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The Classical Field Limit of Scattering Theory for Non-Relativistic Many-Boson Systems. I

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Abstract. We study the classical field limit of non-relativistic many-boson theories in space dimension $n \geqq 3$. When $\hbar \to 0$, the correlation functions, which are the averages of products of bounded functions of field operators at different times taken in suitable states, converge to the corresponding functions of the appropriate solutions of the classical field equation, and the quantum fluctuations are described by the equation obtained by linearizing the field equation around the classical solution. These properties were proved by Hepp [6] for suitably regular potentials and in finite time intervals. Using a general theory of existence of global solutions and a general scattering theory for the classical equation, we extend these results in two directions: (1) we consider more singular potentials, (2) more important, we prove that for dispersive classical solutions, the $\hbar \to 0$ limit is uniform in time in an appropriate representation of the field operators. As a consequence we obtain the convergence of suitable matrix elements of the wave operators and, if asymptotic completeness holds, of the S-matrix.

1. Introduction and Statement of the Problem

Since the early days of quantum mechanics it has been a natural question to compare the classical and quantum mechanical descriptions of physical systems. One of the oldest and by now best known relations between the two theories goes back to Ehrenfest [1]. Only recently however was this relation put on a firm mathematical basis by Hepp [6] who proved that in the limit $\hbar \to 0$ the matrix elements of bounded functions of quantum observables between suitable $\hbar$-dependent coherent states tend to classical values evolving according to the appropriate classical equation. Furthermore he proved that the quantum mechanical fluctuations evolve according to the equation obtained by linearizing the quantum mechanical evolution equation around the classical solution. However his analysis is limited to finite time intervals and therefore does not

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provide any information on the connection between the classical and quantum mechanical scattering theories. For systems of finitely many particles such a connection was provided recently by Yajima [11].

In this and subsequent papers we shall study the connection between classical and quantum mechanical scattering theories for non-relativistic many-boson systems. The main result is that, under suitable circumstances, the $\hbar \to 0$ limit of quantities similar to those considered by Hepp is uniform in time. In particular this extends his results to the S-matrix.

In order to state the problem in more precise terms, we first analyse the general structure of the quantum theory in connection with its classical limit. Our results can and will be described more explicitly only at the end of this analysis, which takes up most of the present section. It is inspired by that of Hepp and the earlier one of Gross [6, 5]. It is phrased in a way that is suitable for many-boson systems although it applies to more general situations. In this discussion we restrict our attention to the algebraic aspects of the problem and do not make any attempt at mathematical rigor. We treat $\hbar$ as a free parameter. Since nature gives us only one physical value for it, this means of course that we are imbedding the actual quantum theory in a one-parameter family of quantum theories indexed by $\hbar$.

We consider a system described by a family of quantum variables $a = \{a_i\}_{i \in I}$ satisfying the CCR

$$[a_i, a_j] = 0, \qquad [a_i, a_j^*] = \delta_{ij}. \tag{1.1}$$

The variables $a_\hbar = \{a_{\hbar i}\}_{i \in I}$ expected to have classical limits are related to the a's by

$$a_{\hbar i} = \hbar^\mu a_i, \qquad i \in I, \tag{1.2}$$

for some $\mu > 0$. There are reasons why μ should take the value 1/2, but they are irrelevant for the discussion that follows. We shall come back to this point later. In all this paper, we shall make the following convention: an operator A which is a function of both the a's and the a^*'s (for instance a polynomial) will be denoted simply by $A(a)$. This should be remembered especially when taking commutators.

In the Heisenberg picture, the time evolution of the $a_\hbar$'s is given by the equation

$$i\hbar \dot{a}_\hbar(t) = [a_\hbar(t), H_\hbar(a_\hbar(t))], \tag{1.3}$$

where the Hamiltonian $H_\hbar$ is assumed to have the form

$$H_\hbar(a_\hbar) = \hbar^\delta H(a_\hbar) \tag{1.4}$$

for some real δ; the function $H(\cdot)$ is assumed to have no explicit $\hbar$ dependence and to be suitably regular, for instance polynomial. It may however have an explicit time dependence, which is omitted for brevity. The Eq. (1.3) is solved by

$$a_\hbar(t) = U(t, t_0; a(t_0))^* a_\hbar(t_0) U(t, t_0; a(t_0)), \tag{1.5}$$

where $U(t, t_0; a(t_0))$ is the unitary group satisfying $U(t_0, t_0; a(t_0)) = \mathbb{1}$ and the equation

$$i\hbar \frac{d}{dt} U(t, t_0; a(t_0)) = H_\hbar(a_\hbar(t_0)) U(t, t_0; a(t_0)). \tag{1.6}$$

