $U(\gamma)$ denotes the set of all $\gamma' \in U$ with $\gamma' \subset \gamma$ (in particular $U(G) = U$). In the following, all the momenta which we introduce belong to the Euclidean space $E^r = \mathbb{R}^{r-1} + i\mathbb{R}$.

Definition 2c

i) For any subgraph γ of G, we denote by $\mathcal{N}_\gamma$ the set of vertices of γ, by $\mathcal{L}_\gamma$ (resp. X_γ) the set of internal (resp. external) lines of γ; $\mathcal{N}_\gamma$ (resp. $\mathcal{L}_\gamma$) is a subset of $\mathcal{N}$ (resp. $\mathcal{L}$). We also call $n_\gamma = |X_\gamma|$ the number of external lines of γ, and $m(\gamma)$ the number of independent loops of γ (note that $m(G) = m, n_G = n, \mathcal{N}_G = \mathcal{N}$ etc ...).

ii) With γ, one can always associate two sets of variables, K^γ and k^γ, which play the same role as the variables K and k for $G(K^G = K, k^G = k)$.

– K^γ is the set of "external" r-momenta of γ, namely:

$$K^\gamma = \{K_j^\gamma;\ j \in X_\gamma;\ \sum_{j \in X_\gamma} K_j^\gamma = 0\} \tag{2.20}$$

it varies in a space $\mathscr{E}^{r(n_\gamma - 1)}_{(K^\gamma)}$.

– $k^\gamma = \{k_1^\gamma, \ldots, k_{m(\gamma)}^\gamma\}$ is a set of independent "internal" (or "integration") r-momenta of γ; varies in a space $E^{rm(\gamma)}_{(K^\gamma)}$.

The choice of the variables k^γ contains a large arbitrariness, but satisfies the following requirement.

For every vertex $v \in \mathcal{N}_\gamma$ (resp. internal line $i \in \mathcal{L}_\gamma$), there exists a *linear* mapping $\tilde{\lambda}_v^\gamma$ (resp. $\tilde{\lambda}_i^\gamma$) from $\mathscr{E}^{r(n_\gamma - 1)}_{(K^\gamma)} \times E^{rm(\gamma)}_{(k^\gamma)}$ to $\mathscr{E}_v$ (resp. $\mathscr{E}_i$) such that the corresponding substitutions:

$$(K^\gamma, k^\gamma) \xrightarrow{\tilde{\lambda}_v^\gamma} K^v(K^\gamma, k^\gamma)$$

$$(K^\gamma, k^\gamma) \xrightarrow{\tilde{\lambda}_i^\gamma} l_i(K^\gamma, k^\gamma)$$

be solutions of the following system:

$$-\sum_{\alpha \in X_v} K_\alpha^v(K^\gamma, k^\gamma) = 0 \qquad \text{(see (2.3))}$$

$$-K_{\alpha(i)}^v(K^\gamma, k^\gamma) = \varepsilon_{iv} l_i(K^\gamma, k^\gamma), \tag{2.21}$$

3 The intersection of graphs is understood as being taken on the set of vertices and on the set of internal and external lines.

if $\alpha = \alpha(i)$ is an internal line of $\gamma(i \in \mathscr{L}_\gamma)$.

$$- K^v_{\alpha(j)}(K^\gamma, k^\gamma) = K^\gamma_j, \tag{2.22}$$

if $\alpha = \alpha(j)$ is an external line of $\gamma(j \in X_\gamma)$.

Note that $\tilde{\lambda}^G_v = \lambda_v$, $\tilde{\lambda}^G_i$, $\lambda^G_i = \lambda_i$ (see definition 2.a.).

iii) It has been shown by Zimmermann [7] through a constructive procedure that it is possible to define simultaneously for all subgraphs γ of G (including G) the variables (K^γ, k^γ) and the corresponding mappings $\tilde{\lambda}^\gamma_i$, $\tilde{\lambda}^\gamma_v$ in such a way that the following properties hold:

—for every couple (γ, γ') with $\gamma' \subset \gamma$ (where γ can be G itself), there exists a linear mapping $\beta^\gamma_{\gamma'}$ from $\mathscr{E}^{r(n_\gamma - 1)}_{(K^\gamma)} \times E^{rm(\gamma)}_{(k^\gamma)}$ to $\mathscr{E}^{r(n_{\gamma'} - 1)}_{(K^{\gamma'})} \times E^{rm(\gamma')}_{(k^{\gamma'})}$ which has the following form:

$$\beta^\gamma_{\gamma'}: \quad K^{\gamma'} = K^{\gamma'}(K^\gamma, k^\gamma) \tag{2.23}$$

$$k^{\gamma'} = k^{\gamma'}(k^\gamma) \tag{2.24}$$

$- \forall (\gamma, \gamma', \gamma'')$ with $\gamma'' \subset \gamma' \subset \gamma$, one has:

$$\beta^\gamma_{\gamma''} = \beta^{\gamma'}_{\gamma''} \circ \beta^\gamma_{\gamma'} \tag{2.25}$$

The important point here is that $k^{\gamma'}$ is a function of k^γ but does not depend on K^γ. Following the terminology of [7], we say that such a set of variables $\{(K_\gamma, k_\gamma), \forall \gamma \subset G\}$ is an *admissible set of basic momenta.* The choice of this set still contains a large arbitrariness. Such a choice will play a crucial role in the definition of R_G and in the proof of the convergence of the integral (2.19). We shall suppose that we have chosen once for all a certain admissible set in the following sections 3 and 4; (for a proof of the fact that H_G is independent of the choice of any admissible set of basic momenta see [12]).

Definition 2d

i) For every $\gamma \subset G$, the mapping β^G_γ allows to reexpress (K^γ, k^γ) as a function of (K^γ, k) through the linear substitution $k^\gamma = k^\gamma(k)$ (see (2.24)). (K^γ, k) varies in a space $\underline{\mathscr{E}}^{rN_\gamma}_{(K^\gamma, k)} = \mathscr{E}^{r(n_\gamma - 1)}_{(K^\gamma)} \times E^{rm(\gamma)}_{(k)} (N_\gamma = m(\gamma) + n_\gamma - 1)$.

Then with every couple γ, γ' such that $\gamma' \subset \gamma$, we can associate a linear mapping $s^\gamma_{\gamma'}$ from $\underline{\mathscr{E}}^{rN_\gamma}_{(K^\gamma, k)}$ to $\underline{\mathscr{E}}^{rN_{\gamma'}}_{(K^{\gamma'}, k)}$ which is defined as follows:

$$(K^\gamma, k) \overset{s^\gamma_{\gamma'}}{\longleftrightarrow} (K^{\gamma'} = K^{\gamma'}(K^\gamma, k^\gamma(k)), k) \tag{2.26}$$

In (2.26) the substitutions are those defined by $\beta^\gamma_{\gamma'}$, and β^G_γ (see (2.23), (2.24)). It is easy to check that due to formula (2.25), the set $\{s^\gamma_{\gamma'}; \forall \gamma, \gamma' \subset G\}$ satisfy the similar property:

$$\text{if } \gamma'' \subset \gamma' \subset \gamma : s^\gamma_{\gamma''} = s^{\gamma'}_{\gamma''} \circ s^\gamma_{\gamma'} \tag{2.27}$$

ii) With each mapping $\tilde{\lambda}^\gamma_v$ (resp. $\tilde{\lambda}^\gamma_i$) introduced in Definition 2b ii), we can associate a mapping λ^γ_v (resp. λ_i) from $\underline{\mathscr{E}}^{rN_\gamma}_{(K^\gamma, k)}$ to $\mathscr{E}_v$ (resp. $\mathscr{E}_i$) through the following formulae:

$$(K^\gamma, k) \overset{\lambda^\gamma_v}{\longrightarrow} K^v = K^v(K^\gamma, k^\gamma(k)) \tag{2.28a}$$

$$(K^\gamma, k) \overset{\lambda^\gamma_i}{\longrightarrow} K^i = l^i(K^\gamma, k^\gamma(k)), \tag{2.28b}$$

In (2.28a) (2.28b), the substitutions are those defined by $\tilde{\lambda}^\gamma_v, \tilde{\lambda}^\gamma_i, \beta_\gamma$.

Remark. We can verify the following relation between the above mappings:

$$\forall \mu \subset \gamma \text{ and } j \in \mathscr{L}_\mu \lambda_j^\gamma = \lambda_j^\mu s_\mu^\gamma \tag{2.29}$$

Definition 2e. Being given the sets of functions $H^{(n_v)}, H^{(2)}$ of Definition 2b, we shall now associate with them for every subgraph γ of G a function $I_\gamma(K^\gamma, k)$ on $\mathscr{E}^{rN_\gamma}_{(K^\gamma,k)}$ which is similar to I_G (see formula (2.15)).

We put:

$$I_\gamma(K^\gamma, k) = \prod_{v \in \mathscr{N}_\gamma} \hat{H}^{(n_v)}(K^v(K^\gamma, k^\gamma(k))) \times \prod_{i \in \mathscr{L}_\gamma} H_i^{(2)}(l_i(K^\gamma, k^\gamma(k))) \tag{2.30}$$

By writing:

$$\hat{H}^{(n_v)}(K^v(K^\gamma, k^\gamma(k)) = (\lambda_v^{\gamma *} \hat{H}^{(n_v)})(K^\gamma, k))$$

and

$$H_i^{(2)}(l_i(K^\gamma, k^\gamma(k)) = (\lambda_i^{\gamma *} H_i^{(2)})(K^\gamma, k)$$

and applying Lemma 1.7 to the mappings $\lambda_i^\gamma, \lambda_v^\gamma$, we deduce from the assumption $\hat{H}^{(n_v)} \in \sum_{r(n_v-1)}^{\mu_v}$ and $H_i^{(2)} \in \sum_r^{\mu_i}$ the following lemma whose proof is exactly similar to that of Lemma 2.1. (one now applies Proposition 1.3.b. and d. to the product (2.30)).

Lemma 2.2. *For every $\gamma \subset G$, the corresponding function $I_\gamma(K^\gamma, k)$ belongs to the class $\mathscr{A}_{rN^\gamma}^{(\alpha_\gamma,\sigma_\gamma,\omega_\gamma)}$ which is determined as follows:*

a) $$\sigma_\gamma = (\bigcap_{i \in \mathscr{L}_\gamma} \sigma_i^\gamma) \tag{2.31}$$

where

$$\sigma_i^\gamma = \{S \subset E_{(k)}^{rm}; S \not\subset \operatorname{Ker} \lambda_i^\gamma\} \tag{2.32}$$

b) $$\omega_\gamma = \{S_\gamma \subset \mathscr{E}^{rN_\gamma}_{(K^\gamma,k)} : S_\gamma \not\subset \operatorname{Ker} \lambda_i^\gamma \, \forall i \in \mathscr{L}_\gamma; \pi(S_\gamma) \in \sigma_\gamma\} \tag{2.33}$$

c) $\forall S_\gamma \subset \mathscr{E}^{rN^\gamma}_{(K^\gamma,k)}$, *one has:*

$$\alpha_\gamma(S_\gamma) = \sum_{i; S_\gamma \not\subset \operatorname{Ker} \lambda_i^\gamma} \mu_i + \sum_{v; S_\gamma \not\subset \operatorname{Ker} \lambda_v^\gamma} \mu_v \tag{2.34}$$

2.3. *The Renormalized Integrand R_G*

Definition 2f. We first need to recall the following notions which are relative to forests of subgraphs in G.

i) If γ is an element of a given forest $U(G)$, we shall introduce the set $\mathscr{M}_\gamma(U) = \{\gamma_a; 1 \leqq a \leqq c_\gamma\}$ of all subgraphs $\gamma_a \in U(\gamma)$ which are maximal in γ. We also consider the associated "reduced graph" $\bar{\gamma}$ of γ, which is obtained from γ by contracting each γ_a to a single vertex in γ. (see [7]). We then call $m(\bar{\gamma})$ the number of independent loops of $\bar{\gamma}$. In view of the definition of $m(\gamma)$ (see Definition 2b)), we obviously have:

$$m(\gamma) = m(\bar{\gamma}) + \sum_{1 \leqq a \leqq c} m(\gamma_a) \tag{2.35}$$

Note that the definition of $\bar{\gamma}$ depends on the forest $U = U(G)$. When several forests

U are involved, we shall use the more precise notation $\bar{\gamma}(U)$ in order to avoid any ambiguous meaning (it will be the case in section 3).
ii) With the reduced graph $\bar{G}$ (resp. $\bar{\gamma}$) of G (resp. γ) in the given forest $U(G)$, we associate the following functions $I_{\bar{G}(U)}$ (resp. $I_{\gamma(U)}$):

$$I_{\bar{G}(U)}(K,k) = \prod_{v\in\mathcal{N}\bar{G}} H^{n_v}(K^v(K,k)) \prod_{i\in\mathcal{L}\bar{G}} H^{(2)}_{(i)}(l_i(K,k)) \tag{2.36}$$

$$(\text{resp.}: I_{\bar{\gamma}(U)}(K^\gamma,k) = \prod_{v\in\mathcal{N}_{\bar{\gamma}}} H^{n_v}(K^v(K^\gamma,k^\gamma(k)) \prod_{i\in\mathcal{L}_{\bar{\gamma}}} H^{(2)}_{(i)}(l_i(K^\gamma,k^\gamma(k)) \tag{2.37}$$

Here we have used the notations: $\mathcal{L}_{\bar{\gamma}} = \mathcal{L}_\gamma \setminus \bigcup_{\gamma_a\in\mathcal{M}_\gamma(U)} \mathcal{L}_{\gamma_a}$, $\mathcal{N}_{\bar{\gamma}} = \mathcal{N}_\gamma \setminus \bigcup_{\gamma_a\in\mathcal{M}_\gamma(U)} \mathcal{N}_{\gamma_a}$.
By applying again Lemma 1.7 and Proposition 1.3b and d, we then obtain a property which is analogous to Lemmas 2.1 and 2.2.

