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Analyticity of Solutions of the $O(N)$ Nonlinear σ-Model

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Abstract. We consider continuous weak solutions of the Euler-Lagrange equations associated with the Euclidean d-dimensional $O(N)$ nonlinear σ-model. We show for arbitrary N and arbitrary d that such solutions with locally square integrable gradient are real analytic.

1. Introduction

We consider solutions of the d-dimensional ($d \geqq 2$) Euclidean $O(N)$ non-linear σ-model, i.e. stationary points of the Lagrangian

$$L(n) = \sum_{\alpha=1}^{d} \sum_{l=1}^{N} (\partial_\alpha n_l)^2 \equiv (\nabla n)^2 \tag{1.1}$$

where $\partial_\alpha = \frac{\partial}{\partial x_\alpha}$, and $n \in \mathbb{R}^N$ satisfies the constraint

$$n^2 \equiv |n|^2 \equiv (n, n) := \sum_{l=1}^{N} n_l^2 = 1 . \tag{1.2}$$

Stationary points n of L such that $L(n)$ is locally L^1 are (weak) solutions of the Euler-Lagrange equations associated with (1.1)

$$\Delta n_l + L(n) n_l = 0 \qquad l = 1 \dots N . \tag{1.3}$$

(A detailed proof of this fact along the lines of the usual variational argument has been given in [1] where also the class of variations was specified.)

Since the left hand side of (1.3) is an elliptic operator, one may expect weak solutions to show some regularity, i.e. to be C^k (k times continuously differentiable) for some k. There is an extensive literature on elliptic regularity, and we quote only some results relevant for (1.3):

In 1929, Lewy [2] gave a lucid proof of Bernstein's theorem that in two dimensions ($d=2$), every C^3-solution of a nonlinear elliptic equation with analytic

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coefficients is, in fact, real analytic. Three years later, Hopf proved [3] the corresponding result for C^2-solutions in any number of dimensions. Finally, Petrowsky [4] generalized Hopf's result to systems ($N>1$) of elliptic partial differential equations; for a recent proof of Petrowsky's result, see [10].

All of this work concerns classical solutions. The first result on weak solutions was proven by Morrey [5] who showed that, in two dimensions, any Hölder continuous solution of (1.3) is C^2 (and hence analytic). Though his results apply to slightly more general elliptic systems, too, the method of proof does not generalize easily to higher dimensions. The next step was taken in the late fifties and culminated in the result of Ladyzhenskaya and Ural'tseva ([6]; see also the earlier work quoted therein on which they rely): Any bounded, locally square integrable solution of a single elliptic equation is analytic. Note that the corresponding result for systems of equations is false: For $d=N(>2)$, (1.3) has the weak solution $n=|x|^{-1}x$.

Under the additional assumption that $|n|<1$, Hildebrandt and Widman proved in [7] that any bounded locally square integrable solution of the system (1.3) is Hölder continuous, for $d=2$ (which implies analyticity by Morrey's result). Note that the assumption $|n|<1$ is incompatible with (1.2). The result in [7] was generalized in [8] to any number of dimensions. It is again inapplicable because of the assumption $|n|<1$; furthermore, for arbitrary d, we have been unable to find in the literature the generalization of Morrey's result that Hölder continuity implies analyticity.

Thus, no regularity result known to us seems to be directly applicable to the system (1.3) with condition (1.2). As the example given above shows, one cannot expect arbitrary bounded solutions of (1.3) to be analytic. We will show, however, the following theorem in Sect. 4:

Theorem. *Let n be a continuous solution of* (1.3) *fulfilling* (1.2), *and let ∇n be locally square integrable. Then n is real analytic.*

2. L^p-Properties of the Gradient

As a first step in proving regularity, we show in this section that the gradient of weak solutions of (1.3), i.e. of vector functions n fulfilling

$$\int_\Omega \partial_\alpha n \partial_\alpha \varphi d^d x = \int_\Omega (\nabla n)^2 n \varphi d^d x \tag{2.1}$$

(summation convention!) for all regions Ω and all test functions $\varphi \in \mathscr{D}(\Omega)$, is in $L^p_{\mathrm{loc}}(\Omega)$ for every $p \geqq 1$, provided n is continuous. As ∇n was initially assumed to be locally L^2, this is an improvement only for $p>2$. We will use (1.2) as it simplifies the proofs and mention only that the regularity results would hold without assuming it. We will estimate ∇n by approximating it by difference quotients defined as follows:

