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Small Perturbations of C^*-Dynamical Systems

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Abstract. It is shown that if δ is the generator of a strongly continuous one-parameter group of *-automorphisms of a C^*-algebra A and δ' is an unbounded *-derivation of A with the same domain as δ, then $\delta+\alpha\delta'$ is also a generator for all sufficiently small real numbers α.

The perturbation theory of strongly continuous one-parameter contraction semi-groups $\{e^{tT}: t \geqq 0\}$ on Banach spaces shows that several features of these systems are stable under relatively bounded perturbations [6, 8]. For example if T' is a dissipative operator with the same domain $\mathscr{D}$ as T, then $T+T'$ is the generator of some contraction semi-group, provided that

$$\|T'x\| \leqq \alpha\|x\| + \beta\|Tx\|$$

for all x in $\mathscr{D}$, for some constants α and $\beta<1$.

In the C^*-algebraic model of a quantum dynamical system, the time evolution is represented by a strongly continuous one-parameter group of *-automorphisms $\{e^{t\delta}: t \in \mathbb{R}\}$ of a C^*-algebra A, where the generator δ is a closed unbounded *-derivation. Longo [7] has shown that in this case, any *-derivation δ' with the same domain is automatically relatively bounded with respect to δ. In this note it will be shown that δ' is also necessarily dissipative, and therefore $\delta+\alpha\delta'$ is a generator for all sufficiently small α (cf. [4, Sect. 5]).

Longo's result also applies if δ is any closed *-derivation (not necessarily a generator), and he asked whether δ' is then necessarily closable. For commutative C^*-algebras, an affirmative answer to this problem was given in [3, Theorem 5.3]. The proof there involved showing that any (maximal) closed ideal containing a and $\delta(a)$ also contains $\delta'(a)$. Since the maximal ideals in a commutative C^*-algebra are of codimension 1 and have zero intersection, this enabled a very specific description of δ' to be given in terms of δ. For non-commutative C^*-algebras, it will be shown here that $\delta'(a)$ again belongs to the closed ideal generated by a and $\delta(a)$, and a partial answer to Longo's question will be given. All the results of this

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note are obtained as corollaries of Theorem 3, the proof of which makes use of Longo's theorem, functional calculus in the domain of δ and the Hahn-Banach separation theorem.

Let δ be an (unbounded) *-derivation of a C^*-algebra A, defined on a dense *-subalgebra $\mathscr{D}$ of A. We recall the following definitions from [1, 10, 11]. An operator a in the self-adjoint part $\mathscr{D}^s$ of $\mathscr{D}$ is *δ-well-behaved* (resp. *strongly δ-well-behaved*) if $\phi(\delta(a))=0$ for some (resp. for all) states ϕ of A with $|\phi(a)|=\|a\|$. The derivation δ is *well-behaved* if every operator in $\mathscr{D}^s$ is δ-well-behaved; δ is *quasi well-behaved* if there is a dense open subset of $\mathscr{D}^s$ (in the relative topology) consisting of δ-well-behaved operators. A closed ideal J in A is *δ-invariant* if δ maps $\mathscr{D}\cap J$ into J. Then δ induces a *-derivation δ_J of A/J with dense domain $\pi_J(\mathscr{D})$, given by

$$\delta_J(\pi_J(a))=\pi_J(\delta(a)) \qquad (a\in\mathscr{D})$$

where π_J is the quotient mapping of A onto A/J. The derivation δ is *pseudo well-behaved* if for each non-zero a in A there is a δ-invariant ideal J, not containing a, such that δ_J is quasi well-behaved. Note that the generator of any one-parameter *-automorphism group is well-behaved.

The following lemma is known, but since different authors have used different, but equivalent, definitions and terminology, we state it here for convenience.

Lemma 1. *Let δ be a *-derivation of A with dense domain $\mathscr{D}$. The following are equivalent:*

(i) *δ is well-behaved.*

(ii) *For any a in $\mathscr{D}$, there is a non-zero functional ϕ in A^* such that $\phi(a)=\|a\|\,\|\phi\|$ and* $\operatorname{Re}\phi(\delta(a))=0$.

(iii) $\operatorname{Re}\phi(\delta(a))=0$ *for any a in $\mathscr{D}$ and ϕ in A^* such that $\phi(a)=\|a\|\,\|\phi\|$.*

(iv) *Every operator in $\mathscr{D}^s$ is strongly δ-well-behaved.*

(v) $\|a+\alpha\delta(a)\|\geqq\|a\|$ *for all α in $\mathbb{R}$ and a in $\mathscr{D}$.*

Proof. The implications (i)$\Rightarrow$(ii) and (ii)$\Rightarrow$(iii) were proved in [11, Proposition 2.19] and [2, Corollary 3] respectively, (iii)$\Rightarrow$(iv) and (iv)$\Rightarrow$(i) are trivial, and (ii)$\Leftrightarrow$(v) follows from [5, Theorem V.9.5].