Lemma 2.3. *Let $\bar{\gamma}$ be the reduced graph of γ in a forest $U(G)$. Then the corresponding function $I_{\bar{\gamma}(U)}(K^\gamma,k)$ belongs to the class $\mathcal{A}^{(\alpha_{\bar{\gamma}},\sigma_{\bar{\gamma}},\omega_{\bar{\gamma}})}_{rN_\gamma}$ which is defined as follows:*

a) $$\sigma_{\bar{\gamma}} = \Big(\bigcap_{i\in\mathcal{L}_{\bar{\gamma}}} \sigma_i^\gamma\Big) \tag{2.38a}$$

where σ_i^γ, are defined by (2.32)

b) $$\omega_{\bar{\gamma}} = \{S_\gamma \subset \mathcal{E}^{rN_\gamma}_{(K^\gamma,k)} : S_\gamma \not\subset \operatorname{Ker}\lambda_i^\gamma \cdot \forall i \in \mathcal{L}_{\bar{\gamma}}\,;\, \pi(S_\gamma)\in\sigma_{\bar{\gamma}}\} \tag{2.38b}$$

c) $\forall S_\gamma \subset \mathcal{E}^{rN_\gamma}_{(K^\gamma,k)}$ *one has:*

$$\alpha_{\bar{\gamma}}(S_\gamma) = \sum_{\substack{i\in\mathcal{L}_{\bar{\gamma}};\\ S_\gamma\not\subset\operatorname{Ker}\lambda_i^\gamma}} \mu_i + \sum_{\substack{v\in\mathcal{N}_{\bar{\gamma}};\\ S_\gamma\not\subset\operatorname{Ker}\lambda_v^\gamma}} \mu_v \tag{2.39}$$

Definition 2g
i) The sets $\{\mu_v\,; v\in\mathcal{N}\}$ and $\{\mu_i\,; i\in\mathcal{L}\}$ being given, we associate with every subgraph γ of G the following number which we call the "*dimension of* γ" (relative to the latter sets):

$$d(\gamma) = \sum_{v\in\mathcal{N}_\gamma} \mu_v + \sum_{i\in\mathcal{L}_\gamma} \mu_i + rm(\gamma) \tag{2.40}$$

If we similarly put:

$$d(\bar{\gamma}) = \sum_{v\in\mathcal{N}_{\bar{\gamma}}} \mu_v + \sum_{i\in\mathcal{L}_{\bar{\gamma}}} \mu_i + rm(\bar{\gamma}) \tag{2.41}$$

for the reduced graph $\bar{\gamma}$ of γ (relative to a certain forest $U(G)$ containing γ), we have the obvious relation:

$$d(\gamma) = d(\bar{\gamma}) + \sum_{1\leq a\leq c_\gamma} d(\gamma_a) \tag{2.42}$$

ii) For every function $F(K^\gamma,k)$ on $\mathcal{E}^{rN_\gamma}_{(K^\gamma,k)}$, $(t^{d(\gamma)}F)(K^\gamma,k)$ will denote the Taylor expansion of F of degree $d(\gamma)$ with respect to K^γ at $K^\gamma=0$ (this function is intrinsically defined: see Sect. I.3). Of course this definition only holds if $d(\gamma)\geq 0$ which is not necessarily the case (if this is the case γ is called a "renormalization part" as in [7]). If $d(\gamma)<0$ we put by definition $t^{d(\gamma)}F=0$.

Remark. The mapping $(K^\gamma,k)\to(K^\gamma,k^\gamma=k^\gamma(k))$ defined by (2.24) commutes with the operation $t^{d(\gamma)}$, namely if $F(K^\gamma, k) = \tilde{F}(K^\gamma, k^\gamma(k))$, then

$$(t^{d(\gamma)}F)(K^\gamma, k) = (t^{d(\gamma)}\tilde{F})(K^\gamma, k^\gamma)\big|_{k^\gamma = k^\gamma(k)}$$

Definition 2h. We propose the following generalisation of Zimmermann's renormalized integrand. Let $\mathscr{U}$ be the set of all the forests of G. Then we put:

$$R_G(K,k) = \sum_{U \in \mathscr{U}} F_U(K,k) \tag{2.43}$$

where:

$$\left.\begin{aligned} \forall U \in \mathscr{U} : F_U(K,k) &= Y_G^{(U)}(K,k) && \text{if } U \text{ is normal} \\ &= (-t^{d(G)})\, Y_G^{(U)}(K,k) && \text{if } U \text{ is full} \end{aligned}\right\} \tag{2.44}$$

The function $Y_G^{(U)}(K,k)$ and all the auxiliary functions $Y_\gamma^{(U)}(K^\gamma, k)$ corresponding to every $\gamma \in U$ are recursively defined as follows:

$$Y_\gamma^{(U)}(K^\gamma, k) = I_{\bar{\gamma}(U)}(K^\gamma, k) \prod_{1 \leqq a \leqq c_\gamma} (s_a^*(-t^{d(\gamma_a)})\, Y_{\gamma_a}^{(U)})(K^\gamma, k). \tag{2.45}$$

In (2.45), $I_{\bar{\gamma}}$ is the function introduced by (2.37), and s_a^* is a short notation for $(s_{\gamma_a}^\gamma)^*$, namely the inverse image operation induced by the linear mapping $s_{\gamma_a}^\gamma$ defined through (2.26) (Definition 2d (i)). (Indeed, $-t^{d(\gamma_a)}Y_{\gamma_a}$ is a function of (K^{γ_a}, k); s_a^* denotes the substitution: $(K^{\gamma_a}, k) = (K^{\gamma_a}(K^\gamma, k), k)$.

To see that (2.44) actually provides a complete recursive definition of F_U, it is sufficient to notice that if γ is a minimal subgraph of U(*i.e.* $c_\gamma = 0$), then: $Y_\gamma^{(U)}(K^\gamma, k) = I_{\bar{\gamma}(U)}(K^\gamma, k) = I_\gamma(K^\gamma, k)$.

So the recursion works by inclusion (of subgraphs) inside the forest, starting from the minimal subgraphs and ending at $\gamma = G$.

Remarks

i) Due to the definition of $t^{d(\gamma)}$; if γ is not a renormalization part ($d(\gamma) < 0$), it yields no *contribution* to the formulae (2.43), (2.44), (2.45).

ii) Formulae (2.43), (2.44), (2.45) are the exact analogs of Zimmermann's integrand R_G^F (see [7]) although in the latter the sets of variables (K^γ, k^γ) instead of (K^γ, k) had been used in the corresponding expressions (namely for the operations s and $t^{d(\gamma)}$); however, our last remark at the end of Definition 2g, shows that this slight change in the presentation causes no perturbation in the algorithm which defines R_G (or R_G^F).

From the analyticity of $\hat{H}^{n_v}, \hat{H}_i^{(2)}$ in the regions $\mathscr{E}_v, \mathscr{E}_i$ (Definition 2bi) we can now derive the following property:

Lemma 2.3. *$R_G(K,k)$ is a (real) analytic function on $\underline{\mathscr{E}}_{(K,k)}^{(rN)}$. This comes from the fact that all the operations (including the linear mappings s_a^* and the Taylor expansions $t^{d(\gamma_a)}$) involved in the above definition of R_G preserve the analyticity property.*

The above analyticity property allows us to take $E_{(k)}^{rm}$ as an integration contour (inside the analyticity domain of $R_G(K,k)$). In order to show that the Renormalized Convolution product defined by (2.19) makes sense, we now have to prove the absolute convergence of this integral; this will be possible if we ensure that the conditions of the Power Counting Theorem of Weinberg (Lemma 1.2.) are satisfied; this is the scope of the two next sections.

3. The "Complete Forest" Formula for $\mathbf{R}_G$ with Respect to a Nested Set $\mathscr{F}$ of Subspaces of $\mathbf{E}^{rm}$

As a first step towards the proof of the integrability of the function $R_G(K,k)$ with respect to k it is necessary to transform the original expression of R_G (see Definition 2h) and establish some combinatorial results which are similar to those of [7] and make an essential use of the notion of "complete forest". Here however, we need to introduce the notion of complete forest not only with respect to a given subspace S of $E^{rm}_{(k)}$ (as in [7]), but more generally with respect to an arbitrary nested set of subspaces of $E^{rm}_{(k)}$.

Definitions 3a (see also [7])
1. An internal line i of γ is called constant (resp. variable) in a subspace $S \subset E^{rm}_{(k)}$ relatively to γ, if the corresponding subspace $\{K^\gamma = 0, k \in S\}$ of $\mathscr{E}^{rN_\gamma}_{(K^\gamma,k)}$ is contained (resp. is not contained) in Ker λ_i^γ (see Definition 2dii.).
2. A G-forest U is called complete with respect to a subspace S of $E^{rm}_{(k)}$ if:
i) $G \in U$
ii) For every $\gamma \in U$ the internal lines of the reduced graph $\bar{\gamma}(U)$ are either all variable in S relatively to γ or all constant in S relatively to γ.
3. For a G-forest U which is complete with respect to S one defines two sets of subgraphs $W^S(U)$ and $B^S(U)$ by the following conditions:
  a) $W^S(U)$ is the set of all $\gamma \in U$ for which all the lines in $\mathscr{L}_{\gamma(U)}$ are constant in S relatively to γ.
  b) $B^S(U)$ is the set of all $\tau \in U$ satisfying:

$$\tau \notin W^S(U); \quad \exists \gamma \in W^S(U) \text{ such that } \tau \in \mathscr{M}_\gamma(U)$$

Lemma 3.1. *Let U be a complete forest with respect to a subspace $S \subset E^{rm}_{(k)}$ and let $\gamma \in W^S(U)$, and $\mu \in \mathscr{M}_\gamma(U)$. Then the following properties hold:*
a) $s^\gamma_\mu(\{(K^\gamma k);\, K^\gamma = 0,\, k \in S\}) = \{(K^\mu k);\, K^\mu = 0,\, k \in S\}$
b) *A line $i \in \mathscr{L}_\mu$ is constant in S relatively to μ, if and only if it is constant in S relatively to γ.*

Proof. According to (2.20) we have:
$K^\mu = \{K^\mu_j; j \in X_\mu;\ \sum_{j \in X_\mu} K^\mu_j = 0\}$. Now the equations which define s^γ_μ (see (2.23), (2.24), (2.26)) have to satisfy the system (2.21), (2.22) which implies (since $X_\mu \subset X_\gamma \cup \mathscr{L}_{\bar{\gamma}(U)}$),

$$\forall j \in X_\mu : K^\mu_j(K^\gamma, k^\gamma(k)) = \begin{cases} K^\gamma_j & \text{if } j \in X_\gamma \\ \pm\, l_j(K^\gamma, k^\gamma(k)) & \text{if } j \in \mathscr{L}_\gamma \end{cases}$$

But since $\gamma \in W^S(U)$, we have: $l_j(K^\gamma; k^\gamma(k)) = 0$ for $K^\gamma = 0$, $k \in S$, and this entails:

$$\forall j \in X_\mu;\ K^\mu_j(K^\gamma, k^\gamma(k)) = 0 \ \text{ for } K^\gamma = 0,\ k \in S, \text{ which proves } a).$$

Moreover, let us apply relation 2.29):

$\forall i \in \mathscr{L}_\mu ; \lambda_i^\mu \circ s_\mu^\gamma = \lambda_i^\gamma$. Then from the above result $a)$ we deduce that:

$\forall i \in \mathscr{L}_\mu, \lambda_i^\mu(\{(K^\mu, k); K^\mu = 0, k \in S\}) = \lambda_i^\gamma(K^\gamma, k); K^\gamma = 0, k \in S\})$

and this entails b). q.e.d.

One can state the following property of R_G, which is similar to the one proved for R_G^F in [7].

Lemma 3.2. *Being given a subspace* S, R_G *admits the following corresponding expression:*

$$R_G = \sum_U X_U^{(S)} \tag{3.1}$$

in which the sum extends to all the forests which are complete with respect to S *and where*

$$X_U^{(S)} = (1 - t^{d(G)}) Y_G^{(U,S)} \tag{3.2}$$

$Y_G^{(U,S)}$ *and the following auxiliary functions* Y^γ *are defined recursively (for every* $\gamma \subset G$) *by*

$$Y_\gamma = I_{\bar{\gamma}} \prod_{\gamma_a \in \mathscr{M}_\gamma(U)} s_a^* f_{\gamma_a} Y_{\gamma_a} \tag{3.3}$$

and

$$\left.\begin{array}{ll} f_{\gamma_a} = (1 - t^{d(\gamma_a)}) & \text{if } \gamma_a \in B^S(U) \\ f_{\gamma_a} = -t^{d(\gamma_a)} & \text{if } \gamma_a \notin B^S(U) \end{array}\right\} \tag{3.4}$$

We omit the proof of this lemma which goes exactly as that of [7] for R_G^F; indeed, it will be generalized in Proposition 3.1 below.

Definition 3b. Let U be a complete forest with respect to an arbitrary subspace $S \subset E_{(k)}^{rm}$. For a given graph $\gamma \in U$ we define the integer $M^{(S)}(\gamma)$ by:

$$M^{(S)}(\gamma) = r \sum_{\substack{\mu \in U(\gamma) \\ \mu \notin W(U)}} m(\bar{\mu}(U)) \tag{3.5}$$

With $m(\bar{\mu}(U))$ given by Definition 2fi).

In (3.5) the summation extends over all $\mu \in U$ such that $\mu \subset \gamma$ and $\mu \notin W(U)$; so two cases occur:

$$\text{— if } \gamma \notin W(U), \text{ then}: M^{(S)}(\gamma) = rm(\bar{\gamma}(U)) + \sum_{\gamma_a \in \mathscr{M}_\gamma(U)} M^{(S)}(\gamma_a) \tag{3.6}$$

$$\text{— if } \gamma \in W(U), \text{ then}: M^{(S)}(\gamma) = \sum_{\gamma_a \in \mathscr{M}_\gamma(U)} M^{(S)}(\gamma_a) \tag{3.7}$$

Remark. It can easily be seen that the dimension $h(S)$ of S satisfies the inequality:

$$M^{(S)}(G) \geqq h(S) \tag{3.8}$$

Definitions 3c

1. We now consider an arbitrary set of subspaces $S^{(j)} \subset E_{(k)}^{rm} j = 1 \dots L$ which satisfy: $S^{(1)} \subset S^{(2)} \subset \dots \subset S^{(L)}$. We denote by $\mathscr{F}$ this nested set of subspaces.

2. Let $\mathscr{U}(S^{(j)})$ be the set of all the complete forests of G with respect to the subspace $S^{(j)} \in \mathscr{F}(j = 1 \ldots L)$.

Following Definition 3a.3, we can associate with such a forest $U \in \mathscr{U}(S^{(j)})$, the sets of subgraphs $W^{S(j)}(U)$, $B^{S(j)}(U)$; for simplicity we shall put: $W^{S(j)}(U) = W^{(j)}(U)$, $B^{S(j)}(U) = B^{(j)}(U)$; all these sets are subsets of U.

Starting from any forest $U^{(1)} \in U(S^{(1)})$, it will be possible to associate with it a unique minimal forest which contains it and which is complete simultaneously with respect to all subspaces $S^{(j)} \in \mathscr{F}$: this will be achieved by a recursion procedure which makes use of successive completions as follows:

Let $U^{(j)}$ be a forest which is complete with respect to *all* subspaces $S^{(l)}$, such that $l \leqq j$. By using the prescription of [7][4], we construct the completion forest $U^{(j+1)}$ of $U^{(j)}$ with respect to $S^{(j+1)}$ through the following formula:

$$U^{(j+1)} = U^{(j)} \cup \mathscr{A}^{(j+1)}(U^{(j)}) \tag{3.9}$$

We recall briefly the definition of $\mathscr{A}^{(j+1)}(U^{(j)})$ as given in [7]. We consider the set $\tilde{W}^{(j+1)}(U^{(j)})$ of subgraphs in $U^{(j)}$ such that at least one line in $\mathscr{L}_{\bar{\gamma}(U^{(j)})}$ is constant in $S^{(j+1)}$ relatively to γ; for every subgraph $\gamma \in \tilde{W}^{(j+1)}(U^{(j)})$, we then call $s(\gamma, U^{(j)})$ the subset of *all* lines in $\mathscr{L}_{\bar{\gamma}(U^{(j)})}$ which are constant in $S^{(j+1)}$; the set $\mathscr{A}^{(j+1)}(U^{(j)})$ is then defined by:

$$\mathscr{A}^{(j+1)}(U^{(j)}) = \begin{cases} \tau \subset G;\ \tau \notin U^{(j)};\ \exists \gamma \in \tilde{W}^{(j+1)}(U^{(j)}) \text{ such that } \tau \text{ is a connected} \\ \text{component of } \gamma \backslash s(\gamma, U^{(j)}) \end{cases} \tag{3.10}$$

Note that if $U^{(j)}$ belongs to $\mathscr{U}(S^{(j+1)})$, then $\mathscr{A}^{(j+1)}(U^{(j)})$ is empty.