Choose an orthonormal basis $\{e_\alpha\}$ of $\mathbb{R}^d$ and put, for any function f on $\mathbb{R}^d$ and $0 \neq h \in \mathbb{R}$.

$$\Delta_\alpha(h) f := h^{-1}[E_\alpha(h) - 1] f \qquad \alpha = 1 \dots d \tag{2.2}$$

where we introduced the translation operators

$$[E_\alpha(h)f](x) := f(x+he_\alpha) \qquad \alpha = 1 \dots d. \tag{2.3}$$

We note the formula on "summation by parts"

$$\int [\Delta_\alpha(h)f] g d^d x = -\int f[\Delta_\alpha(-h)g] d^d x \tag{2.4}$$

for $f \in \mathscr{D}'(\mathbb{R}^d)$, $g \in \mathscr{D}(\mathbb{R}^d)$, and the product rule

$$\Delta_\alpha(f \cdot g) = f\Delta_\alpha(g) + (\Delta_\alpha f)E_\alpha g. \tag{2.5}$$

The next, well-known lemma relates difference quotients and derivatives:

2.1. Lemma. *Let $\Omega \in \mathbb{R}^d$ be a region and Ω' strictly interior to Ω. Suppose $f \in L^p(\Omega)$ and $1 < p < \infty$.*

(i) *If $\partial_\alpha f \in L^p(\Omega)$, then $\Delta_\alpha(h)f$ is strongly bounded in $L^p(\Omega')$ for small $h \in \mathbb{R}$.*

(ii) *If $\Delta_\alpha(h)f$ is strongly bounded in $L^p(\Omega)$, then $\partial_\alpha f \in L^p(\Omega)$, and $\Delta_\alpha(h)f$ converges strongly in $L^p(\Omega')$ to $\partial_\alpha f$, for $h \to 0$.*

Proof. (i) it follows from

$$\Delta_\alpha(h)f = h^{-1} \int_0^h \partial_\alpha f(x+te_\alpha)dt \tag{2.6}$$

by using the triangle inequality as generalized to integrals that

$$\|\Delta_\alpha(h)f\|_{L^p(\Omega')} \leqq \sup_{0 \leqq t \leqq h} \|E_\alpha(t)\partial_\alpha f\|_{L^p(\Omega')} \leqq \|\partial_\alpha f\|_{L^p(\Omega)}$$

for all h such that $x + he_\alpha \in \Omega$ if $x \in \Omega'$.

(ii) Since

$$\|(E_\alpha(h) - 1)f\|_{L^p(\Omega)} \leqq K|h|$$

with K independent of h, the distributions $T_h \in \mathscr{D}'(\Omega)$ defined by

$$T_h(\varphi) := h^{-1} \int [(E_\alpha(h) - 1)f](x)\varphi(x) d^d x$$

obey the estimate

$$|T_h(\varphi)| \leqq K \|\varphi\|_{L^q(\Omega)}$$

where $q^{-1} = 1 - p^{-1}$, by the Hölder inequality. For $h \to 0$, T_h converges on $\mathscr{D}'(\Omega)$ to $T = \partial_\alpha f$, by (2.4), and the limit distribution obeys again

$$|T(\varphi)| \leqq K \|\varphi\|_{L^q(\Omega)}.$$

Hence, T can be uniquely extended to a continuous linear functional on $L^q(\Omega)$ and is thus in $L^p(\Omega)$, by the duality of $L^q(\Omega)$ and $L^p(\Omega)$ which proves $\partial_\alpha f \in L^p(\Omega)$.

To show the strong convergence, note that (2.6) implies

$$\|\partial_\alpha f - \Delta_\alpha(h)f\|_{L^p(\Omega')} \leqq \sup_{0 \leqq t \leqq h} \|(E_\alpha(t) - 1)\partial_\alpha f\|_{L^p(\Omega')}$$

which tends to zero by the strong continuity of the translation operators in $L^p(\Omega)$. □

We will later want to insert into (2.1) as test vectors φ functions of n which are not known a priori to be in $\mathscr{D}$. But note that any weak solution of (1.3) fulfils (2.1) with $\varphi\in\mathring{H}_1^2(\Omega)$, i.e. with functions φ which are, together with their derivatives, in $L^2(\Omega)$ and vanish close to the boundary of Ω. To be precise, define the $\|\cdot\|_2^{(1)}$-norm by

$$\|\varphi\|_2^{(1)}:=\|\varphi\|_2+\sum_{\alpha=1}^{d}\|\partial_\alpha\varphi\|_2\,.$$