We want to relate the set of operators $a_\hbar(t)$ with a family of $\hbar$-independent, time-dependent c-number variables

$$\varphi(t)=\{\varphi_i(t)\}_{i\in I} \tag{1.7}$$

to be thought of as their classical limits. It is therefore natural to expand the Hamiltonian in power series of $a_\hbar-\varphi$ and $a_\hbar^*-\bar{\varphi}$ in a neighborhood of φ, $\bar{\varphi}$ (where $\bar{\varphi}$ is the complex conjugate of φ):

$$\begin{aligned} H(a_\hbar)=H(\varphi)&+H_1(a_\hbar-\varphi)+H_2(a_\hbar-\varphi)\\ &+H_{\geqq 3}(a_\hbar-\varphi), \end{aligned} \tag{1.8}$$

where the functions H_1, H_2 and $H_{\geqq 3}$ have total degree 1, 2 and $\geqq 3$ respectively in the variables $a_\hbar-\varphi$ and $a_\hbar^*-\bar{\varphi}$. They have an additional dependence on φ and possibly on time (which we have omitted for brevity), but no explicit $\hbar$ dependence. We now define

$$H'_k(a)=[a,H_k(a)], \qquad k=1,2,\geqq 3. \tag{1.9}$$

It is clear from (1.1) that the functions H'_k are $\hbar$-independent, that H'_1 is a c-number, and that $H'_2(a)$ is linear in the variables a, a^*. The Eq. (1.3) can now be rewritten as

$$i\dot{\varphi}+i(\dot{a}_\hbar-\dot{\varphi})=\hbar^{\delta+2\mu-1}(H'_1+H'_2(a_\hbar-\varphi)+H'_{\geqq 3}(a_\hbar-\varphi)), \tag{1.10}$$

where the time dependence has been omitted for brevity.

In order to obtain a non trivial classical limit, we assume $\delta=1-2\mu$. We then choose for φ a solution of the equation

$$i\dot{\varphi}=H'_1, \tag{1.11}$$

which is the classical evolution equation associated with the hamiltonian H. The Eq. (1.10) then becomes

$$i(\dot{a}-\dot{\varphi}_\hbar)=H'_2(a-\varphi_\hbar)+\hbar^{-\mu}H'_{\geqq 3}(\hbar^\mu(a-\varphi_\hbar)), \tag{1.12}$$

where

$$\varphi_\hbar=\hbar^{-\mu}\varphi. \tag{1.13}$$

In order to study the classical limit of the evolution Eq. (1.3) with initial condition $a_\hbar(t_0)$ at time $t=t_0$, we define new variables $b(t)$ by

$$b(t)=C(a(t_0),\varphi_\hbar(t_0))^*(a(t)-\varphi_\hbar(t))\,C(a(t_0),\varphi_\hbar(t_0)), \tag{1.14}$$

where

$$C(a,\alpha)\equiv\exp\left[\sum_i(a_i^*\alpha_i-a_i\bar{\alpha}_i)\right] \tag{1.15}$$

for any $\alpha=\{\alpha_i\}_{i\in I}$. The operators $C(a,\alpha)$ are the Weyl operators. They are unitary and satisfy

$$C(a,\,\alpha)^*\,a\,C(a,\alpha)=a+\alpha. \tag{1.16}$$

When applied to the vacuum, they yield coherent states. Using (1.16), one can rewrite the definition (1.14) as

$$b(t)=C(a(t_0),\varphi_\hbar(t_0))^*C(a(t),\varphi_\hbar(t))a(t)$$
$$\cdot C(a(t),\varphi_\hbar(t))^*C(a(t_0),\varphi_\hbar(t_0)). \tag{1.17}$$

The initial value problem is now reduced to finding a family of operators $b(t)$ satisfying the CCR, the initial condition

$$b(t_0)=a(t_0) \tag{1.18}$$

and the equation

$$i\dot{b}=H'_2(b)+\hbar^{-\mu}H'_{\geqq 3}(\hbar^\mu b). \tag{1.19}$$

The second term in the R.H.S. of (1.19) is $O(\hbar^\mu)$ because $H'_{\geqq 3}(a)$ has degree at least two in the a and a^*'s. We now face a well defined perturbation problem, namely to prove that when $\hbar\to 0$, the solution of (1.19) with initial condition (1.18) tends in a suitable sense to the solution of the equation

$$i\dot{b}=H'_2(b) \tag{1.20}$$

with the same initial condition (1.18). The Eq. (1.20) describes the evolution of the quantum fluctuations in the limit $\hbar\to 0$. It is the equation obtained by linearizing the Eq. (1.3) around the classical solution $\varphi(t)$. In sufficiently regular situations the solution $a_2(t)$ of (1.20) with initial condition (1.18) is given by