Lemma 3.3. *Let $U^{(j)}$ belong to all sets $\mathscr{U}(S^{(l)})$, with $l \leqq j$. Then the forest $U^{(j+1)}$ (see (3.9)), the set $\mathscr{A}^{(j+1)}(U^{(j)})$ (see (3.10)) and the various sets $W^{(l)}(U^{(j)})(l \leqq j)$ satisfy the following properties:*

i) $U^{(j+1)} \in \mathscr{U}(S^{(l)})$, for every $l \leqq j+1$;

moreover

$$W^{(j+1)}(U^{(j+1)}) = \tilde{W}^{(j+1)}(U^{(j)}) \subset W^{(j)}(U^{(j)}) \tag{3.11}$$

$$\text{ii)}\ \mathscr{A}^{(j+1)}(U^{(j)}) \subset B^{(j+1)}(U^{(j+1)}), \tag{3.12}$$

$$\text{iii)}\ \forall l \leqq j,\ W^{(l)}(U^{(j+1)}) = \mathscr{A}^{(j+1)}(U^{(j)}) \cup W^{(l)}(U^{(j)}) \tag{3.13}$$

Proof

a) The fact that $U^{(j+1)}$ is a forest has been proved in [7].
Let $\gamma \in U^{(j+1)}$; three cases are possible:

$$\gamma \in \tilde{W}^{(j+1)}(U^{(j)}), \quad \gamma \in U^{(j)} \backslash \tilde{W}^{(j+1)}(U^{(j)}), \quad \gamma \in \mathscr{A}^{(j+1)}(U^{(j)}).$$

By construction if $\gamma \in U^{(j)} \backslash \tilde{W}^{(j+1)}(U^{(j)})$, all the lines in $\mathscr{L}_{\bar{\gamma}}$ are variable in $S^{(j+1)}$, if $\gamma \in \tilde{W}^{(j+1)}(U^{(j)})$, all the lines in $\mathscr{L}_{\bar{\gamma}(U^{(j+1)})}$ are constant (since in $s(\gamma, U^{(j)})$); if $\gamma \in \mathscr{A}^{(j+1)}(U^{(j)})$ there exists a $\gamma' \in \tilde{W}^{(j+1)}(U^{(j)})$ such that $\gamma \in \mathscr{M}_{\gamma'}(U^{(j+1)})$, and all the lines in $\mathscr{L}_{\bar{\gamma}(U^{(j+1)})}$ are variable in $S^{(j+1)}$ relatively to γ'; but the argument of Lemma

4 See in [7] Lemmas (4.1) ... (4.5)

3.1 then shows that all these lines are variable relatively to γ. All these remarks show that $U^{(j+1)} \in \mathscr{U}(S^{(j+1)})$, with the associated sets:

$W^{(j+1)}(U^{(j+1)}) = \tilde{W}^{(j+1)}(U^{(j)})$ and $B^{(j+1)}(U^{(j+1)})$ containing $\mathscr{A}^{(j+1)}(U^{(j)})$. The inclusion $\tilde{W}^{(j+1)}(U^{(j)}) \subset W^{(j)}(U^{(j)})$ is trivial since for every $\gamma \notin W^{(j)}(U^{(j)})$, every line in $\mathscr{L}_{\bar{\gamma}(U^{(j)})}$ is variable in $S^{(j)}$ and therefore in $S^{(j+1)}$, so that $\gamma \notin \tilde{W}^{(j+1)}(U^{(j)})$.

b) Let us now study the forest $U^{(j+1)}$ with respect to any subspace $S^{(l)}$, with $l \leqq j$:

If $\gamma \in U^{(j)}$, then all the lines in $\mathscr{L}_{\bar{\gamma}(U^{(j+1)})}$ belong to $\mathscr{L}_{\bar{\gamma}(U^{(j)})}$, and since $U^{(j)} \in \mathscr{U}(S^{(l)})$, they are either all constant in $S^{(l)}$, or all variable in $S^{(l)}$.

If $\gamma \in \mathscr{A}^{(j+1)}(U^{(j)})$, let $\gamma' \in \tilde{W}^{(j+1)}(U^{(j)})$ such that $\gamma \in \mathscr{M}_{\gamma'}(U^{(j+1)})$. The inclusion $S^{(l)} \subset S^{(j+1)}$ implies that every line in $\mathscr{L}_{\gamma'(U^{(j+1)})}$ is constant in $S^{(l)}$. Now since $\mathscr{L}_{\bar{\gamma}'(U^{(j+1)})} \subset \mathscr{L}_{\bar{\gamma}'(U^{(j)})}$ and that $U^{(j)} \in \mathscr{U}(S^{(l)})$, we deduce that all lines in $\mathscr{L}_{\bar{\gamma}'(U^{(j)})}$ are constant in $S^{(l)}$ (relatively to γ'). But $\mathscr{L}_{\bar{\gamma}(U^{(j+1)})} \subset \mathscr{L}_{\bar{\gamma}'(U^{(j)})}$ (by construction), and then in view of Lemma 3.1, all the lines in $\mathscr{L}_{\bar{\gamma}(U^{(j+1)})}$ are constant in $S^{(l)}$, also relatively to γ. So we have proved that $U^{(j+1)} \in \mathscr{U}(S^{(l)})$ and at the same time that $\mathscr{A}^{(j+1)}(U^{(j)}) \subset W^{(l)}(U^{(j+1)})$. We have moreover proved (see the beginning of the above argument) that:

if $\gamma \in U^{(j)}$, then $\gamma \in W^{(l)}(U^{(j+1)})$ if and only if $\gamma \in W^{(l)}(U^{(j)})$.

Adding up these results yield Eq. (3.13). q.e.d.

Through an obvious recursion we obtain the following more general statement:

Lemma 3.4. *Let $\mathscr{F}$ be the set of nested subspaces as defined in 3c.1. Given an index $j(1 \leqq j \leqq L)$ and a forest $U^{(j)} \in \mathscr{U}(S^{(l)})$, $\forall l \leqq j$, one can construct a forest $U^{(L)}$ given by:*

$$U^{(L)} = U^{(j)} \cup \left(\bigcup_{m=j}^{L-1} \mathscr{A}^{(m+1)}(U^{(m)}) \right) \tag{3.14}$$

where each $U^{(m+1)}$ (resp. $\mathscr{A}^{(m+1)}(U^{(m)})$ is defined recursively by (3.9) (resp. 3.10).
$U^{(L)}$ is called *the completion forest of $U^{(j)}$ with respect to $\mathscr{F}$* and satisfies the following properties:

$$\forall S^{(m)} \in \mathscr{F} \cdot U^{(L)} \in \mathscr{U}(S^{(m)}), \tag{3.15}$$

$$\forall l \leqq j,\ W^{(l)} \cdot (U^{(L)}) = \left(\bigcup_{m=j}^{L-1} \mathscr{A}^{(m+1)}(U^{(m)}) \right) \cup W^{(l)}(U^{(j)}) \tag{3.16}$$

Remark. We note that in general the sets $\mathscr{A}^{(m+1)}(U^{(m)}) m = j \dots L-1$ in (3.14) are non empty; so there are in general several forests $U^{(j)} \in \mathscr{U}(S^{(l)})(l \leqq j)$ which admit the same completion forest $U^{(L)} \in \mathscr{U}(S^{(l)})$; $l \leqq L$. This enables us to classify all the forests $U^{(1)} \in \mathscr{U}(S^{(1)})$ into equivalence classes defined as follows:

A class C_U is the set of all forests $U^{(1)} \in \mathscr{U}(S^{(1)})$ which have the same completion forest $\mathscr{U}$ with respect to $\mathscr{F}$. The precise construction of such classes will be given in Lemma 3.7.

Lemma 3.5. *For some $j \leqq L$, let $U^{(j)} \in \mathscr{U}(S^{(l)})$, $\forall l \leqq j$, and let U be the completion forest of $U^{(j)}$ with respect to $\mathscr{F}$, then:*

$$\forall l \leqq j \quad B^{(l)}(U^{(j)}) = B^{(l)}(U). \tag{3.17}$$

Proof. From Lemma 3.4 we have:

$$U = U^{(j)} \cup \left(\bigcup_{m=j}^{L-1} \mathscr{A}^{(m+1)}(U^{(m)}) \right) \tag{3.14}$$

and

$$\bigcup_{m=j}^{L-1} \mathscr{A}^{(m+1)}(U^{(m)}) \text{ satisfies (3.16).}$$

Let $\gamma \in U^{(j)}$; in view of (3.14) $\gamma \in U$; if $\gamma \neq G$, there exist subgraphs $\mu \in U^{(j)}$ and $\tau \in U$ such that $\gamma \in \mathscr{M}_\mu(U^{(j)})$, $\gamma \in \mathscr{M}_\tau(U)$, $\gamma \subset \tau \subset \mu$. Necessarily if $\tau \neq \mu$ then:

$$\tau \in \bigcup_{m=j}^{L-1} \mathscr{A}^{(m+1)}(U^{(m)}). \tag{3.18}$$

Two cases are possible: a) $\gamma \in B^{(l)}(U^{(j)})$.

Then $\gamma \notin W^{(l)}(U^{(j)})$, and in view of (3.14) and (3.16), this implies (since $\gamma \in U^{(j)}$); $\gamma \notin W^{(l)}(U)$. By assumption, $\mu \in W^{(l)}(U^{(j)})$ and so either $\tau = \mu \in W^{(l)}(U^{(j)})$ or $\tau \neq \mu$, with $\tau \in \bigcup_{m=j}^{L-1} \mathscr{A}^{(m+1)}(U^{(m)})$ in both cases, (3.16) entails that $\tau \in W^{(l)}(U)$. We conclude that $\gamma \in B^{(l)}(U)$.

b) $\gamma \notin B^{(l)}(U^{(j)})$. If $\gamma \in W^{(l)}(U^{(j)})$, formula (3.16) entails that $\gamma \in W^{(l)}(U)$ so $\gamma \notin B^{(l)}(U)$. If $\gamma \notin W^{(l)}(U^{(j)})$, necessarily $\mu \notin W^{(l)}(U^{(j)})$ and that means $\mathscr{L}_{\tilde{\mu}(U^{(j)})}$ contains only variable lines in $S^{(l)}$ and consequently in $S^{(L)}$; due to the construction of U, this entails that there cannot be any $\tau \neq \mu$, with $\gamma \subset \tau \subset \mu$ and $\tau \in W^{(l)}(U)$; so finally $\gamma \notin B^{(l)}(U)$.

Definition 3d. Let $\mathscr{F} = \{S^{(j)}; 1 \leqq j \leqq L\}$ be a nested set of subspaces of $E_{(k)}^{rN}$, as defined in 3c1.

1. A forest U of G is called *complete with respect to* $\mathscr{F}$ if it is complete with respect to each subspace $S^{(j)}$ in $\mathscr{F}$, namely:

$$\forall j,\ 1 \leqq j \leqq L : U \in \mathscr{U}(S^{(j)})$$

Following Definition 3a.3, we can associate with such a forest U, the sets of subgraphs $W^{S^{(j)}}(U)$, $B^{S^{(j)}}(U)$ (for every $j \leqq L$): for simplicity we shall put: $W^{S^{(j)}}(U) = W^{(j)}(U)$, $B^{S^{(j)}}(U) = B^{(j)}(U)$; all these sets are subsets of U.

2. If U is complete with respect to $\mathscr{F}$, we call $\mathscr{B}^{\mathscr{F}}(U)$ the set of all subgraphs γ in U which belong to at least one set $B^{(j)}(U)$ (for some integer $j \leqq L$); namely:

$$\mathscr{B}^{\mathscr{F}}(U) = \bigcup_{1 \leqq j \leqq L} B^{(j)}(U) \tag{3.19}$$

Lemma 3.6. *Let $\mathscr{F}$ be a nested set $\{S^{(i)}; 1 \leqq i \leqq L\}$ and U be a complete forest with respect to $\mathscr{F}$.*

i) *Let $\gamma \in U$; if there is some integer $i \leqq L$ such that $\gamma \in W^{(i)}(U)$, then:*

$$\forall l \leqq i : \gamma \in W^{(l)}(U)$$

ii) *If $\gamma \in \mathscr{B}^{\mathscr{F}}(U)$ and if i is the minimal integer ($i \leqq L$) such that $\gamma \in B^{(i)}(U)$, then:*

$$\forall l \geqq i, \quad \gamma \notin W^{(l)}(U)$$
$$\forall l < i, \quad \gamma \in W^{(l)}(U).$$

Proof
i) If $\gamma\in W^{(i)}(U)$, γ has only constant lines on $S^{(i)}$, and since $S^{(l)}\subset S^{(i)}$ for every $l\leqq i$, γ also has only constant lines on all these subspaces $S^{(l)}$; this proves *i*).
ii) Let $\gamma\in B^{(i)}(U)$; then $\gamma\notin W^{(i)}(U)$ and in view of i), $\gamma\notin W^{(l)}(U)$ for every $l\geqq i$. Moreover there exists a graph γ in $W^{(i)}(U)$ such that γ be a maximal subgraph of γ' in U; but in view of *i*), we also have:

$$\forall l<i,\ \gamma'\in W^{(l)}(U). \tag{3.20}$$

Then if i is the *smallest* integer such that $\gamma\in B^{(i)}(U)$, γ cannot have variable lines in $S^{(l)}$ such that $l<i$ (indeed if it were the case, one would have $\gamma\in B^{(l)}(U)$ in view of (3.20). So necessarily $\gamma\in W^{(l)}(U)$, for every $l<i$.

Definitions 3e. Let $\mathscr{F}=\{S^{(0)},S^{(1)}\dots,S^{(L)};S^{(0)}\subset S^{(1)}\subset\dots\subset S^{(L)}\}$ and for every j $(0\leqq j\leqq L)$, let us put: $\mathscr{F}_j=\{S^0,\dots,S^j\}$. $(\mathscr{F}_L=\mathscr{F})$. We call $\mathscr{U}(\mathscr{F}_j)$ the set of all forests which are complete with respect to the nested set $\mathscr{F}_j$. It is natural to say that any full forest is complete with respect to the nested set $\mathscr{F}_0=\{S^{(0)}\}$, since such a forest contains G and has (in a trivial way) all its lines constant in $\{S^{(0)}\}$; so the set of full forests of G is identical with $\mathscr{U}(\mathscr{F}_0)$, and we have the following obvious inclusion relations:

$$\mathscr{U}(\mathscr{F})\subset\dots\subset\mathscr{U}(\mathscr{F}_j)\subset\dots\subset\mathscr{U}(\mathscr{F}_1)\subset\mathscr{U}(\mathscr{F}_0).$$

For any given forest U in $\mathscr{U}(\mathscr{F})$ and every $j(0\leqq j\leqq L)$ we define $C_U^{(j)}$ as the class of all forests $U^{(j)}$ in $\mathscr{U}(\mathscr{F}_j)$ whose completion forest with respect to $\mathscr{F}$ (in the sense of Lemma 3.4) is equal to U.