Then $\mathring{H}_1^2(\Omega)$ is the completion of $\mathscr{D}(\Omega)$ in the $\|\cdot\|_2^{(1)}$-norm. We have the following estimate:

2.2. Lemma. *For any continuous solution n of* (2.1), (1.2), *any $y\in\Omega$, and any $\varepsilon>0$, there is a ball $K_\varrho(y)$ around y of radius $\varrho=\varrho(\varepsilon)$ such that*

$$\int_{K_\varrho(y)}(\nabla n)^2\xi^2d^dx\leqq\varepsilon\int_{K_\varrho(y)}(\nabla\xi)^2d^dx$$

for all $\xi\in\mathring{H}_1^2(K_\varrho(y))$.

Proof. Choose as a test vector in (2.1)

$$\varphi(x):=[n(x)-n(y)]\xi^2(x)$$

and use the continuity of n to fix ϱ such that

$$4|n(x)-n(y)|<4\delta^2<1$$

for $x\in K_\varrho=K_\varrho(y)$. Then

$$\int_{K_\varrho}(\partial_\alpha n,\partial_\alpha n)\xi^2d^dx=-2\int_{K_\varrho}(\partial_\alpha n,n-n(y))\xi\partial_\alpha\xi d^dx$$
$$+\int_{K_\varrho}(\nabla n)^2(n,n-n(y))\xi^2d^dx\,.$$

Hence

$$\int_{K_\varrho}(\nabla n)^2\xi^2d^dx\leqq\delta\left\{2\cdot\int_{K_\varrho}|\partial_\alpha n|\,|\xi\partial_\alpha\xi|d^dx+\int_{K_\varrho}(\nabla n)^2\xi^2d^dx\right\}$$

where we used (1.2). But

$$2|\xi\partial_\alpha n|\,|\partial_\alpha\xi|\leqq(\nabla n)^2\xi^2+(\nabla\xi)^2$$

and therefore

$$(1-2\delta)\int_{K_\varrho}(\nabla n)^2\xi^2d^dx\leqq\delta\int_{K_\varrho}(\nabla\xi)^2d^dx$$

from which the statement follows if δ is chosen such that

$$\varepsilon=\delta\cdot(1-2\delta)^{-1}\,.$$ □

As a preliminary step, we prove

2.3. Theorem. *Let n be a weak continuous solution of* (2.1) *which fulfils* (1.2). *Then*

(i) $\nabla n \in L^4(\Omega')$;

(ii) $\partial_\alpha \partial_\beta n \in L^2(\Omega') \quad 1 \leqq \alpha, \beta \leqq d$

for all precompact regions Ω' strictly interior in Ω.

Proof

1. Choose a unit vector e_β and insert into (2.1)

$$\varphi := \Delta_\beta(-h)\psi$$

where $\psi \in \mathring{H}_1^2(\Omega)$ and $|h|$ is so small that $\varphi \in \mathring{H}_1^2(\Omega)$. Then use (2.4) to get

$$\int_\Omega (\Delta_\beta \partial_\alpha n)\partial_\alpha \psi d^d x = \int_\Omega \Delta_\beta[(\nabla n)^2 n]\psi d^d x .$$

Now put

$$\psi := (\Delta_\beta n)\zeta^2$$

with $\zeta \in \mathring{H}_1^2(\Omega)$. Since Δ_β commutes with ∂_α,

$$\int_\Omega (\nabla(\Delta_\beta n))^2 \zeta^2 d^d x \leqq \int_\Omega |(\Delta_\beta[(\nabla n)^2 n], \Delta_\beta n)|\zeta^2 d^d x + 2\int_\Omega |\zeta \partial_\alpha \zeta(\partial_\alpha \Delta_\beta n, \Delta_\beta n)| d^d x . \tag{2.7}$$

For the second integral we use Young's inequality

$$2|a|\,|b| \leqq \delta |a|^2 + \delta^{-1}|b|^2 \tag{2.8}$$

(valid for all $\delta > 0$) to bound the integrand by

$$\delta(\nabla(\Delta_\beta n))^2 \zeta^2 + \delta^{-1}(\nabla\zeta)^2(\Delta_\beta n)^2 .$$