Lemma 2. *Let J be a closed ideal in A and $\mathscr{D}$ be the dense domain of a closed *-derivation of A. Then $J\cap\mathscr{D}$ is dense in J.*

Proof. Let a be a self-adjoint operator in J. For any $\varepsilon>0$, there exists b in $\mathscr{D}^s$ such that $\|a-b\|<\varepsilon/3$. Let f be a C^2-function on $\mathbb{R}$ such that

$$\begin{aligned} |f(t)-t| &< \frac{2\varepsilon}{3} && (t\in\mathbb{R}) \\ f(t) &= 0 && \left(|t|\leqq\frac{\varepsilon}{3}\right). \end{aligned}$$

Then $f(b)$ belongs to $\mathscr{D}$ [11, Theorem 3.6]. Also $\|\pi_J(b)\| < \varepsilon/3$, so $\pi_J(f(b)) = f(\pi_J(b)) = 0$. Thus $f(b)$ belongs to J. Furthermore

$$\|a - f(b)\| \leqq \|a - b\| + \|b - f(b)\| < \varepsilon.$$

Since J is self-adjoint [9, Corollary 1.17.3], the lemma follows.

In Theorem 3 and the subsequent corollaries, δ will be a closed *-derivation of A with dense domain $\mathscr{D}$, and δ' will be another *-derivation of A with domain $\mathscr{D}$.

Theorem 3. *Let J be a closed ideal in A, a be an operator in $\mathscr{D}^s$ and α be the smallest point of the spectrum of $\pi_J(a)$. Suppose that $\phi(\delta(a)) = 0$ whenever ϕ is a state of A annihilating J with $\phi(a) = \alpha$. Then $\phi(\delta'(a)) = 0$ for all such states ϕ.*

Proof. By adjoining a unit 1 to A, extending δ and δ' by putting $\delta(1) = \delta'(1) = 0$ if necessary, and replacing a by $a - \alpha 1$, we may assume that A has a unit lying in $\mathscr{D}$, and that $\alpha = 0$, so $\pi_J(a) \geqq 0$. Let p be the spectral projection of a in the W^*-algebra A^{**} corresponding to the interval $(-\infty, 0]$ and let q be the central projection of A^{**} *such that $A^{**}(1-q)$ is the weak* closure of J in A^{**} [9, Proposition 1.10.5].* Then $papq = 0$, since $pap \leqq 0$ and $aq \geqq 0$. For any state ψ of A with $\psi(pq) > 0$, let $\tilde{\psi}$ be the state defined by

$$\tilde{\psi}(b) = \psi(pq)^{-1}\psi(pbpq) \qquad (b \in A).$$

Then $\tilde{\psi}$ annihilates J and $\tilde{\psi}(a) = 0$, so $\tilde{\psi}(\delta(a)) = 0$. Thus $p\delta(a)pq = 0$.

Let g_1, g_2 and h be real-valued C^2-functions on $\mathbb{R}$ such that

$$\begin{aligned} g_1(0) = g_2(0) &= 0 \\ g_1(t)g_2(t) &= t \qquad (|t| \geqq 1) \\ h(t) &= 0 \qquad (t \leqq 1) \\ h(t) &= 1 \qquad (t \geqq 2) \end{aligned}$$

and put $f(t) = t - g_1(t)g_2(t)$. Then f is a C^2-function of compact support, and $f(nt)h(rt) = 0$ for t in $\mathbb{R}$ and integers n and r with $n \geqq r \geqq 0$. Let $a_n = n^{-1}f(na)$, so that $\|a_n\| \to 0$ as $n \to \infty$. It is shown in [11, Sect. 3] that a_n, $g_1(na)$, $g_2(na)$ and $h(ra)$ belong to $\mathscr{D}$, and if $\hat{f}$ is the Fourier transform of f and

$$\gamma = (2\pi)^{-\frac{1}{2}}\|\delta(a)\| \int_{-\infty}^{\infty} |s\hat{f}(s)|\,ds$$

then γ is finite, and

$$\delta(a_n) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty}\int_0^1 s\hat{f}(s)e^{insta}\delta(a)e^{ins(1-t)a}\,dt\,ds$$

so that $\|\delta(a_n)\| \leqq \gamma$. Since $qpe^{iua} = e^{iua}pq = pq$ for all real numbers u, and $p\delta(a)pq = 0$, it follows that $p\delta(a_n)pq = 0$.