It is clear that $C_U^{(L)}$ reduces to $\{U\}$ and that the following inclusion relations hold:

$$\{U\}\subset C_U^{(L-1)}\subset\dots\subset C_U^{(j)}\subset\dots\subset C_U^{(0)}.$$

We shall now completely characterize the classes $C_U^{(j)}$ which are associated with U, by means of the following sets:

$$\mathscr{B}^{\mathscr{F}_j}(U)=\bigcup_{0\leqq l\leqq j}B^{(l)}(U).$$

Lemma 3.7. *Let U in $\mathscr{U}(\mathscr{F})$. Then for every integer j, with $1\leqq j\leqq L$, the corresponding class $C_U^{(j)}$ is the set of* all *forests $U^{(j)}$ which satisfy the inclusion relations:*

$$(U\backslash\mathscr{B}^{\mathscr{F}}(U))\cup\mathscr{B}^{\mathscr{F}_j}(U)\subset U^{(j)}\subset U \tag{3.21a}$$

The class $C_U^{(0)}$ is the set of all forests $U^{(0)}$ which satisfy the inclusion relations:

$$U\backslash\mathscr{B}^{\mathscr{F}}(U)\subset U^{(0)}\subset U. \tag{3.21b}$$

Proof
1. For a fixed $j(0\leqq j\leqq L)$, let $U^{(j)}$ be a given forest in $C_U^{(j)}$. Then let $\{U^{(m)};j\leqq m\leqq L\}$ be the increasing sequence of forests associated with $U^{(j)}$ through Lemma 3.4: $U^{(m)}\in C_U^m$

$$U^{(m+1)}=U^{(m)}\cup\mathscr{A}^{(m+1)}(U^{(m)})$$

and

$$U = U^{(L)} = U^{(j)} \cup \left(\bigcup_{m=j}^{L-1} \right) \mathscr{A}^{(m+1)}(U^{(m)}). \tag{3.22}$$

Then by applying Lemmas 3.3 and 3.5 to each forest $U^{(m)}$, we can write for $j \leqq m \leqq L$:

$$\mathscr{A}^{(m+1)}(U^{(m)}) \subset B^{(m+1)}(U^{(m+1)}) = B^{(m+1)}(U), \tag{3.23}$$

and from (3.22) and (3.23) we obtain:

$$U^{(j)} = U \setminus \bigcup_{m=j}^{L-1} \mathscr{A}^{(m+1)}(U^{(m)}) \supset U \setminus \bigcup_{m=j+1}^{L} B^{(m)}(U^{(m)}) \supset U \setminus \mathscr{B}^{\mathscr{F}}(U) \tag{3.24}$$

Moreover, by applying Lemma 3.5 to $U^{(j)}$ we also obtain

$$\forall l \leqq j : B^{(l)}(U) = B^{(l)}(U^{(j)}) \subset U^{(j)} \subset U \tag{3.25}$$

From (3.24) and (3.25), we deduce that $U^{(j)}$ satisfies the inclusion relations (3.21.a) if $1 \leqq j \leqq L$ (resp. (3.21.b) if $j = 0$).

2. We shall prove that conversely, if $U^{(j)}$ satisfies the inclusion relations (3.21.a) (resp. (3.21.b) then $U^{(j)} \in C_U^{(j)}$. This statement holds trivially for $j = L$ (since necessarily $U^{(L)} = U$); let us make the recursive assumption that it holds for $j = J + 1$, and prove it for $j = J$.

Being given forest $U^{(J)}$ which satisfies (3.21.a) (resp. (3.21.b)), we put:

$$\underline{U}^{(J+1)} = U^{(J)} \cup B^{(J+1)}(U) \tag{3.26}$$

then the latter clearly satisfies the inclusion relations:

$$(U \setminus \mathscr{B}^{\mathscr{F}}(U)) \cup \mathscr{B}^{\mathscr{F}_{J+1}}(U) \subset \underline{U}^{(J+1)} \subset U \tag{3.27}$$

and from our recursive assumption : $\underline{U}^{(J+1)} \in C_U^{(J+1)}$.

Then by applying Lemma 3.5 to $U^{(J+1)}$ and using (3.26) we can write:

$$\underline{U}^{(J+1)} \setminus B^{(J+1)}(\underline{U}^{(J+1)}) \subset U^{(J)} \subset \underline{U}^{(J+1)}$$

But in view of Lemma 3.3., this shows that $\underline{U}^{(J+1)}$ is the completion forest of $U^{(J)}$ with respect to $S^{(J+1)}$.

It remains to show that for every $l \leqq J$, $U^{(J)}$ is necessarily complete with respect to $S^{(l)}$.

Since all the graphs γ of $U^{(j)}$ belong to $\underline{U}^{(j+1)}$ these graphs can always be classified as follows, with respect to a given subspace $S^{(l)} (l \leqq J)$:

a) $\gamma \notin W^{(l)}(\underline{U}^{(J+1)})$: this implies $\gamma \notin W^{(J+1)}(U^{(J+1)})$ so there *is no* maximal subgraph γ_a of γ in $\underline{U}^{(J+1)}$ which belongs to $B^{(J+1)}(\underline{U}^{(J+1)})$, namely to $B^{(J+1)}(U)$ thus in view of (3.26), every maximal subgraph of γ in $\underline{U}^{(J+1)}$ must belong to $U^{(J)}$ and therefore we have : $\bar{\gamma}(\underline{U}^{(J)}) = \bar{\gamma}(\underline{U}^{(J+1)})$. But since $\gamma \notin W^{(l)}(\underline{U}^{(J+1)})$ we can conclude that all the lines in $\mathscr{L}_{(\gamma(U^{(J)}))} (= \mathscr{L}_{(\bar{\gamma}(U^{(J+1)}))})$ are variable in $S^{(l)}$ (relatively to γ).

b) $\gamma \in W^{(l)}(\underline{U}^{(J+1)})$: we shall prove that every line i in $\mathscr{L}_{(\gamma(U^{(J)}))}$, is constant in $S^{(l)}$ relatively to γ. Let τ be the (unique) subgraph in $U^{(J+1)}$ such that $i \in \mathscr{L}(\bar{\tau}(\underline{U}^{(J+1)}))$.

Then either $\tau = \gamma$, and then i is constant in $S^{(l)}$ relatively to γ, or $\tau \subset \gamma$, with $\tau \neq \gamma$; in the latter case, τ belongs to $\underline{U}^{(J+1)} \setminus U^{(J)}$, namely to the set $\mathscr{A}^{(J+1)}(U^{(J)})$ (since $\underline{U}^{(J+1)}$ is the completion of $U^{(J)}$ with respect to $S^{(J+1)}$; by Lemma 3.3, τ is thus a *maximal subgraph* of γ in $\underline{U}^{(J+1)}$. Moreover, from Lemma 3.5 applied to $\underline{U}^{(J+1)}$ and relation (3.21.a) satisfied by $U^{(J)}$ (for $j = J$), we obtain:

$$B^{(l)}(U^{(J+1)}) = B^{(l)}(U) \subset U^{(J)}$$

and since $\tau \notin U^{(j)}$ $\tau \notin B^{(l)}(\underline{U}^{(J+1)})$.

Since $\gamma \in W^{(l)}(\underline{U}^{(j+1)})$, we then deduce that $\tau \in W^{(l)}(\underline{U}^{(J+1)})$, and that the line i is constant in $S^{(l)}$, relatively to τ and also (due to Lemma 3.1) relatively to γ.

So we have proved that $U^{(J)}$ is complete with respect to each subspace $S^{(l)}$, with $l \leqq J$; namely $U^{(J)} \in \mathscr{U}(\mathscr{F}_j)$. Since $\underline{U}^{(J+1)}$ is the completion forest of $U^{(J)}$ with respect to $S^{(J+1)}$, and that $\underline{U}^{(J+1)} \in C_U^{(J+1)}$, we have thus proved that $U^{(J)} \in C_U^{(J)}$.

To end our recursion argument, we just remark that in the last step (from $j=1$ to $j=0$), only the first part of the above argument is applied (namely $\underline{U}^{(1)}$ is the completion of $U^{(0)}$ with respect to $S^{(1)}$). q.e.d.

The purpose of the end of this section is to prove the following property for the renormalized integrand R_G.

Proposition 3.1. *Given any nested set* $\mathscr{F} = \{S^{(j)}; 1 \leqq j \leqq L\}$ *there exists a corresponding expression of* R_G *which is defined as follows:*

$$R_G(K,k) = \sum_{U \in \mathscr{U}(\mathscr{F})} \tilde{X}_U(K,k) \tag{3.28}$$

Each term $\tilde{X}_U$ *(for* $U \in \mathscr{U}(\mathscr{F})$*) is given by:*

$$\tilde{X}_U = (1 - t^{d(G)})\, \tilde{Y}_G^{(U)} \tag{3.29}$$

and $\tilde{Y}_G^{(U)}$ *is defined together with the set of auxiliary functions* $\{\tilde{Y}_O^{(U)}; \forall \gamma \in U\}$ *by the following recursion formula:*

$$\tilde{Y}_\gamma^{(U)} = I_{\bar{\gamma}(U)} \prod_{\gamma_a \in \mathscr{M}_\gamma(U)} s_a^* \tilde{f}_a^{(U)}\, \tilde{Y}_{\gamma_a}^{(U)} \tag{3.30}$$

where:

$$\tilde{f}_a^{(U)} = (1 - t^{d(\gamma_a)}) \quad \text{if } \gamma_a \in \mathscr{B}^{\mathscr{F}}(U) \tag{3.31}$$

$$\tilde{f}_a^{(U)} = -t^{d(\gamma_a)} \quad \text{if } \gamma_a \notin \mathscr{B}^{\mathscr{F}}(U) \tag{3.32}$$

To prove this proposition we need some auxiliary definitions and properties.

Definition 3f. Given a forest $U \in \mathscr{U}(\mathscr{F})$ we consider a forest $U^{(0)} \subset U$. Let $\gamma \in U$;
i) a subgraph $\mu \in U(\gamma) \cap U^{(0)}$ is called *maximal with respect to* $U^{(0)}$ if there is no $\tilde{\mu} \in U(\gamma) \cap U^{(0)}$ ($\tilde{\mu} \neq \gamma$) such that $\mu \in U(\tilde{\mu})$.
ii) we define the following subset of $U(\gamma)$:

$$V^0(\gamma) = \{\mu \in U(\gamma) : \mu \text{ maximal with respect to } U^{(0)}\} \tag{3.33}$$

iii) With γ we associate the following function:

$$I_\gamma^{(U,U^{(0)})}(K^\gamma k) = \prod_{\substack{v \in \mathscr{N} \setminus (\cup \mathscr{N}_\mu) \\ \mu \in V^0(\gamma)}} H^{n_v}(K^\gamma, k) \prod_{\substack{i \in \mathscr{L}_\gamma \setminus \cup \mathscr{L}_\mu \\ \mu \in V^0(\gamma)}} H_i^{(2)}(K^\gamma, k) \tag{3.34}$$

We state the following:

Lemma 3.8

a) $$I_\gamma^{(U,U^{(0)})} = I_{\bar{\gamma}(U)} \prod_{\substack{\mu\in\mathscr{M}_\gamma(U)\\ \mu\notin U^{(0)}}} s_\mu^\gamma I_\mu^{(U,U^{(0)})} \tag{3.35}$$

b) *for every* $\gamma\in U^{(0)}: V^0(\gamma) = \mathscr{M}_\gamma(U^{(0)})$ (3.36a)

$$I_\gamma^{(U,U^{(0)})} = I_{\gamma(U^{(0)})} \tag{3.36b}$$

a) From Definition (2.37) of $I_{\bar{\gamma}(U)}$ and (3.34) of $I_{\bar{\mu}}^{(U,U^{(0)})}$ and taking into account Property (2.29) we obtain:

$$I_{\bar{\gamma}(U)} \prod_{\substack{\mu\in\mathscr{M}_\gamma(U)\\ \mu\notin U^0}} s_\mu^{\gamma*} I_{\bar{\mu}}^{(U,U^{(0)})} = \prod_{\substack{v\in\mathscr{N}\gamma\backslash(\cup\mathscr{N}\gamma_a)\\ \gamma_a\in\mathscr{M}_\gamma(U)}} H^{n_v}(K^\gamma,k) \prod_{\substack{i\in\mathscr{L}\gamma\backslash(\cup\mathscr{L}\gamma_a)\\ \gamma_a\in\mathscr{M}_\gamma(U)}} H_{(i)}^{(2)}(K^\gamma,k)$$
$$\prod_{\substack{\mu\in\mathscr{M}_\gamma(U)\\ \mu\notin U^{(0)}}} \left[\prod_{\substack{v\in\mathscr{N}\mu\backslash\cup\mathscr{N}\mu\\ \mu'\in V^0(\mu)}} H^{n_v}(K^\gamma,k) \prod_{\substack{i\in\mathscr{L}\gamma\backslash\cup\mathscr{L}\mu'\\ \mu'\in V^0(\mu)}} H^{(2)}(K^\gamma,k)\right] = \prod_{\substack{v\in\mathscr{N}\gamma\backslash(\cup\mathscr{N}\gamma')\\ \gamma'\in V^0(\gamma)}} H^{n_v}$$
$$\prod_{\substack{i\in\mathscr{L}\gamma\backslash(\cup\mathscr{L}\gamma')\\ \gamma'\in V^0(\gamma)}} H_{(i)}^{(2)} = I_{\bar{\gamma}}^{(U,U^{(0)})}(K^\gamma,k)\,;$$ q.e.d.

(b) Property (3.36.a) is trivially verified from Definitions 2f.i) and 3f.ii). If $\gamma\in U^{(0)}$ then by comparison of Definition 3f.iii) of $I_{\bar{\gamma}}^{(U,U^{(0)})}$ with Definition (2.36) of $I_{\bar{\gamma}(U^{(0)})}$ and taking into account 3.36.a we verify 3.36.b.

Definition 3.8. For every subset $J\subset\mathscr{B}^{\mathscr{F}}(U)$ we write:

$$U = J\cup(U\backslash\mathscr{B}^{\mathscr{F}}(U))\cup(\mathscr{B}^{\mathscr{F}}\backslash J)$$

and define the forest $U_{(J)}^{(0)}\subset U$ by:

$$U_{(J)}^{(0)} = (U\backslash\mathscr{B}^{\mathscr{F}}(U))\cup J \tag{3.37}$$

Remark. From Lemma 3.7 when J varies in $\mathscr{B}^{\mathscr{F}}(U)\,U_{(J)}^{(0)}$ varies in $C_U^{(0)}$ that means: the sets J are in one to one correspondence with the forests $U^{(0)}$ which constitute the class $C_U^{(0)}$.