We choose $2\delta < 1$ for later purposes to get

$$(1-\delta)\int_\Omega (\nabla(\Delta_\beta n))^2 \zeta^2 d^d x \leqq \delta^{-1}\int_\Omega (\Delta_\beta n)^2 (\nabla\zeta)^2 d^d x + \int_\Omega \zeta^2((E_\beta+1)\partial_\alpha n, \Delta_\beta \partial_\alpha n) \cdot (E_\beta n, \Delta_\beta n) d^d x + \int_\Omega (\nabla n)^2 (\Delta_\beta n)^2 \zeta^2 d^d x \tag{2.9}$$

where we used the product rule (2.5) for the first integral on the right of (2.7). In the second integral of (2.9), we use Young's inequality again to bound the integrand by

$$\zeta^2[\delta(\nabla(\Delta_\beta n))^2 + \delta^{-1}\{(E_\beta+1)(\nabla n)^2\}(\Delta_\beta n)^2]$$

so that

$$(1-2\delta)\int_\Omega (\nabla(\Delta_\beta n))^2 \zeta^2 d^d x \leqq \delta^{-1}\int_\Omega (\nabla\zeta)^2 (\Delta_\beta n)^2 d^d x + \delta^{-1}\int_\Omega \{(E_\beta+2)(\nabla n)^2\}(\Delta_\beta n)^2 \zeta^2 d^d x . \tag{2.10}$$

2. To estimate the second integral on the right of (2.10), we use Lemma 2.2. Fix a point $y \in \Omega$ and choose $\varepsilon > 0$ so that $6\varepsilon\delta^{-1} + 2\delta < 1$ which is possible since $2\delta < 1$. Replace, in Lemma 2.2, ξ by $\psi \Delta_\beta n$. Then there is a ball $K_\varrho(y) \subset \Omega$ so that for all $\psi \in \mathring{H}_1^2(K_\varrho)$,

$$\int_{K_\varrho} (\nabla n)^2 (\Delta_\beta n)^2 \psi^2 d^d x \leqq \varepsilon \int_{K_\varrho} [\nabla(\psi \Delta_\beta n)]^2 d^d x$$
$$\leqq 2\varepsilon \int_{K_\varrho} [(\nabla\psi)^2 (\Delta_\beta n)^2 + \psi^2 (\nabla(\Delta_\beta n))^2] d^d x. \tag{2.11}$$

This takes care of the terms not containing the translation operator E. To estimate the translated terms, note that (1.3) is translation invariant; hence, Lemma 2.2 holds for $E_\beta(\nabla n)^2$ as well with possibly a different circle $K_\mu(y)$, and (2.11) remains true if $(\nabla n)^2$ on the left is replaced by $E_\beta(\nabla n)^2$ and integration extending over K_μ. Put $\sigma = \min(\mu, \varrho)$. Then choose $\psi \in \mathring{H}_1^2(K_\sigma)$ in (2.11) and $\zeta = \psi$ in (2.10):

$$(1 - 2\delta - 6\varepsilon\delta^{-1}) \int_{K_\sigma} (\nabla n)^2 (\Delta_\beta n)^2 \psi^2 d^d x \leqq \delta^{-1}(1 + 6\varepsilon) \int_{K_\sigma} (\nabla\psi)^2 (\Delta_\beta n)^2 d^d x. \tag{2.12}$$

Choose $\psi \equiv 1$ on $K_\lambda(y)$ where $\lambda < \sigma$. Then, since $\nabla n \in L^2(\Omega)$, the integral on the right in (2.12) is bounded independently of h, by Lemma 2.1(i). By the same lemma, $\Delta_\beta n$ converges almost everywhere to ∇n so that Fatou's lemma implies $|\nabla n| \in L^4(K_\lambda)$. Since $y \in \Omega$ was arbitrary, the precompactness of Ω' implies $|\nabla n| \in L^4(\Omega')$. Furthermore, (2.10) shows that all second derivatives $\partial_\alpha \partial_\beta n \in L^2(\Omega')$. □

We are now ready to prove the announced L^p-properties:

2.4. Theorem. *Let $\Omega \in \mathbb{R}^d$ be a region, and Ω' a precompact region strictly interior to Ω. Assume that the gradient ∇n of any weak continuous solution is in $L^2_{\mathrm{loc}}(\Omega)$. Then $\nabla n \in L^p_{\mathrm{loc}}(\Omega')$ for any $p \geqq 1$.*

Proof

1. By the last theorem we may replace φ in (2.1) by $\partial_\beta \varphi$ for any $\varphi \in \mathring{H}_1^2(\Omega)$ and integrate the first term by parts:

$$\int_\Omega \partial_\alpha \partial_\beta n \partial_\alpha \varphi d^d x + \int_\Omega (\nabla n)^2 n \partial_\beta \varphi d^d x = 0. \tag{2.13}$$