Let x be a weak* limit point of the bounded sequence $\delta(a_n)$ in A^{**}, so that $pxpq = 0$. Since $\delta(a_n h(ra)) = 0$ $(n \geqq r)$ and $\|a_n\| \to 0$, $xh(ra) = 0$. But, as $r \to \infty$, $h(ra)$ converges ultrastrongly to $1 - p$, so $x = xp = pxp$. Thus $xq = 0$.

By Lemma 2 and the Kaplansky Density Theorem, the unit ball of $J \cap \mathscr{D}$ contains a net $\{y_\lambda : \lambda \in \Lambda\}$ converging ultrastrongly to $1-q$. For any triple $\mu = (m, n, \lambda)$ in $\mathbb{N} \times \mathbb{N} \times \Lambda$, let

$$b_\mu = a_n(1-y_\lambda)$$
$$b'_\mu = \delta(a_n)(1-y_\lambda)$$
$$b''_\mu = a_n \delta(y_\lambda).$$

Take $\varepsilon > 0$, and let

$$\Lambda' = \{\mu = (m, n, \lambda) \in \mathbb{N} \times \mathbb{N} \times \Lambda : \|b''_\mu\| \leqq 2^{-m}, \|a_n \delta'(y_\lambda)\| \leqq \varepsilon\}.$$

In the product ordering, Λ' is directed upwards, the nets b_μ and $b''_\mu (\mu \in \Lambda')$ are norm-convergent to 0, while $\{b'_\mu : \mu \in \Lambda'\}$ has $xq = 0$ as a weak* limit point in A^{**}. Since $\delta(b_\mu) = b'_\mu - b''_\mu$, $(0, 0)$ belongs to the weak closure of $\{(b_\mu, \delta(b_\mu)) : \mu \in \Lambda'\}$ in the Banach space $A \oplus A$. By the Hahn-Banach separation theorem [5. Corollary V.2.14], there is a sequence a'_r in the convex hull of $\{b_\mu : \mu \in \Lambda'\}$ such that $\|a'_r\| \to 0$ and $\|\delta(a'_r)\| \to 0$. Since δ' is relatively bounded with respect to δ [7, Corollary 2], $\|\delta'(a'_r)\| \to 0$.

Let ϕ be a state annihilating J with $\phi(a) = 0$. Since ϕ induces a state of A/J, the spectral theory of $\pi_J(a)$ shows that $\phi(g_1(na)^2) = \phi(g_2(na)^2) = 0$. But $a_n = a - n^{-1} g_1(na) g_2(na)$, so

$$\delta'(a - b_\mu) = n^{-1} \delta'(g_1(na)) g_2(na) + n^{-1} g_1(na) \delta'(g_2(na)) + \delta'(a_n) y_\lambda + a_n \delta'(y_\lambda).$$

Hence

$$|\phi(\delta'(a - b_\mu))| = |\phi(a_n \delta'(y_\lambda))| \leqq \varepsilon$$

for all μ in Λ', so $|\phi(\delta'(a - a'_r))| \leqq \varepsilon$. Letting first r tend to ∞ and then ε tend to 0, it follows that $\phi(\delta'(a)) = 0$.

Corollary 4. *Let a be an operator in $\mathscr{D}$, and J be a closed ideal in A containing a and $\delta(a)$. Then $\delta'(a)$ belongs to J.*

Proof. Since J is self-adjoint, it suffices to assume that $a = a^*$. The hypotheses of Theorem 3 are satisfied (with $\alpha = 0$), so $\phi(\delta'(a)) = 0$ for any state ϕ annihilating J. Hence $\delta'(a)$ belongs to J.

Corollary 5. *Let J be a δ-invariant ideal in A. Then J is δ'invariant and any strongly δ_J-well-behaved operator in $\pi_J(\mathscr{D}^s)$ is strongly δ'_J-well-behaved.*

Proof. The first assertion is immediate from Corollary 4. The second follows easily from Theorem 3 using the correspondence between states of A annihilating J and states of A/J.

Corollary 6. *If δ is (quasi, resp. pseudo) well-behaved, then δ' is (quasi, resp. pseudo) well-behaved and closable.*

Proof. Since any open set of δ_J-well-behaved operators in $\pi_J(\mathscr{D}^s)$ consists of strongly δ_J-well-behaved operators [2, Proposition 7] and any pseudo well-behaved derivation is closable [1, Proposition 6], the assertion follows from Corollary 5.

Corollary 7. *If δ is the generator of a strongly continuous one-parameter group of *-automorphisms of A, then for all real numbers α with $|\alpha|$ sufficiently small, $\delta+\alpha\delta'$ is also a generator.*

Proof. This follows immediately from Corollary 6, Lemma 1, [7, Corollary 2] and [8, p. 244].

In view of Corollary 6 and the closability problem raised by Longo [7], it would be of interest to find examples of closed *-derivations which are not pseudo well-behaved (cf. [1, p. 266]).

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