Lemma 3.9.

a) *The function* $\tilde{Y}_G^{(U)}$ *defined by* (3.30) (*when* $\gamma = G$) *satisfies*:

$$\tilde{Y}_G^{(U)} = \sum_{J\subset\mathscr{B}^{\mathscr{F}}(U)} Y_G^{(U,J)} \tag{3.38}$$

where $Y_\gamma^{(U,J)}\,\forall\gamma\in U(G)$ *is given by the recurrent formula*:

$$Y_\gamma^{(U,J)} = I_{\bar{\gamma}(U)} \prod_{\gamma_a\in\mathscr{M}_\gamma(U)} s_a^* f_{\gamma_a}^{(J)} Y_{\gamma_a}^{(U,J)} \tag{3.39}$$

with
$$\left.\begin{aligned} f_{\gamma_a}^{(J)} &= -t^{d(\gamma_a)} && \text{if } \gamma_a\in(U\backslash\mathscr{B}^{\mathscr{F}}(U))\cup J\\ f_{\gamma_a}^{(J)} &= 1 && \text{if } \gamma_a\in\mathscr{B}^{\mathscr{F}}\backslash J\end{aligned}\right\} \tag{3.40}$$

the summation in (3.38) *runs overall subsets* J *of* $\mathscr{B}^{\mathscr{F}}(U)$.

b) *Let* $U^{(0)} = U^{(0)}(J)$ *be defined by* (3.37) *then*:

$$\text{If } \gamma \in U^{(0)}, \quad Y_\gamma^{(U,J)} = Y_\gamma^{(U^{(0)})} \tag{3.41a}$$

$$\text{If } \gamma \notin U^{(0)}, \quad Y_\gamma^{(U,J)} = I_{\bar{\gamma}}^{(U,U^{(0)})} \prod_{\mu \in V^0(\gamma)} s_\mu^*(-t^{d(\mu)}) Y_\mu^{(U^{(0)})} \tag{3.41b}$$

Here the function $Y_\gamma^{(U^{(0)})}$ *corresponds to* γ *as a subgraph of the forest* $U^{(0)}(G)$ *and it is given by Definition* (2.45).

Proof

a) Let us consider formula (3.30) for $\gamma = G$:

$$\tilde{Y}_G^{(U)} = I_G^{(U)} \prod_{\gamma \in \mathcal{M}_G(U)} s_\gamma^{G*} \tilde{f}_\gamma^{(U)} \tilde{Y}_\gamma^{(U)}. \tag{3.42}$$

If we work out the factors $(1 - t^{d(\mu)})$ occurring for each $\mu \in U(G)$ such that $\mu \in \mathcal{B}^{\mathcal{F}}(U)$, then formula (3.42) yields Eqs. (3.38), (3.39), (3.40).

b) To prove Properties (3.41.a) (3.41.b) we suppose that the latter hold for every subgraph belonging to the set $\mathcal{M}_\gamma(U)$; from this recurrent hypothesis we can write Eqs. (3.39) (3.40) for every $\gamma \in U$ as follows:

$$Y_\gamma^{(U,J)} = I_{\bar{\gamma}(U)} \prod_{\mu \in (\mathcal{M}_\gamma(U) \cup U^{(0)})} s_\mu^{\gamma*}(-t^{d(\mu)}) Y_\mu^{(U^{(0)})} \prod_{\substack{\gamma' \in \mathcal{M}_\gamma(U) \\ \gamma' \notin U^{(0)}}} s_{\gamma'}^{\gamma*} I_{\bar{\gamma}'}^{(UU^{(0)})}$$
$$\left(\prod_{\mu' \in V^0(\gamma')} s_{\mu'}^{\gamma'*}(-t^{d(\mu')}) Y_{\mu'}^{(U^{(0)})} \right) \tag{3.43}$$

We notice that in (3.43) the products between different kinds of disjoint maximal subgraphs of $U(\gamma)$ commute. Using then the composition property $s_{\gamma'}^{\gamma*} s_{\mu'}^{\gamma'*} = s_{\mu'}^{\gamma*}$ (Eq. 2.27) and Lemma 3.8a we obtain from (3.43):

$$Y_\gamma^{(U,J)} = I_{\bar{\gamma}}^{(U,U^{(0)})} \prod_{\mu \in V^0(\gamma)} s_\mu^{\gamma*}(-t^{d(\mu)}) Y_\mu^{(U^{(0)})} \tag{3.44}$$

So if $\gamma \notin U^{(0)}$ the last equation proves (3.41.b); if $\gamma \in U^{(0)}$ we use properties 3.36.a)b) of Lemma 3.8 and finally obtain from (3.44) and Definition (2.45):

$$Y_\gamma^{(U,J)} = I_{\bar{\gamma}(U^{(0)})} \prod_{\gamma_a \in \mathcal{M}_\gamma(U^{(0)})} s_{\gamma_a}^{\gamma*}(-t^{d(\gamma_a)}) Y_{\gamma_a}^{(U^{(0)})} = Y_\gamma^{(U^{(0)})} \tag{3.41a}$$

q.e.d.

Proof of Proposition 3.1. The Definition (2.44) of R_G can be reexpressed as follows:

$$R_G = \sum_{U^{(0)} \in \mathcal{U}(\mathcal{F}_0)} (1 - t^{d(G)}) Y_G^{(U^{(0)})} \tag{3.45}$$

or by regrouping together the terms which correspond to forests $U^{(0)}$ in the same classes $C_U^{(0)}$ with U in $\mathcal{U}(\mathcal{F})$:

$$R_G = \sum_{U \in \mathcal{U}(\mathcal{F}_1)} \left[\sum_{U^{(0)} \in C_U^{(0)}} (1 - t^{d(G)}) Y_G^{(U^{(0)})} \right] \tag{3.46}$$

In order to show (3.28) (3.29) we are clearly led to prove that the quantities $\tilde{Y}_G^{(U)}$

defined by (3.30) (3.31) (3.32) are related with the $Y_G^{(U^{(0)})}$ through the following formula:

$$\forall U \in \mathscr{U}(\mathscr{F}), \quad \tilde{Y}_G^{(U)} = \sum_{U^{(0)} \in C_U^{(0)}} Y_G^{(U^{(0)})} \tag{3.47}$$

From Lemma 3.9a) we have:

$$\tilde{Y}_G^{(U)} = \sum_{J \subset \mathscr{B}^{\mathscr{F}}(U)} Y_G^{(U,J)} \tag{3.38}$$

and by Lemma 3.9 b) Eq. (3.41.a) for $\gamma = G$ yields (every forest $U^{(0)} \in C_U^{(0)}$ is a full forest so $G \in U^{(0)}$:

$$Y_G^{(U,J)} = Y_G^{(U^{(0)})} \tag{3.48}$$

By inserting Eq. (3.48) in (3.38) and taking into account the remark after Definition 3 g, we finally obtain (3.47). q.e.d.

Remark. The "complete forest formula" of Zimmermann [7] appears as the special case of Proposition 3.1 which corresponds to $\mathscr{F} = \{S^{(0)}, S^{(1)}\}$ ($\mathscr{B}^{\mathscr{F}}(U)$ then reduces to $B^{(S)}(U)$).

Definition 3h. For every $\gamma \in U$ we define the subset $\sigma_\gamma^{\mathscr{F}} \subset \mathscr{F}$ as follows:

$$\sigma_\gamma^{\mathscr{F}} = \{S^{(j)} \in \mathscr{F} : \gamma \notin W^{(j)}(U)\} \tag{3.49}$$

we can then prove.

Proposition 3.2. *Let $\gamma_a \in \mathscr{M}_\gamma(U)$; then*

a) *if $\gamma_a \in \mathscr{B}^{\mathscr{F}}(U)$ then $\sigma_{\gamma_a}^{\mathscr{F}} \supset \sigma_\gamma^{\mathscr{F}}$ and $\sigma_{\gamma_a}^{\mathscr{F}} \neq \sigma_\gamma^{\mathscr{F}}$*

b) *if $\gamma_a \notin \mathscr{B}^{\mathscr{F}}(U)$ then $\sigma_\gamma^{\mathscr{F}} \supset \sigma_{\gamma_a}^{\mathscr{F}}$*

Proof

a) If $\gamma_a \in \mathscr{B}^{\mathscr{F}}(U)$, then in view of Lemma 3.6(ii), there is an integer $i \leqq L$ such that:

$$\gamma_a \in B^{(i)}(U) \tag{3.50a}$$

$$\forall l \leqq i \qquad \gamma_a \notin W^{(l)}(U) \tag{3.50b}$$

$$\forall l < i \qquad \gamma_a \in W^{(l)}(U) \tag{3.50c}$$

Formula (3.50a) and (3.50b) show (in view of Definition 3h) that:

$$\sigma_{\gamma_a}^{\mathscr{F}} = \{S^{(i)} \in \mathscr{F}\,;\ i \leqq l \leqq L\} \tag{3.51}$$

Now since γ_a is a maximal subgraph of γ in U, we conclude from (3.50a) that $\gamma \in W^{(i)}(U)$; Lemma 3.6(i) then implies that:

$$\forall l \leqq i \qquad \gamma \in W^{(l)}(U)$$

Then in view of Definition 3h we can say that:

$$\sigma_\gamma^{\mathscr{F}} \subset \{S^{(l)} \in \mathscr{F}\,; i+1 \leqq l \leqq L\} \tag{3.52}$$

Comparing (3.51) with (3.52) yields the desired result a).

b) Let $\gamma_a \notin \mathscr{B}^{\mathscr{F}}(U)$. We have the two following cases:
b.1) $\forall S^{(i)} \in \mathscr{F}, \gamma_a \in W^{(i)}(U)$. That means $\sigma^{\mathscr{F}}_{\gamma_a} = \emptyset$ and property b) is trivially obtained.
b.2) Let l be the minimal integer $(1 \leqq l \leqq L)$ for which $\gamma_a \notin W^{(l)}(U)$. Lemma 3.5(i) implies that:

$$\forall j \geqq l \quad \gamma_a \notin W^{(j)}(U) \tag{3.53}$$

and from Definition 3d.3, we have:

$$\sigma^{\mathscr{F}}_{\gamma_a} = \{S^{(j)} \in \mathscr{F}; j \geqq l\}.$$

Let us show that necessarily if $j \geqq l$, then $\gamma \notin W^{(j)}(U)$; indeed since γ_a is maximal in γ and since γ_a has variable lines in $S^{(j)}$ (in view of (3.53)), the relation $\gamma \in W^{(j)}(U)$ would imply $\gamma_a \in B^{(j)}(U)$, which is not possible since we assumed that $\gamma_a \notin \mathscr{B}^{\mathscr{F}}(U)$. So we have proved that if $S^{(j)} \in \sigma^{\mathscr{F}}_{\gamma_a}$ then $\gamma \notin W^{(j)}(U)$, or in other words $S^{(j)} \in \sigma^{\mathscr{F}}_{\gamma}$, and b) is thus proved.

Definition 3.i.1. With every $\mu_0 \in \mathscr{B}^{\mathscr{F}}(U) \cap U(\gamma)$ we associate a nested sequence of subgraphs, denoted by $\mathscr{A}^{(\mu_0)}_{\gamma}$:

$$\mathscr{A}^{(\mu_a)}_{\gamma} = \left\{ \begin{matrix} \mu_j \in U(\gamma) \cap \mathscr{B}^{\mathscr{F}}(U) : \mu_{j+1} \supset \mu_j \mu_j \in \mathscr{M}_{j+1}(U) \forall j = 0, \ldots r-1 \\ \mu_r \in \mathscr{M}_{\gamma}(U) \end{matrix} \right\} \tag{3.54}$$

The above set $\mathscr{A}^{(\mu_a)}_{\gamma}$ can also be empty. We then define:

$$\mathscr{B}_{\gamma}(U) = \{\mu \in U(\gamma) \cap \mathscr{B}^{\mathscr{F}}(U) : \exists \text{ a sequence } \mathscr{A}^{(\mu)}_{\gamma} \neq \emptyset\} \tag{3.55}$$

Definition 3i.2. For every $\gamma \in U$ $U \in \mathscr{U}(\mathscr{F})$ we define the following sets of subspaces:

$$\hat{\sigma}_{\gamma} = \{S \subset E^{rm}_{(k)} : \exists S^{(j)} \in \sigma^{\mathscr{F}}_{\gamma} \text{ such that } S^{(j)} \subset S\} \tag{3.56}$$

$$\hat{\omega}_{\gamma} = \left\{ S_{\gamma} \subset \mathscr{E}^{rN_{\gamma}}_{(K^{\gamma},k)} : S_{\gamma} \not\subset \operatorname{Ker} \lambda^{\gamma}_i \ \forall i \in \mathscr{L}_{\bar{\gamma}} \cup \Big(\bigcup_{\mu \in \mathscr{B}_{\gamma}(U)} \mathscr{L}_{\bar{\mu}} \Big); \pi(S_{\gamma}) \in \hat{\sigma}_{\gamma} \right\} \tag{3.57}$$

We prove the following:

Lemma 3.10.
i) *For every* $\gamma_a \in \mathscr{M}_{\gamma}(U)$:

$$\begin{aligned} &\hat{\sigma}_{\gamma_a} \supset \hat{\sigma}_{\gamma} \text{ and } \hat{\sigma}_{\gamma_a} \neq \hat{\sigma}_{\gamma} && \forall \gamma_a \in \mathscr{B}^{\mathscr{F}}(U) \\ &\hat{\sigma}_{\gamma_a} \subset \hat{\sigma}_{\gamma} && \forall \gamma_a \notin \mathscr{B}^{\mathscr{F}}(U) \end{aligned} \tag{3.58}$$

ii) *The couple* $(\hat{\sigma}_{\gamma} \hat{\omega}_{\gamma})$ *is an admissible couple in* $\mathscr{E}^{rN_{\gamma}}_{(K^{\gamma},k)}$.

Proof
i) From Definitions (3.56), (3.49) and in view of Proposition 3.2 (for $\sigma^{\mathscr{F}}_{\gamma_a}$) we obtain easily this proof.
ii) The Properties b), c), d) of Definition 1d follow easily from Definitions (3.56), (3.57). We then show the validity of Property a) of Definition 1d i.e.:

$$\hat{\sigma}_{\gamma} \subset \hat{\omega}_{\gamma} \tag{3.59}$$

Let $S \in \hat{\sigma}_{\gamma}$; by Definition (3.56) that means $S \not\subset \operatorname{Ker} \lambda^{\gamma}_i \ \forall i \in \mathscr{L}_{\bar{\gamma}}$ and in view of Definition (3.57) we have to prove also that $S \not\subset \operatorname{Ker} \lambda^{\gamma}_l \ \forall l \in \mathscr{L}_{\bar{\mu}}$ for every $\mu \in \mathscr{B}_{\gamma}(U)$.