Now choose $N > 0$, define

$$b_N(x) := \min\{(\nabla n)^2(x), N\}$$

and insert into (2.13)

$$\varphi := (b_N)^s \partial_\beta n \zeta^2$$

for an arbitrary integer s and $\zeta \in \mathring{H}_1^2(\Omega)$. Taking into account

$$(n, \partial_\beta n) = 0$$

because of $(n, n) = 1$, this yields, if we sum over β,

$$\sum_{\alpha,\beta} \left\{ \int_\Omega [(\partial_\alpha \partial_\beta n)^2 b_N^2 \zeta^2 + (\partial_\alpha \partial_\beta n, \partial_\beta n) s b_N^{s-1} \partial_\alpha b_N \zeta^2 + 2(\partial_\alpha \partial_\beta n, \partial_\beta n) \zeta \partial_\alpha \zeta b_N^s] d^d x \right.$$
$$\left. + \int_\Omega (\nabla n)^2 (n, \partial_\beta \partial_\beta n) b_N^s \zeta^2 d^d x \right\} = 0. \tag{2.14}$$

Use (1.3) in the last term; for the second term, note that $\partial_\alpha b_N$ is non-zero only if $(\nabla n)^2 \leqq N$, i.e. if $b_N = (\nabla n)^2$ so that, for those points,

$$\partial_\alpha b_N = 2(\partial_\alpha \partial_\gamma n, \partial_\gamma n). \tag{2.15}$$

This yields

$$\int_\Omega \left\{ \left[\sum_{\alpha,\beta} (\partial_\alpha \partial_\beta n)^2 \right] b_N^s \zeta^2 + \frac{s}{2} b_N^{s-1} |\nabla b_N|^2 \zeta^2 \right\} d^d x$$
$$\leqq \int_\Omega |\nabla n|^4 b_N^s \zeta^2 d^d x + \varepsilon \int_\Omega \sum_{\alpha,\beta} (\partial_\alpha \partial_\beta n)^2 b_N^s \zeta^2 d^d x + \varepsilon^{-1} \int_\Omega (\nabla n)^2 (\nabla \zeta)^2 b_N^s d^d x \tag{2.16}$$

where we used Young's inequality for the third term in (2.14).

2. We now fix a point $y \in \Omega$ and a number $0 < \delta < \min\{s^{-1}, 2^{-1}(1-\varepsilon)\}$. By Lemma 2.2 we can find a ball $K_\varrho(y)$ so that

$$\int_{K_\varrho} (\nabla n)^2 \xi^2 d^d x \leqq \delta \int_{K_\varrho} (\nabla \xi)^2 d^d x \tag{2.17}$$

for all $\xi \in \mathring{H}_1^2(K_\varrho)$. We replace ξ by $\xi \partial_\beta n$ so that

$$\int_{K_\varrho} |\nabla n|^4 \xi^2 d^d x \leqq 2\delta \int_{K_\varrho} [(\nabla \xi)^2 (\nabla n)^2 + \sum_{\alpha,\beta} (\partial_\alpha \partial_\beta n)^2 \xi^2] d^d x$$

and choose

$$\xi = b_N^{s/2} \eta$$

with $\eta \in \mathring{H}_1^2(K_\varrho)$:

$$\int_{K_\varrho} |\nabla n|^4 b_N^s \eta^2 d^d x \leqq 2\delta \left[\int_{K_\varrho} \sum_{\alpha,\beta} (\partial_\alpha \partial_\beta n)^2 \eta^2 b_N^s d^d x \right.$$
$$\left. + \frac{s^2}{4} \int_{K_\varrho} (\nabla n)^2 b_N^{s-2} |\nabla b_N|^2 \eta^2 d^d x + \int_{K_\varrho} (\nabla n)^2 b_N^s |\nabla n|^2 d^d x \right].$$

In the second term, $(\nabla n)^2$ can again be replaced by b_N because of the support properties of ∇b_N. Choose $\zeta = \eta$ in (2.16) and insert the last inequality into it, transferring the first two terms to the left:

$$\int_{K_\varrho} \sum_{\alpha,\beta} (\partial_\alpha \partial_\beta n)^2 b^s \eta^2 d^d x \leqq (1 - \varepsilon - 2\delta)^{-1} (\varepsilon^{-1} + 2\delta) \int_{K_\varrho} (\nabla \eta)^2 (\nabla n)^2 b_N^s d^d x.$$

We have dropped the second term on the left containing $|\nabla b_N|^2$ since it is positive $(\varepsilon < (2s)^{-1})$.