$$a_2(t)=U_2(t,t_0;a(t_0))^*a(t_0)U_2(t,t_0;a(t_0)), \tag{1.21}$$

where $U_2(t,t_0;a(t_0))$ is the unitary group satisfying $U_2(t_0,t_0;a(t_0))=\mathbb{1}$ and the equation

$$i\frac{d}{dt}U_2(t,t_0;a(t_0))=H_2(a(t_0))U_2(t,t_0;a(t_0)). \tag{1.22}$$

Note that $U_2(t,t_0;a(t_0))$ is independent of $\hbar$ in so far as $a(t_0)$ is. Similarly the solution $b(t)$ of (1.19) with initial condition (1.18) is given by

$$b(t)=W(t,t_0;a(t_0))^*a(t_0)W(t,t_0;a(t_0)), \tag{1.23}$$

where $W(t,t_0;a(t_0))$ is the unitary group

$$W(t,t_0;a(t_0))=\exp[i\omega_\hbar(t,t_0)]U(t,t_0;a(t_0))$$
$$\cdot C(a(t),\varphi_\hbar(t))^*C(a(t_0),\varphi_\hbar(t_0)) \tag{1.24}$$
$$=\exp[i\omega_\hbar(t,t_0)]C(a(t_0),\varphi_\hbar(t))^*U(t,t_0;a(t_0))$$
$$\cdot C(a(t_0),\varphi_\hbar(t_0)) \tag{1.25}$$

with

$$\omega_\hbar(t,t_0)$$
$$=\hbar^{-2\mu}\int_{t_0}^{t}d\tau\{H(\tau,\varphi(\tau))-\mathrm{Re}\langle\varphi(\tau),H'_1(\tau,\varphi(\tau))\rangle\}. \tag{1.26}$$

In order to pass from (1.24) to (1.25), we have used (1.5). In (1.26) we have written the dependence of H'_1 on φ and the possible explicit dependence of H and H'_1 on time. The scalar product $\langle\cdot,\cdot\rangle$ is defined by

$$\langle\alpha,\beta\rangle=\sum_{i\in I}\bar{\alpha}_i\beta_i\,. \tag{1.27}$$

The choice of the phase (1.26) in (1.24) ensures that the operator $W(t,t_0;a(t_0))$ satisfies the equation

$$\begin{aligned} &i\frac{d}{dt}W(t,t_0;a(t_0))\\ &=(H_2(a(t_0))+\hbar^{-2\mu}H_{\geqq 3}(\hbar^{\mu}a(t_0)))\,W(t,t_0;a(t_0))\,. \end{aligned} \tag{1.28}$$

The computation showing that $W(t,t_0;a(t_0))$ satisfies (1.28) is straightforward and will be omitted here. It will be performed in a special case of interest in Sect. 3 (see the proof of Proposition 3.1).

Now the previous perturbation problem is reduced to comparing $W(t,t_0;a(t_0))$ with $U_2(t,t_0;a(t_0))$. In favorable cases,

$$\underset{\hbar\to 0}{s\text{-lim}}\,W(t,t_0;a(t_0))=U_2(t,t_0;a(t_0))\,. \tag{1.29}$$

This strong convergence implies that, for any family of bounded suitably regular functions $R_i(a)$ and for any family of times $\{t_i\}$ $i=1,2,\ldots,m$,

$$\begin{aligned} &\underset{\hbar\to 0}{s\text{-lim}}\,C(a(t_0),\varphi_\hbar(t_0))^*\prod_{i=1}^{m}R_i(a(t_i)-\varphi_\hbar(t_i))\,C(a(t_0),\varphi_\hbar(t_0))\\ &\quad=\prod_{i=1}^{m}R_i(a_2(t_i))\,, \end{aligned} \tag{1.30}$$

where $a_2(t)$ is given by (1.21) and $a(t)$ is given by (1.2) and (1.5) or equivalently through $b(t)$ by (1.14) and (1.23). The convergence (1.30) can be interpreted in terms of correlation functions in coherent states.

In more singular cases, it may happen that U_2 does not exist. One may then expect only convergence of the automorphisms of the CCR algebras defined by (1.19) and (1.20).

So far we have considered the problem of evolution in finite time intervals. Now we turn to scattering theory. This requires a little more structure. We want to compare the evolution of the system with a simpler evolution called the free evolution. Therefore we assume that we are given a free hamiltonian

$$H_{0\hbar}(a_\hbar)=\hbar^{1-2\mu}H_0(a_\hbar)\,, \tag{1.31}$$

where $H_0(\cdot)$ is quadratic and time-independent. The free evolution is then represented by the one-parameter unitary group of operators

$$U_0(t;a)=\exp\{-i\hbar^{-1}tH_{0\hbar}(a_\hbar)\}=\exp\{-itH_0(a)\}\,. \tag{1.32}$$