Let us consider such a subgraph $\mu_0 \in \mathscr{B}_\gamma(U)$; following Definition 3i.1. there exist in $U(\gamma)$ a sequence $\mathscr{A}_\gamma^{(\mu_0)}$ of subgraphs of γ defined by (3.54). By Lemma 3.10.i) we obtain that:

$$\hat{\sigma}_{\mu_0} \supset \hat{\sigma}_{\mu_1} \supset \dots \hat{\sigma}_{\mu_r} \supset \hat{\sigma}_\gamma \text{; with } \hat{\sigma}_{\mu_0} \neq \hat{\sigma}_{\mu_1}$$

so in view of Definition (3.56) there exist at least one subspace $\underline{S} \subset S$ such that:

$$\underline{S} \not\subset \operatorname{Ker} \lambda_l^{\mu_0} \qquad \forall l \in \mathscr{L}_{\bar{\mu}_0} \tag{3.60}$$

$$\underline{S} \subset \left(\bigcap_{i \in \mathscr{L}_{\bar{\gamma}}} \operatorname{Ker} \lambda_i^\gamma \right) \cap \left(\bigcap_{l \in \mathscr{L}_{\bar{\mu}_j}} \operatorname{Ker} \lambda_{l_j}^{\mu_j} \right) \quad j = 1 \dots r \tag{3.61}$$

or equivalently

$$\gamma \in W^{\underline{S}}(U),\ \mu_j \in W^{\underline{S}}(U), j = 1 \dots r \tag{3.62}$$

We can rewrite formula (3.61) equivalently as follows:

$$\begin{aligned} &\lambda_j^\gamma(K^\gamma = 0, k)\big|_{k \in \underline{S}} = 0 \qquad \forall j \in \mathscr{L}_{\bar{\gamma}} \\ &\lambda_{l_j}^{\mu_j}(K^{\mu_j} = 0, k)\big|_{k \in \underline{S}} = 0 \quad \forall l_j \in \mathscr{L}_{\bar{\mu}_j}, j = 1 \dots r \end{aligned} \tag{3.63}$$

We shall now show that (3.60) and (3.63) imply that $\forall l \in \mathscr{L}_{\bar{\mu}_0} S \not\subset \operatorname{Ker} \lambda_l^\gamma$. Indeed, let us suppose that the contrary holds, namely that for one $l \in \mathscr{L}_{\bar{\mu}_0}$ one has $\underline{S} \subset \operatorname{Ker} \lambda_l^\gamma$ which means:

$$\lambda_l^\gamma(K^\gamma = 0, k)\big|_{k \in S} = 0, \quad l \in \mathscr{L}_{\bar{\mu}_0} \tag{3.64}$$

From the relations (2.29) between the mappings $\lambda^{\mu_0}, s_{\mu_j}^{\mu_{J+1}}, j = 0, \dots r, s_{\mu_0}$ we obtain:

$$\lambda_l^\gamma = \lambda_l^{\mu_0} \circ s_{\mu_0}^\gamma = \lambda_l^{\mu_0} \circ s_{\mu_r}^\gamma \circ s_{\mu_{r-1}}^{\mu_r} \circ \dots s_{\mu_0}^{\mu_1}, \ l \in \mathscr{L}_{\bar{\mu}_0} \tag{3.65}$$

Inserting (3.64) in (3.65) we have:

$$\lambda_l^{\mu_0}(K^{\mu_0}(K^{\mu_1}(\dots(K^\gamma = 0, k) \dots, k)\big|_{k \in S} = 0 \tag{3.66}$$

Let us show that

$$\forall j (0 \leqq j \leqq r),\ K^{\mu_j}(K^{\mu_{j+1}} = 0, k)\big|_{k \in S} = 0 \tag{3.67}$$

(where μ_{r+1} denotes γ).
This results from Lemma 3.1.a) which, in view of (3.62) can be applied to each subgraph $\mu_{j+1}(0 \leqq j \leqq r)$.
In view of (3.67) Eq. (3.66) yields:

$$\lambda_l^{\mu_0}(K^{\mu_0} = 0, k)\big|_{k \in S} = 0 \quad \text{which means:}$$

$$S \subset \operatorname{Ker} \lambda_l^{\mu_0} \text{ for the considered line } l \in \mathscr{L}_{\bar{\mu}_0} \tag{3.68}$$

But (3.68) is contrary to the hypothesis (3.60) and we conclude that statement (3.64) is not true. It follows that:

$$\underline{S} \not\subset \operatorname{Ker} \lambda_l^\gamma, \quad \forall l \in \mathscr{L}_{\bar{\mu}_0} \tag{3.69}$$

The inclusion $\underline{S} \subset S$ allows us to obtain from (3.69) that also $S \not\subset \operatorname{Ker} \lambda_l^\gamma, \forall l \in \mathscr{L}_{\bar{\mu}_0}$ q.e.d.

Lemma 3.11.

i) $\hat{\sigma}_\gamma \subset \sigma_{\bar{\gamma}}$; $\quad \hat{\sigma}_\gamma = \hat{\sigma}_\gamma \cap \left(\bigcap_{\mu \in \mathscr{B}_\gamma(U)} \hat{\sigma}_\mu \right)$

ii) $\hat{\omega}_\gamma \subset \sigma_\gamma$

Proof

i) From Definitions (2.38.a) and (3.56) we obtain that $\hat{\sigma}_\gamma \subset \sigma_{\bar{\gamma}}$. In view of definition (3.i.1) for every $\mu \in \hat{\mathscr{B}}_\gamma(U)$ there exists a sequence $\mathscr{A}_\gamma^{(\mu)} \neq \varnothing$ (3.54) therefore by Lemma (3.10.i) we have: $\hat{\sigma}_\mu \supset \hat{\sigma}_{\mu_1} \supset \ldots \hat{\sigma}_{\mu_r} \supset \hat{\sigma}_\gamma$; so it follows: $\hat{\sigma}_\gamma = \hat{\sigma}_\gamma \cap \left(\bigcap_{\mu \in \mathscr{B}_\gamma(U)} \hat{\sigma}_\mu \right)$

ii) In view of Definitions (3.57), (2.38.b) and the first property if we verify that $\hat{\omega}_\gamma \subset \omega_{\bar{\gamma}}$.

Definitions 3i.3. For every $\gamma_a \in \mathscr{M}_\gamma(U)$, $U \in \mathscr{U}(\mathscr{F})$ we define the following sets:

a) If $\gamma_a \in \mathscr{B}_\gamma(U)$

$$\omega_{\gamma_a}^{(\gamma)} = \{ S_{\gamma_a} \subset \underline{\mathscr{E}}_{(K^{\gamma_a},k)}^{rN_{\gamma_a}} : \pi_a(S_{\gamma_a}) \in \hat{\sigma}_\gamma,\ S_{\gamma_a} \in \hat{\omega}_{\gamma_a} \} \tag{3.70}$$

b) If $\gamma_a \notin \mathscr{B}_\gamma(U)$

$$\omega_{\gamma_a}^{(\gamma)} = \{ S_{\gamma_a} \subset \underline{\mathscr{E}}_{(K^{\gamma_a},k)}^{rN_{\gamma_a}} : \pi_a(S_{\gamma_a}) \in \hat{\sigma}_\gamma \} \tag{3.71}$$

Lemma 3.12

i) $(\omega_{\gamma_a}^{(\gamma)}, \hat{\sigma}_\gamma)$ *is an admissible couple in* $\underline{\mathscr{E}}_{(K^{\gamma_a},k)}^{rN_{\gamma_a}}$

ii) *For every* $S_\gamma \in \hat{\omega}_\gamma$, $\quad S_{\gamma_a} \equiv s_a S_\gamma \in \omega_{\gamma_a}^{(\gamma)}$

iii) *For every* $\gamma_a \in \mathscr{B}_\gamma(U) \quad \omega_{\gamma_a}^{(\gamma)} \subset \hat{\omega}_{\gamma_a}$

Proof

i) In view of Definitions (3.70) (resp. (3.71)) and (3.56) together with Property (3.58) the sets $(\omega_{\gamma_a}^{(\gamma)}, \hat{\sigma}_\gamma)$ satisfy the requirements a), b), c) of Definition 1d.

ii) Let $S_\gamma \in \hat{\omega}_\gamma$ and $S_{\gamma_a} = s_a S_\gamma$. By Property (1.18) and Definition (3.57) we have:

$$\pi_a(S_{\gamma_a}) = \pi(S_\gamma) \in \hat{\sigma}_\gamma \tag{3.72}$$

In view of the Definition (3.71) Property (3.72) proves that $S_{\gamma_a} \in \omega_{\gamma_a}^{(\gamma)}$ in the case $\gamma_a \notin \mathscr{B}_\gamma(U)$.

Now from Property (2.29) $\lambda_i^\gamma = \lambda_i^{\gamma_a} s_a$ it follows directly that $S_{\gamma_a} \not\subset \operatorname{Ker} \lambda_i^{\gamma_a}$ if and only if $S_\gamma \subset \operatorname{Ker} \lambda_i^\gamma$. Taking into account this property together with Definition (3.57) of $\hat{\omega}_\gamma$ we obtain:

$$S_{\gamma_a} \not\subset \operatorname{Ker} \lambda_i^{\gamma_a} \qquad \forall i \in \mathscr{L}_{\bar{\gamma}} \cup \left(\bigcup_{\mu \in \mathscr{B}_\gamma(U)} \mathscr{L}_{\bar{\mu}} \right) \tag{3.73}$$

If $\gamma_a \in \mathscr{B}_\gamma(U)$ then:

$$\mathscr{L}_{\bar{\gamma}_a} \cup \left(\bigcup_{\mu' \in \mathscr{B}_{\gamma_a}(U)} \mathscr{L}_{\bar{\mu}'} \right) \subset \left(\bigcup_{\mu \in \mathscr{B}_\gamma(U)} \mathscr{L}_{\bar{\mu}} \right) \tag{3.74}$$

From (3.73) (3.74) and by Definition (3.57) of $\hat{\omega}_{\gamma_a}$ we obtain $S_{\gamma_a} \in \hat{\omega}_{\gamma_a}$. The latter

property together with (3.72) prove (in view of Definition (3.70)) that $S_{\gamma_a} \in \omega_{\gamma_a}^{(\gamma)}$ q.e.d.
iii) By comparison of Definition (3.70) with (3.57) (of $\hat{\omega}_{\gamma_a}$) and in view of Property (3.58) $\hat{\sigma}_\gamma \subset \hat{\sigma}_{\gamma_a}$, we obtain that $\omega_{\gamma_a}^{(\gamma)} \subset \hat{\omega}_{\gamma_a}$ q.e.d.

4. Integrability of the Renormalized Integrand $\mathbf{R}_G$

This section is devoted to the proof of our main theorem:

Theorem 4.1
a) For every fixed value of K in $\mathscr{E}_K^{r(n-1)}$, the renormalized integrand $R_G(K,k)$ (defined by formulae (2.43), (2.44) *and all the derivatives $D_K R_G$ of the latter belong to a class $A_{rm}^{(\alpha)}$ whose asymptotic indicatrix α satisfies the following condition:*

$$\sup_{\{S' \neq \{0\}, S' \subset E_{(k)}^{rm}\}} (\alpha(S') + h(S')) = -1 \tag{4.1}$$

where $h(S') \leqq rm$ denotes the dimension of the subspace S′.
b) The integral

$$H_G^{ren}(K) = \int_{E_{(k)}^{rm}} R_G(K,k) d_{rm} k \tag{4.2}$$

is absolutely convergent. The function H_G^{ren} that it defines belongs to the class $C^\infty(\mathscr{E}_{(K)}^{r(n-1)})$.

The proof of this theorem is based on the possibility of writing decomposition formulae of the type (3.28) for R_G, corresponding to a variety of nested sets $\mathscr{F} = \{S^{(1)} \subset S^{(2)} \subset \ldots \subset S^{(m)}\}$; for each such decomposition (3.28), the various terms $\tilde{X}_U$, with $U \in \mathscr{U}(\mathscr{F})$ will be proved (in Proposition 4.2) to belong to appropriate classes $A_{rN}^{\alpha}(U)$. But since $\tilde{X}_U$ is itself defined through recursive formulae (see (3.29), (3.30)) which involve auxiliary functions $Y_\gamma(\gamma \in U(\mathscr{F}))$ it will be first necessary to prove (in Proposition 4.1) that each of these functions Y_γ belongs itself to an appropriate class $\mathscr{A}_{rN_\gamma}^{(\alpha_\gamma, \hat{\sigma}_\gamma, \hat{\omega}_\gamma)}$.

Proposition 4.1. *Let $\mathscr{F} = \{S^{(1)} \ldots S^{(m)}\}$ be a nested set of subspaces in $E_{(k)}^{rm}$ and U be a forest in $\mathscr{U}(\mathscr{F})$ $(\tilde{m} \leq m)$.*

Then for any subgraph $\gamma \in U$, the function $\tilde{Y}_\gamma(K^\gamma k)$ in formula (3.30) *belongs to a class $\mathscr{A}_{rN_\gamma}^{(\alpha_\gamma, \hat{\sigma}_\gamma, \hat{\omega}_\gamma)}$ of admissible Weinberg functions with the following properties:*

a) $\hat{\sigma}_\gamma = \hat{\sigma}_\gamma(U, \mathscr{F})$, $\hat{\omega}_\gamma$ *are defined by formulae* (3.56) (3.57)
b) *i)* $\forall S^{(j)} \in \mathscr{F}$ *such that* $S^{(j)} \notin \hat{\sigma}_\gamma$, *the asymptotic coefficient* $\alpha(S^{(j)})$ *satisfies*:

$$\textit{either}: \alpha_\gamma(S^{(j)}) = 0 \qquad \textit{if } \forall \mu \in U(\gamma) \qquad S^{(j)} \notin \hat{\sigma}_\mu \tag{4.3}$$

$$\textit{or} \quad : \alpha_\gamma(S^{(j)}) \leqq -M^{(j)}(\gamma) - 1 \tag{4.4}$$

if there is at least one $\mu \in U(\gamma)$ with $S^{(j)} \in \hat{\sigma}_\mu$

ii) $\forall S^{(j)} \in \mathscr{F}$ *such that* $S^{(j)} \in \hat{\sigma}_\gamma$ *the asymptotic coefficient with respect to every subspace* $S_\gamma \in \hat{\omega}_\gamma$ *with* $\pi(S_\gamma) = S^{(j)}$, *satisfies*:

$$\alpha_\gamma(S_\gamma) \leqq d(\gamma) - M^{(j)}(\gamma) \tag{4.5}$$

For the proof of this statement we shall use the recurrence hypothesis that Proposition (4.1) holds for every function $\tilde{Y}_{\gamma_a}$, with $\gamma_a \in \mathscr{M}_\gamma(U)$. We shall need to prove the following auxiliary Lemmas 4.1., 4.2., 4.3.