3. We use the same procedure as in 2 to estimate the integral of $(\nabla n)^2 b_N^s$. Insert $\xi = b_N^{s/2} \eta$ in (2.17):

$$\int_{K_\varrho} (\nabla n)^2 b_N^s \eta^2 d^d x \leqq 2\varepsilon \left[\int_{K_\varrho} \{ s^2 b_N^{s-2} |\nabla b_N|^2 \eta^2 + b_N^s (\nabla \eta)^2 \} d^d x \right|$$
$$\leqq 2\varepsilon \left[\int_{K_\varrho} \left\{ 4 s^2 \eta^2 b_N^{s-1} \sum_{\alpha,\beta} (\partial_\alpha \partial_\beta n)^2 + b_N^s (\nabla \eta)^2 \right\} d^d x \right]$$

where we used, on the support of ∇b_N, Eq. (2.15):

$$|\nabla b_N|^2 = 4 \sum_{\alpha,\beta} |(\partial_\alpha \partial_\beta n, \partial_\alpha n)|^2 \leqq 4 \sum_{\alpha,\beta} (\partial_\alpha \partial_\beta n)^2 (\partial_\alpha n)^2$$
$$\leqq 4 \sum_{\alpha,\beta} (\partial_\alpha \partial_\beta n)^2 \cdot \sum_\alpha (\partial_\alpha n)^2 = 4 b_N \sum_{\alpha,\beta} (\partial_\alpha \partial_\beta n)^2$$

Now replace s by $s+1$ and use the inequality proven in 2:

$$\int_{K_\varrho} (\nabla n)^2 b_N^{s+1} \eta^2 d^d x \leqq 2\varepsilon \int_{K_\varrho} b_N^{s+1} (\nabla\eta)^2 d^d x + K_{s,\varepsilon} \int_{K_\varrho} (\nabla\eta)^2 (\nabla n)^2 b_N^s d^d x$$
$$\leqq \hat{K}_{s,\varepsilon} \int_{K_\varrho} b_N^s (\nabla n)^2 (\nabla\eta)^2 d^d x . \tag{2.18}$$

4. From the last inequality we conclude by induction that $|\nabla n|^k$ exists for all k. This is true for $k \leqq 4$, by the last theorem. If it is true for k one can take $s=k-1$ in (2.18) and let $N \to \infty$ in the right hand side to bound the integral on the left independently of N. Fatou's lemma then shows that $\int_{K_\varrho(y)} |\nabla n|^{k+1} d^d x$ exists. Since $y \in \Omega$ was arbitrary, the statement follows. □

We remark that Theorem 2.4 can be proven rather quickly in $d=2$ dimensions by appealing to the Sobolev inequality (see Theorem 3.5.5 in [5])

$$\|n\|_{L^r(\Omega')} \leqq C(d,p,\Omega') \left(\|n\|_{L^p(\Omega')} + \sum_{\alpha=1}^{d} \|\partial_\alpha n\|_{L^p(\Omega')} \right) \tag{2.19}$$

which is valid for $1 \leqq p < d$, $\frac{1}{r} = \frac{1}{p} - \frac{1}{d}$. Apply (2.19) for $d=2$ and $p<2$ to $\partial_\beta n$ and then use Lemma 3.2 (ii) below (which will be proven independently of Theorem 2.4) together with Theorem 2.3 (ii) to conclude that $\partial_\beta n \in L^r(\Omega')$ for $\frac{1}{r} = \frac{1}{p} - \frac{1}{2}$ and all p with $1 \leqq p < 2$, i.e. $\partial_\beta n \in L^r(\Omega')$ for all r, $1 \leqq r < \infty$.

3. Differentiability of Weak Solutions

In this section we show that all distributional derivatives of a continuous solution n of (2.1) are infinitely often differentiable. It is enough to prove, for all k, that the derivative of order k is locally L^2; it is then, in particular, locally L^1 so that, by integrating locally over $\mathbb{R}^d$, the derivative of order $k-d$ is continuous (in fact even absolutely continuous).