Lemma 4.1. *$\mathscr{F}$ and $U \in \mathscr{U}(\mathscr{F})$ being given, the function $I_{\bar{\gamma}}(K^{\gamma}, k)$ belongs to a class $\mathscr{A}_{4N_{\gamma}}^{(\alpha_{\bar{\gamma}}, \hat{\sigma}_{\gamma}, \hat{\omega}_{\gamma})}$ of admissible Weinberg functions with the following properties:*
i) *For every $S^{(j)} \in \mathscr{F}$ such that $S^{(j)} \notin \hat{\sigma}_{\gamma}$, the corresponding coefficient satisfies:*

$$\alpha_{\bar{\gamma}}(S^{(j)}) = 0 \tag{4.6}$$

ii) *For every $S^{(j)} \in \mathscr{F}$ such that $S^{(j)} \in \hat{\sigma}_{\gamma}$, the coefficient corresponding to every $S_{\gamma} \in \hat{\omega}_{\gamma}$ such that $\pi(S_{\gamma}) = S^{(j)}$,* satisfies:

$$\alpha_{\gamma}(S_{\gamma}) = d(\bar{\gamma}) - rm(\bar{\gamma}). \tag{4.7}$$

Proof. From Lemma 2.3 we obtain that: $I_{\bar{\gamma}}(K^{\gamma}, k) \in \mathscr{A}_{rN_{\gamma}}^{(\alpha_{\bar{\gamma}}, \sigma_{\bar{\gamma}}, \omega_{\gamma})}$ with $\sigma_{\bar{\gamma}} \omega_{\bar{\gamma}}$ and $\alpha_{\bar{\gamma}}$ defined by formulae (2.38) a) b) and (2.39). Now the following properties have been proved in Sect. 3 (Lemmas 3.11, 3.10): $\hat{\sigma}_{\gamma} \subset \sigma_{\bar{\gamma}}$ $\hat{\omega}_{\bar{\gamma}} \subset \omega_{\bar{\gamma}}$ and $(\hat{\omega}_{\gamma}, \hat{\sigma}_{\gamma})$ is an admissible couple in $\mathscr{E}_{(K^{\gamma},k)}^{rN_{\gamma}}$, so that in view of Proposition 1.3.c. we also have:

$I_{\bar{\gamma}} \in \mathscr{A}^{(\alpha_{\bar{\gamma}}, \hat{\sigma}_{\gamma}(U, \mathscr{F}), \hat{\omega}_{\gamma})}$.

It remains to check (by using formula (2.39) and the Definition (2.41) of $d(\bar{\gamma})$ that:

– $\forall S_{\gamma} \in \hat{\omega}_{\gamma}$

one has:

$$\alpha_{\bar{\gamma}}(S) = \sum_{v \in \mathscr{N}_{\bar{\gamma}}} \mu_v + \sum_{i \in \mathscr{L}_{\bar{\gamma}}} \mu_i = d(\bar{\gamma}) - rm(\bar{\gamma}) \tag{4.8}$$

– $\forall S = S^{(j)} \in \mathscr{F}$ $(1 \leqq j \leqq \tilde{m})$, such that $S^{(j)} \notin \hat{\sigma}_{\gamma}(U, \mathscr{F})$, one has $\alpha_{\bar{\gamma}}(S) = 0$. (4.9)

The latter result comes from the fact that for such an $S^{(j)}$, $\gamma \in W_j(U)$, which means that:

$$S^{(j)} \subset \left(\bigcap_{v \in \mathscr{N}_{\bar{\gamma}}} \operatorname{Ker} \lambda_v^{\gamma} \right) \cap \left(\bigcap_{i \in \mathscr{L}_{\bar{\gamma}}} \operatorname{Ker} \lambda_i^{\gamma} \right)$$ [5]

therefore formula (2.13) applies to $\alpha_{\gamma}(S^{(j)})$.

Lemma 4.2. *For every subgraph $\gamma_a \in \mathscr{M}_{\gamma}(U)$, with $\gamma_a \mathscr{B}^{\mathscr{F}}(U)$ the function $s_a^*(1 - t^{d(\gamma_a)} \tilde{Y}_{\gamma_a}$ belongs to the class $\mathscr{A}_{rN_{\gamma}}^{(\alpha_{\gamma}(a), \hat{\sigma}_{\gamma}, \hat{\omega}_{\gamma})}$ of admissible Weinberg functions with the following properties:*
i) *$\forall S^{(j)} \in \mathscr{F}$ with $S^{(j)} \in \hat{\sigma}_{\gamma}$ the asymptotic coefficient corresponding to every subspace $S_{\gamma} \in \hat{\omega}_{\gamma}$ with $\pi(S_{\gamma}) = S^{(j)}$ satisfies:*

$$\alpha_{\gamma}^{(a)}(S_{\gamma}) \leqq d(\gamma_a) - M^{(j)}(\gamma_a) \tag{4.10}$$

ii) *$\forall S^{(j)} \in \mathscr{F}$ with $S^{(j)} \notin \hat{\sigma}_{\gamma}$*

either $: \alpha_{\gamma}^{(a)}(S^{(j)}) = 0 = -M^{(j)}(\gamma_a)$ *if $\forall \mu \in U(\gamma_a)$ $S^{(j)} \notin \hat{\sigma}_{\mu}$* (4.11)

or $: \alpha_{\gamma}^{(a)}(S^{(j)}) \leqq -M^{(j)}(\gamma_a) - 1$ *if $\exists$ at least one $\mu \in U(\gamma_a)$ with $S^{(j)} \in \hat{\sigma}_{\mu}$.* (4.12)

Remark. The last equality in (4.11) is a consequence of Definition (3.7) which yields $M^{(j)}(\gamma_a) = 0$ if $\forall \mu \subset \gamma$, $\mu \in W_j(U)$, *i.e.* $S^{(j)} \notin \hat{\sigma}_{\mu}$.

Proof. From the recurrence hypothesis we have that $\tilde{Y}_{\gamma_a} \in \mathscr{A}_{rN_{\gamma_a}}^{(\alpha_{\gamma_a}; \hat{\sigma}_{\gamma_a}, \hat{\omega}_{\gamma_a})}$; from

5 We note that when $S \subset \operatorname{Ker} \lambda_i^{\gamma} \forall i \in \mathscr{L}_{\bar{\gamma}}$ then from the Definitions 2c and 2d we can easily verify that $S \subset \operatorname{Ker} \lambda_v^{\gamma} \forall v \in \mathscr{N}_{\bar{\gamma}}$

(4.3), (4.4), (4.5) the corresponding coefficients α_{γ_a} satisfy:

– for every : $S_{\gamma_a} \in \hat{\omega}_{\gamma_a}$ with $\pi_a(S_{\gamma_a}) = S^{(j)}(S^{(j)} \in \hat{\sigma}_{\gamma_a})$:

$$\alpha_{\gamma_a}(S_{\gamma_a}) = \alpha_{\gamma_a}(S^{(j)}) \leqq d(\gamma_a) - M^{(j)}(\gamma_a) \tag{4.13}$$

$$\text{– for } S^{(j)} \notin \hat{\sigma}_{\gamma_a} : \begin{cases} \alpha_{\gamma_a}(S^{(j)}) \leqq -M^{(j)}(\gamma_a) - 1 & (4.14) \\ \text{if } \exists \text{ at least one } \mu \in U(\gamma_a) \text{ with } S^{(j)} \in \hat{\sigma}_\mu & \\ \alpha_{\gamma_a}(S^{(j)}) = 0 \qquad \text{if } \forall \mu \in U(\gamma_a),\ S^{(j)} \notin \hat{\sigma}_\mu & (4.15) \end{cases}$$

The property $\gamma_a \in \mathscr{B}^{\mathscr{F}}(U)$ yields by Lemma 3.10:

$$\hat{\sigma}_\gamma \subset \hat{\sigma}_{\gamma_a} \tag{4.16a}$$

moreover by Definition 3i.1 we have $\gamma_a \in \mathscr{B}_\gamma(U)$ which by Lemma 3.12 iii) implies:

$$\omega^{(\gamma)}_{\gamma_a} \subset \hat{\omega}_{\gamma_a} \tag{4.16b}$$

We now apply Lemma 1.6 to the function $(1 - t^{d(\gamma_a)})\, \tilde{Y}_{\gamma_a}$; in view of Properties (4.16a, 4.17b) and Lemma 3.12(i), $(\hat{\sigma}_\gamma \hat{\omega}^{\gamma}_{(\gamma_a)})$ (resp. $(\hat{\sigma}_{\gamma_a} \hat{\omega}_{\gamma_a})$) can play the role of $(\sigma' \omega')$ (resp (σ, ω)) in this lemma and we obtainthat $(1 - t^{d(\gamma_a)})\, \tilde{Y}_{\gamma_a} \in \mathscr{A}^{(\tilde{\alpha}_{\gamma_a}, \hat{\sigma}_\gamma, \hat{\omega}_\gamma{}^{(\gamma)}_a)}_{rN_{\gamma_a}}$; the coefficients $\tilde{\alpha}_{\gamma_a}$ are specified as follows:

a) If $S^{(j)} \in \hat{\sigma}_\gamma$ Property a) of Lemma 1.6 yields:

$$\forall S_{\gamma_a} \in \omega^{(\gamma)}_{\gamma_a} \qquad \text{with } \pi(S_{\gamma_a}) = S^{(j)},$$

$$\tilde{\alpha}_{\gamma_a}(S_{\gamma_a}) = \tilde{\alpha}_{\gamma_a}(S^{(j)}) = \alpha_{\gamma_a}(S_{\gamma_a}) \tag{4.17}$$

Moreover by Properties (4.16 a, b) and the hypothesis $S^{(j)} \in \hat{\sigma}_\gamma$ we have $S^{(j)} \in \hat{\sigma}_{\gamma_a}$ and $S_{\gamma_a} \in \hat{\omega}_{\gamma_a}$, too; so, we insert (4.13) in (4.17) to obtain:

$$\tilde{\alpha}_{\gamma_a}(S_{\gamma_a}) \leqq d(\gamma_a) - M^{(j)}(\gamma_a) \tag{4.18}$$

b) If $S^{(j)} \notin \hat{\sigma}_\gamma$ but $S^{(j)} \in \hat{\sigma}_{\gamma_a}$, then Property b) of Lemma 1.6 yields:

$$\tilde{\alpha}_{\gamma_a}(S^{(j)}) = \alpha_{\gamma_a}(S^{(j)}) - d(\gamma_a) - 1 \tag{4.19}$$

Inserting (4.13) in (4.19) we obtain:

$$\tilde{\alpha}_{\gamma_a}(S^{(j)}) \leqq -M^{(j)}(\gamma_a) - 1 \tag{4.20}$$

c) If $S^{(j)} \notin \hat{\sigma}_{\gamma_a}$ Property c) of Lemma 1.6 yields:

$$\tilde{\alpha}_{\gamma_a}(S^{(j)}) = \alpha_{\gamma_a}(S^{(j)}) \tag{4.21}$$

We insert now (4.15) (resp. 4.14) in (4.21) to obtain:

$$\tilde{\alpha}_{\gamma_a}(S^{(j)}) = 0 \qquad \text{if } \forall \mu \in U(\gamma_a)\ S^{(j)} \notin \hat{\sigma}_\mu \tag{4.22}$$

$$\tilde{\alpha}_{\gamma_a}(S^{(j)}) \leqq -M^{(j)}(\gamma_a) - 1 \quad \text{if } \exists \text{ at least one } \mu \in U(\gamma_a) \text{ with } S^{(j)} \in \hat{\sigma}_\gamma . \tag{4.23}$$

Lemmas 3.10 (ii) and 3.12 (i) and (ii) allow us to apply Lemma 1.4 to the function $(1 - t^{d(\gamma_a)})\, \tilde{Y}_{\gamma_a} \in \mathscr{A}^{(\tilde{\alpha}_{\gamma_a}, \hat{\sigma}_\gamma, \omega^{(\gamma)}_{\gamma_a})}_{rN_{\gamma_a}}$, and to the mapping s_a(from $\underline{\mathscr{E}}^{rN_\gamma}_{(K^\gamma,k)}$ to $\underline{\mathscr{E}}^{rN_{\gamma_a}}_{(K^{\gamma_a},k)}$); we obtain that $s_a^*(1 - t^{d(\gamma_a)})\, \tilde{Y}_{\gamma_a}$ is an admissible function on $\underline{\mathscr{E}}^{rN_\gamma}_{(K^\gamma,k)}$ in the class $\mathscr{A}^{(\alpha^{(a)}_\gamma, \hat{\sigma}_\gamma, \hat{\omega}_\gamma)}_{rN_\gamma}$ such that:

$$\forall S_\gamma \subset \underline{\mathscr{E}}^{rN_\gamma}_{(K^\gamma,k)} : \alpha^{(a)}_\gamma(S_\gamma) = \tilde{\alpha}_{\gamma_a}(S_{\gamma_a}) \qquad \text{with } S_{\gamma_a} = s_a(S_\gamma) \tag{4.24}$$

Since the condition $\pi_a(S_{\gamma_a}) = \pi(S_\gamma)$ holds (in view of formula (1.19) it is easy to check that in view of (4.24) the conditions (4.18) (resp. (4.20), (4.22), (4.23)) entail property (i) (resp. (ii)) of the Lemma. q.e.d.

Lemma 4.3. *For every subgraph $\gamma_a \in \mathscr{M}_\gamma(U)$ with $\gamma_a \notin \mathscr{B}(U)$, the function $s_a^*(-t^{d(\gamma_a)})\tilde{Y}_{\gamma_a}$ belongs to a class $\mathscr{A}_{rN_\gamma}^{(\alpha_\gamma^{(a)},\hat{\sigma}_\gamma,\hat{\omega}_\gamma)}$ of admissible Weinberg functions whose asymptotic coefficients satisfy the following properties:*

i) Let $S^{(j)} \in \hat{\sigma}_\gamma$; $\forall S_\gamma \in \hat{\omega}_\gamma$ with $\pi(S_\gamma) = S^{(j)}$

$$\text{either}: \alpha_\gamma^{(a)}(S_j) = d(\gamma_a) = d(\gamma_a) - M^{(j)}(\gamma_a) \quad \text{if } \forall \mu \in U(\gamma_a) \quad S^{(j)} \notin \hat{\sigma}_\mu \tag{4.25}$$

$$\text{or} \quad \alpha_\gamma^{(a)}(S_\gamma) \leqq d(\gamma_a) - M^{(j)}(\gamma_a) \quad \text{if } \exists \text{ at least one } \mu \in U(\gamma_a) \text{ with } S^{(j)} \in \hat{\sigma}_\mu . \tag{4.26}$$

ii) *Let $S^{(j)} \notin \hat{\sigma}_\gamma$; then*

$$\text{either}: \alpha_\gamma^{(a)}(S^{(j)}) = 0 = -M^{(j)}(\gamma_a) \quad \text{if } \forall \mu \in U(\gamma_a) S^{(j)} \notin \hat{\sigma}_\mu \tag{4.27}$$

$$\text{or} \quad : \alpha_\gamma^{(a)}(S^{(j)}) \leqq -M^{(j)}(\gamma_a) - 1 \quad \text{if } \exists \text{ at least one } \mu \in U(\gamma_a) \text{ with } S^{(j)} \in \hat{\sigma}_\mu \tag{4.28}$$

Remark. For the last Eqs. in (4.25) (4.27) see the remark at the end of Lemma 4.2.