We will even prove that all derivatives are in $L(\Omega)$,

$$L(\Omega) := \bigcap_{1<p<\infty} L^p_{\text{loc}}(\Omega) \tag{3.1}$$

for any region $\Omega \in \mathbb{R}^d$. Since (2.1) relates L^p-properties of ∇n and n to those of Δn, we will have to estimate second derivatives in terms of Δn. This can be done by using the known L^p-properties of Riesz transforms R_α, $\alpha = 1 \dots d$. For $f \in L^2(\mathbb{R}^d)$, their Fourier transforms are defined to be

$$(\widetilde{R_\alpha f})(p) := p_\alpha |p|^{-1} \tilde{f}(p) \quad \alpha = 1, \dots, d . \tag{3.2}$$

One can then show that R_α is, in x-space, defined on all L^p spaces for $1<p<\infty$, i.e. for $L(\mathbb{R}^d)$, and is a map into:

3.1. Lemma. *For all* $f \in L(\mathbb{R}^d)$,

$$\|R_\alpha f\|_p \leqq A_p \|f\|_p . \tag{3.3}$$

For the proof, see p. 59 of [9]. The proof is not difficult, but lengthy, and uses interpolation: By Fourier transformation, R_α maps L^2 into L^2 and, in fact, L^2_w into L^2_w (where f is in the weak L^p space $L^p_w(\mu)$ for a measure μ if

$$\mu\{x/|f(x)|>t\}\leqq Ct^{-p}$$

for all $t>0$). One then shows that R_α maps L^1_w into L^1_w and uses the Marcinkiewicz interpolation theorem to conclude that R_α is bounded as a map from L^p to L^p for $1<p<2$. By duality, the same is true for $2<p<\infty$. An immediate consequence is

3.2. Lemma. (i) *For all* $1\leqq\alpha$, $\beta\leqq d$ *and* $f\in\mathcal{D}(\mathbb{R}^d)$,

$$\|\partial_\alpha\partial_\beta f\|_p\leqq A_p\|\Delta f\|_p. \tag{3.4}$$

(ii) *For all* $1\leqq\alpha$, $\beta\leqq d$, $f\in L^p(\Omega)$ *with* ∇f, $\Delta f\in L^p(\Omega)$, *and any strictly interior precompact region* $\Omega'\subset\Omega$,

$$\|\partial_\alpha\partial_\beta f\|_{L^p(\Omega')}\leqq A_p\|\Delta f\|_{L^p(\Omega)}+B_p\sum_{\gamma=1}^{d}\|\partial_\gamma f\|_{L^p(\Omega)}+C_p\|f\|_{L^p(\Omega)}. \tag{3.5}$$

Proof.

(i) As one can immediately show by Fourier transformation,

$$\partial_\alpha\partial_\beta f=-R_\alpha R_\beta\Delta f$$

so that (i) follows by the previous lemma.

(ii) By regularization and (i), (3.4) extends to all f such that $\Delta f\in L^p(\mathbb{R}^d)$. Now replace f in (3.4) by $f\cdot\chi$ where $\chi\in\mathcal{D}(\Omega)$ fulfils $0\leqq\chi\leqq1$ and is identically 1 on Ω'. □

This is enough to prove

3.3. Theorem. *Assume that n is a continuous solution of* (2.1) *with locally square integrable gradient, and that n obeys the constraint* (1.2). *Then all distributional derivatives of n are locally square integrable (and hence infinitely often differentiable).*

Proof. Choose a point $y\in\mathbb{R}^d$ and consider a ball $K_{2R}(y)$ of radius $2R$ around y. We will show that all derivatives $\partial_{\alpha_1}\dots\partial_{\alpha_k}n$ are in $L(K_{R(k)}(y))\supset L(K_R(y))$, for $R(k)=(1+k^{-1})R$, by induction on k. By Theorem 2.4 this is true for $k=1$. Suppose it is true for k. Consider an index set $I:=\{\alpha_1,\dots,\alpha_{k-1}\}$ of $k-1$ indices (empty for $k=1$) and write

$$\partial_I n:=\partial_{\alpha_1}\cdot\ldots\cdot\partial_{\alpha_{k-1}}n$$

for short. By the product rule

$$-\partial_I\Delta n=\partial_I((\nabla n)^2 n)=\sum{}'\left(c_{LMN}\sum_{\alpha,l}\partial_L\partial_\alpha n_l\partial_M\partial_\alpha n_l\partial_N n\right) \tag{3.6}$$

where $\sum'$ denotes the sum over all partitions of I into three sets L, M, N, and c_{LMN} are combinatorial factors arising because we don't require the partitions to be ordered. The right hand side of (3.6) contains at most k derivatives of n and is therefore in $L(K_{R(k)}(y))$ by the induction assumption and Hölder's inequality. Thus,