Proof. From the recurrence hypothesis we have $\tilde{Y}_{\gamma_a} \in \mathscr{A}_{rN_{\gamma_a}}^{(\alpha_{\gamma_a},\hat{\sigma}_{\gamma_a},\hat{\omega}_{\gamma_a})}$. Moreover the property $\gamma_a \notin \mathscr{B}(U)$ yields by Lemma 3.10:

$$\hat{\sigma}_{\gamma_a} \subset \hat{\sigma}_\gamma \tag{4.29}$$

We apply directly Lemma 1.5 to the function $(-t^{d(\gamma_a)})\tilde{Y}_\gamma$; now in view of (4.29) the role of the set $(\sigma'\omega')$ (resp. σ, ω) is played by the admissible couple $(\hat{\omega}_{\gamma_a}^{(\gamma)}, \hat{\sigma}_\gamma)$ (see Lemma 3.12.*i*) (resp. $\hat{\sigma}_{\gamma_a}, \hat{\omega}_{\gamma_a}$) and we obtain that:

$(-t^{d(\gamma_a)})\tilde{Y}_{\gamma_a} \in \mathscr{A}_{rN\gamma_a}^{(\tilde{\alpha}_{\gamma_a},\hat{\sigma}_\gamma,\, \omega_{\gamma_a}^{(\gamma)})}$ with $\tilde{\alpha}_{\gamma_a}$ defined as follows:

a) If $S^{(j)} \in \hat{\sigma}_\gamma$, $S^{(j)} \in \hat{\sigma}_{\gamma_a}$, then $\forall S_{\gamma_a}$ with $\pi_a(S_{\gamma_a}) = S^{(j)}$ we have by Property a) of Lemma 1.5:

$$\tilde{\alpha}_{\gamma_a}(S_{\gamma_a}) = \alpha_{\gamma_a}(S^{(j)}) \tag{4.30}$$

From the recurrence hypothesis (4.5) which applies to $S^{(j)} \in \hat{\omega}_{\gamma_a}$ the Eq. (4.30) yields:

$$\tilde{\alpha}_{\gamma_a}(S_{\gamma_a}) \leqq d(\gamma_a) - M^{(j)}(\gamma_a) \tag{4.31}$$

b) If $S^{(j)} \in \hat{\sigma}_\gamma$ $S^{(j)} \notin \hat{\sigma}_{\gamma_a}$, then $\forall S_{\gamma_a}$ with $\pi_a(S_{\gamma_a}) = S^{(j)}$ by Property b) of Lemma 1.5 we have:

$$\tilde{\alpha}_{\gamma_a}(S_{\gamma_a}) = \alpha_{\gamma_a}(S^{(j)}) + d(\gamma_a) \tag{4.32}$$

From the recurrence hypothesis (4.3), (4.4) the Eq. (4.32) yields:

$$\text{either}: \tilde{\alpha}_{\gamma_a}(S_{\gamma_a}) = d(\gamma_a) \quad \text{if } \forall \mu \in U(\gamma_a) \quad S^{(j)} \notin \hat{\sigma}_\mu \tag{4.33}$$

or $\quad : \tilde{\alpha}_{\gamma_a}(S_{\gamma_a}) \leqq d(\gamma_a) - M^{(j)}(\gamma_a), \quad$ if $\exists$ at least one $\mu \in U(\gamma_a)$ with $S^{(j)} \in \hat{\sigma}_\mu$. (4.34)

c) If $S^{(j)} \notin \hat{\sigma}_\gamma$ by Property c) of Lemma 1.5 we obtain:

$$\tilde{\alpha}_{\gamma_a}(S^{(j)}) = \alpha_{\gamma_a}(S^{(j)}) \tag{4.35}$$

Taking into account (4.29) we also have $S^{(j)} \notin \hat{\sigma}_{\gamma_a}$; so from the recurrence hypothesis (4.3), (4.4) the above Eq. (4.35) yields:

$$\tilde{\alpha}_{\gamma_a}(S^{(j)}) = 0, \qquad \text{if } \forall \mu \in U(\gamma_a) \quad S^{(j)} \notin \hat{\sigma}_\mu \tag{4.36}$$

$$\tilde{\alpha}_{\gamma_a}(S^{(j)}) \leqq -M^{(j)}(\gamma_a) - 1, \qquad \text{if } \exists \text{ at least one } \mu \in U(\gamma_a)\ (\mu \neq \gamma_a) \text{ with } S^{(j)} \in \hat{\sigma}_\mu. \tag{4.37}$$

Lemmas 3.7 (ii) and 3.9 (i) and (ii) allow us to apply Lemma 1.4 to the function $(-t^{d(\gamma_a)})Y_{\gamma_a}$ and to the mapping s_a; as at the end of the proof of Lemma 4.2, we easily check that the Property i) (resp. ii) of the present lemma is directly implied by the above inequalities (4.31), (4.33), (4.34), (resp. (4.36), (4.37) if one again defines the class $\mathscr{A}_{rN_\gamma}^{(\alpha_\gamma^{(a)}, \hat{\sigma}_\gamma, \hat{\omega}_\gamma)}$ by (4.24).

Proof of Proposition. 4.1. We apply the results of Lemmas (4.1), (4.2), (4.3) to each of the factors in formula (3.31):

$$\tilde{Y}_\gamma = I_{\bar{\gamma}} \prod_{1<\alpha<c} s_a^* f_{\gamma_a} \tilde{Y}_{\gamma_a}$$

By Proposition 1.2.b, we obtain that: $\tilde{Y}_\gamma \in \mathscr{A}_{rN_\gamma}^{(\alpha_\gamma, \hat{\sigma}_\gamma, \hat{\omega}_\gamma)}$, where the set of asymptotic coefficients $\alpha_\gamma(S)$ satisfies the following conditions:
i) $\forall S^{(j)} \in \mathscr{F}$ with $S^{(j)} \notin \hat{\sigma}_\gamma$, we take into account Properties (4.6) (4.11) (resp. 4.12), (resp. 4.28) and Proposition 1.2 b, and this yields:

$$\alpha_\gamma(S^{(j)}) = \alpha_\gamma(S^{(j)}) + \sum_a \alpha_\gamma^{(a)}(S^{(j)}) = 0; \quad \text{if } \forall \mu \in U(\gamma)\ S^{(j)} \notin \hat{\sigma}_\mu \tag{4.3}$$

$$\text{or respectively: } \alpha_\gamma(S^{(j)}) \leqq -\sum_a M^{(j)}(\gamma_a) - 1 = -M^{(j)}(\gamma) - 1 \quad \text{if } \exists \text{ at least one } \mu \in U(\gamma) \text{ with } S^{(j)} \in \hat{\sigma}_\mu. \tag{4.4}$$

For writing the equality at the right hand side of (4.3), (4.4) we have made use of Definition 3b and of the fact that $\gamma \in W^{(j)}(U)$.
ii) $\forall S^{(j)} \in \mathscr{F}$ with $S^{(j)} \in \hat{\sigma}_\gamma$ from Properties (4.7), (4.10), (4.25), (resp. (4.26)) and Proposition 1.2 b we have that $\forall S_\gamma \in \hat{\omega}_\gamma$ with $\pi(S_\gamma) = S^{(j)}$:

$$\alpha_\gamma(S_\gamma) = \sum_a \alpha_\gamma^{(a)}(S_\gamma) + \alpha_{\bar{\gamma}}(S_\gamma) \leqq \sum_a d(\gamma_a) - \sum_a M^{(j)}(\gamma_a) + d(\bar{\gamma}) - rm(\bar{\gamma}) \tag{4.38}$$

Let us use the relations:

$$d(\gamma) = \sum_a d(\gamma_a) + d(\bar{\gamma}) \tag{2.42}$$

and

$$M^{(j)}(\gamma) = \sum_a M^{(j)}(\gamma_a) + rm(\bar{\gamma}) \tag{3.6}$$

(the latter trivially follows from Definition (3b) since $\gamma \notin W^{(j)}(U)$). So, (4.38) yields finally:

$$\alpha_\gamma(S_\gamma) \leqq d(\gamma) - M^{(j)}(\gamma) \tag{4.5}$$

q.e.d.

Proposition 4.2. *For every $U \in \mathscr{U}(\mathscr{F})$, the function*

$$\tilde{X}_U = (1 - t^{d(G)})\, \tilde{Y}_G^{(U)} \tag{3.29}$$

and every derivative $D^\nu_{(K)}\tilde{X}_U$ of $\tilde{X}_U$ belong to a class $A_{rN}^{(\alpha(U))}$ of Weinberg functions such that:

$$\forall S^{(j)} \in \mathscr{F},\ \alpha_{(U)}(S^{(j)}) \leqq -h^{(j)} - 1 \tag{4.39}$$

here $h^{(j)}$ means the dimension of the subspace $S^{(j)}$.

Proof. When Proposition 4.1 is applied to $\gamma = G$, it yields:

$\tilde{Y}_G \in \mathscr{A}_{rN}^{(\alpha_G, \hat{\sigma}_G, \hat{\omega}_G)}$ with the properties:

1) $\forall S^{(j)} \in \mathscr{F}$ such that $S^{(j)} \notin \hat{\sigma}_G$:

$$\alpha_G(S^{(j)}) \leqq -M^{(j)}(G) - 1 \tag{4.40}$$

because there is always at least one $\mu \in U(G)$ such that $S^{(j)} \in \hat{\sigma}_\mu$ (since $S^{(j)} \neq \{0\}$).
ii) $\forall S^{(j)} \in \mathscr{F}$ such that $S^{(j)} \in \hat{\sigma}_G$ and $\forall S \in \hat{\omega}_G$ with $\pi(S) = S^{(j)}$:

$$\alpha_G(S) \leqq d(G) - M^{(j)}(G) \tag{4.41}$$

We can now apply Lemma 1.6′ to the function $\tilde{X}_U$ expressed by (3.29) (the cases i) and ii) correspond respectively to the cases b) and a) of Lemma 1.6′); we then obtain that there exists a class $A_{rN}^{(\alpha(U))}$ which contains $\tilde{X}_U$ and all the derivatives $D^\nu_{(K)}\tilde{X}_U$ and which satisfies the conditions:

$$\forall S^{(j)} \in \mathscr{F},\ \alpha_{(U)}(S^{(j)}) \leqq -M^{(j)}(G) - 1 \tag{4.42}$$

Moreover taking into account Definition 3b, we have: $M^{(j)}(G) \geqq h^{(j)}$ where $h^{(j)}$ is the dimension of $S^{(j)}$. It then follows from (4.42) that:

$$\forall S^{(j)} \in \mathscr{F},\ \alpha(S^{(j)}) \leqq -h^{(j)} - 1$$

q.e.d.

Proof of Theorem 4.1. Let $\{L_1, \ldots, L_{\tilde{m}}\}$ be an arbitrary set of independent vectors and W an arbitrary bounded region in $E^{rm}_{(k)}$. With the ordered set $\{L_1, \ldots, L_{\tilde{m}}\}$, we associate a unique nested set of subspaces $\mathscr{F} = \{S^{(1)}, \ldots, S^{(\tilde{m})}\}$ by the following definition:

$$\forall j;\ 1 \leqq j \leqq \tilde{m} : S^{(j)} = \overline{\{L_1, \ldots, L_j\}}$$

We then write the corresponding expression (3.28) of R_G:

$$R_G(K, k) = \sum_{U \in \mathscr{U}(\mathscr{F})} \tilde{X}_U(K, k) \tag{3.28}$$

Then in view of Proposition 4.2 and of Definition 1a, we can say that for each forest U in $\mathscr{U}(\mathscr{F})$, and for each bounded region $\Omega\in\mathscr{E}^{4(n-1)}$ there exist numbers $b_j(U,\Omega)\geqq 1$ $(1\leqq j\leqq\tilde{m})$ and $M_U(\Omega)$ such that $\tilde{X}_U$ satisfies the following bound:

$$\forall K\in\omega:\left|\tilde{X}_U\left(K,\sum_{j=1}^{\tilde{m}}L_j\eta_j\ldots\eta_{\tilde{m}}+C\right)\right|\leqq M_U(\Omega)\prod_{j=1}^{\tilde{m}}\eta_j^{\alpha_U(\{L_1\ldots L_j\})}$$
$$\leqq M_U(\Omega)\prod_{j=1}^{\tilde{m}}\eta_j^{-h(j)-1} \tag{4.43}$$

provided that: $\forall j$, $\eta_j\geqq b_j(U,\omega)$, and $C\in W$.

Let us then put:

$$\left.\begin{aligned}M(\Omega)&=\sum_{U\in\mathscr{U}(\mathscr{F})}M_U(\Omega)\\ b_j(\Omega)&=\sup_{U\in\mathscr{U}(\mathscr{F})}b_j(U,\Omega)\end{aligned}\right\} \tag{4.44}$$

From (3.28), (4.43), (4.44) it follows that:

$$\forall K\in\Omega:\left|R_G\left(K,\sum_{j=1}^{\tilde{m}}L_j\eta_j\ldots\eta_{\tilde{m}}+C\right)\right|\leqq M(K)\prod_{j=1}^{\tilde{m}}\eta_j^{-h(j)-1} \tag{4.45}$$

provided that: $\forall j$, $\eta_j\geqq b_j(\Omega)$ and $C\in W$.

Let us now introduce the class $A_{rm}^{(\alpha)}$ whose asymptotic indicatrix α is defined as follows; for every subspace S in $E_{(k)}^{rm}$ with dimension $h(S)$, one puts:

$$\alpha(S)=-h(S)-1 \tag{4.46}$$

Since formula (4.45) has been established for an arbitrary set of independent vectors $\{L^1,\ldots,L^{\tilde{m}}\}$ and an arbitrary bounded region W in $E_{(k)}^{rm}$ it expresses the fact (see Definition 1a) that for every K in $\mathscr{E}^{r(n-1)}$, $R_G(K,k)$ belongs as a function of k, to the above-defined class $A_{rm}^{(\alpha)}$.

Now, by definition, α satisfies the following property:

$$\sup_S(\alpha(S)+h(S))=-1, \tag{4.1}$$

and therefore R_G satisfies the Weinberg integrability criterion (1.4); thus in view of Lemma 1.2, the absolute convergence of the integral (4.2) is then ensured.

To achieve the proof of Theorem 4.1, it remains to show that the function H_G^{ren} defined by (4.2) is infinitely differentiable on $\mathscr{E}_{(k)}^{r(n-1)}$. This will result from the two following points.

i) In Weinberg's proof of his convergence theorem, (whose details are essentially reproduced here in the proof of our Lemma B.1 in Appendix B, it is clear that the uniformity of the input bounds on the integrand (such as (4.3) for R_G) with respect to the external variables (namely K) varying in a bounded set ω, entails the following property: there exists an *integrable* positive function $g(k)$ on the integration space $E_{(k)}^{rm}$ such that:

$$\forall K\in\Omega\,|R_G(K,k)|\leqq g(k).$$

ii) Every derivative $D_{(K)}^{\nu}\,R_G(K,k)$ also belongs to the class $A_{rm}^{(\alpha)}$ (satisfying condition

(4.1)) and also fulfils a bound of the type (4.3) for K varying in Ω. In fact, for every set $\mathscr{F}$ and every forest U in $\mathscr{U}(\mathscr{F})$, Proposition 2 applies to $D_K^\nu \tilde{X}_U$ as well as to $\tilde{X}_U$, and one just has to use the various decompositions of $D_{(K)}^\nu R_G$ which correspond to those of R_G written above (as an application of formula (3.25)).

By now applying the same argument as above (in i)) to the functions $D_{(K)}^\nu R_G$, we deduce that there exist integrable functions $g_\nu(k)$ such that:

$$\forall K \in \Omega \ \left| D_{(K)}^\nu R_G(K, k) \right| \leqq g_\nu(k)$$

From a known theorem of integration theory, we then conclude that derivation under the sign $\int$ with respect to variables K is licit in formula (4.2), so that

$$H_G^{\text{ren}} \in C^\infty(\mathscr{E}_{(K)}^{r(n-1)}) \qquad \text{q.e.d.}$$

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