$\Delta(\partial_I n)\in L(K_{R(k)}(y))$, and, by Lemma 3.2 (ii), $\partial_\alpha\partial_\beta\partial_I n\in L(\Omega')$ for Ω' strictly interior to $K_{R(k)}(y)$. Choosing $\Omega'=K_{R(k+1)}(y)$ shows that all derivatives of order $k+1$ are in

$$L(K_{R(k+1)}(y))\supset L(K^R(y)). \quad \square$$

4. Analyticity of Solutions

To show the analyticity of solutions of (1.3), one can proceed in at least two ways:

The first method consists in extending (1.3) (or associated equations) into the complex domain; this was done in [2, 3, 5].

The second method consists simply in obtaining bounds for successive derivatives; this has been exploited in [4, 10].

We will follow the second method and just quote a result of [10] as we have not found a significantly shorter proof of analyticity. This result seems to be the farthest reaching and even gives uniqueness results in case successive derivatives grow faster than allowed by analyticity. We define growth classes as follows:

Let M_n be a sequence of positive numbers. Then a function $F: C^\infty(D)\to\mathbb{C}$ where $D\subset\mathbb{R}^\nu$ is open belongs to the class $C\{M_n;D\}$ if to any closed subset $D_0\subset D$ there exist constants H_0, H with

$$|\partial^j F(x)|\leqq H_0 H^j M_j, \qquad x\in D_0 \tag{4.1}$$

where we used multiindex-notation ($\partial^j F=\partial_1^{j_1}\dots\partial_\nu^{j_\nu}F; j=\sum j_i$). Note that $C\{n!;D\}$ is the class of functions analytic in D.

In [10], general elliptic systems of the form

$$\Phi_l(x;u,\nabla u,\nabla^2 u,\dots,\nabla^{2m}u)=0 \quad x\in\Omega\subset\mathbb{R}^d; \quad u\in\mathbb{R}^N; \quad l=1,\dots,N \tag{4.2}$$

are considered, (where e.g. $\nabla^2 u$ stands for the tensor with components $\partial_{\alpha_i}\partial_{\beta_k}u$), and the following theorem is proved:

4.1. Theorem. *Let $u(x)$ be a real solution of the elliptic system* (4.2), *let $\Omega\subset\mathbb{R}^d$ be open and let E be some open set containing*

$$E_1:=\{u(x),\nabla u(x),\dots,\nabla^{2m}u(x)/x\in\Omega\}.$$

Assume that

$$\text{(i)}\ \Phi_l\in C\{M_n;\Omega\times E\} \tag{4.3}$$

and that the M_n satisfy the monotonicity conditions

$$\text{(ii)}\ \binom{n}{i}M_i M_{n-i}\leqq A M_n; \qquad 0\leqq i\leqq n, \qquad n\in\mathbb{N} \tag{4.4}$$

for some $A>0$.

If $u\in C^{2m+\alpha}(\Omega)$, $0<\alpha<1$, then $u\in C\{M_{n-2m+1};\Omega\}$ (where $M_{-i}:=1$ for $i\in\mathbb{N}$).

Proof. Theorem 1 in [10]. $\square$

In the case of the system (1.3), $m=1$ and

$$\Phi_l(n,\nabla n,\nabla^2 n)=\Delta n_l+(\nabla n)^2 n_l$$

so that Φ_l is analytic and hence of class $C\{n!;\Omega\}$. The constants $M_n = n!$ satisfy the monotonicity conditions (4.4) with $A=1$. We even know that $n \in C^\infty(\Omega)$. By the theorem the solution is in the class $C\{(n-1)!;\Omega\}$ and hence (real) analytic there. Furthermore, n obeys the estimate (4.1) so that the nearest complex singularities must be at least at a distance H^{-1} from Ω [where H depends on n and on the closed subset $\Omega_0 \subset \Omega$ for which (4.1) holds]. This proves

4.2. Theorem. *Let n be a weak continuous solution of*

$$\Delta n + (\nabla n)^2 n = 0$$

obeying

$$(n, n) = 1,$$

and assume $\nabla n \in L^2_{\text{loc}}(\Omega)$. *Then n is real analytic, and n can be continued analytically into* $|\text{Im}\, z_l| < C$ *where C depends only on n and on the distance of* $\text{Re}\, z_l$ *to the boundary of* Ω.

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