Note that $U_0(t;a)$ is $\hbar$-independent in so far as a is. It is now convenient to reformulate the evolution problem in the asymptotic picture defined by

$$\tilde{a}(t)=U_0(t;a(t))\,a(t)\,U_0(t;a(t))^* \tag{1.33}$$

and similarly by

$$\tilde{b}(t) = U_0(t; b(t)) b(t) U_0(t; b(t))^*, \tag{1.34}$$

with the operators $b(t)$ given by (1.14). It follows immediately from (1.33), (1.34), and (1.17) that

$$\begin{aligned}\tilde{b}(t) = {} & C(a(t_0), \varphi_\hbar(t_0))^* C(a(t), \varphi_\hbar(t)) \tilde{a}(t) \\ & \cdot C(a(t), \varphi_\hbar(t))^* C(a(t_0), \varphi_\hbar(t_0)).\end{aligned} \tag{1.35}$$

The operators in the asymptotic picture are expected to have limits as t and/or t_0 tend to $\pm\infty$ for fixed $\tilde{a}(t_0) \equiv \tilde{a}_0$ if the Heisenberg operators behave as free operators for large times. Obviously the operators $\tilde{b}(t)$ satisfy the CCR and the initial condition

$$\tilde{b}(t_0) = \tilde{a}_0. \tag{1.36}$$

The evolution Eq. (1.19) becomes

$$i\dot{\tilde{b}} = \tilde{H}'_2(\tilde{b}) - H'_0(\tilde{b}) + \hbar^{-\mu} \tilde{H}'_{\geqq 3}(\hbar^\mu \tilde{b}), \tag{1.37}$$

where

$\tilde{H}'_k$ is defined for $k = 2, \geqq 3$, by

$$\tilde{H}'_k(a) = [a, \tilde{H}_k(a)] \tag{1.37a}$$

with

$$\tilde{H}_k(a) = U_0(t, a)^* H_k(a) U_0(t, a), \tag{1.37b}$$

H'_0 is defined by (1.9) with $k=0$ and the time dependence has been omitted for brevity. The previous perturbation problem now becomes that of proving that when $\hbar \to 0$, the solution of (1.37) with initial condition (1.36) tends in the same sense as above to the solution of the equation

$$i\dot{\tilde{b}} = \tilde{H}'_2(\tilde{b}) - H'_0(\tilde{b}) \tag{1.38}$$

with the same initial condition (1.36). In addition, the limit $\hbar \to 0$ is expected to have some uniformity in t and t_0 for fixed $\tilde{a}_0$, and to commute with the limits where t and/or t_0 tend to $\pm\infty$, provided the latter exist. In sufficiently regular situations, and in particular in those considered in the present paper, where the Eqs. (1.19) and (1.20) have the solution (1.23) and (1.21) respectively, the Eq. (1.38) has the solution

$$\begin{aligned}\tilde{a}_2(t) &= U_0(t; a_2(t)) a_2(t) U_0(t; a_2(t))^* \\ &= \tilde{U}_2(t, t_0; \tilde{a}_0)^* \tilde{a}_0 \tilde{U}_2(t, t_0; \tilde{a}_0),\end{aligned} \tag{1.39}$$

where

$$\tilde{U}_2(t, t_0; \tilde{a}_0)$$

$$= U_0(t_0; a(t_0)) U_2(t, t_0; a(t_0)) U_0(t; a_2(t))^* \tag{1.40}$$

$$= U_0(t; \tilde{a}_0)^* U_2(t, t_0; \tilde{a}_0) U_0(t_0; \tilde{a}_0). \tag{1.41}$$

In order to pass from (1.40) to (1.41) we have used (1.21) and (1.33). The operators $\tilde{U}_2(t, t_0; \tilde{a}_0)$ form a unitary group satisfying $\tilde{U}_2(t_0, t_0; \tilde{a}_0) = \mathbb{1}$ and the equation

$$i\frac{d}{dt}\tilde{U}_2(t, t_0; \tilde{a}_0) = \{\tilde{H}_2(\tilde{a}_0) - H_0(\tilde{a}_0)\}\tilde{U}_2(t, t_0; \tilde{a}_0). \tag{1.42}$$

Similarly the Eq. (1.37) is solved by

$$\tilde{b}(t) = \tilde{W}(t, t_0; \tilde{a}_0)^* \tilde{a}_0 \tilde{W}(t, t_0; \tilde{a}_0), \tag{1.43}$$

where

$$\begin{aligned} &\tilde{W}(t, t_0; \tilde{a}_0) \\ &= U_0(t_0; a(t_0))\, W(t, t_0; a(t_0))\, U_0(t; b(t))^* &(1.44)\\ &= U_0(t; \tilde{a}_0)^*\, W(t, t_0; \tilde{a}_0)\, U_0(t_0; \tilde{a}_0). &(1.45)\end{aligned}$$

In order to pass from (1.44) to (1.45) we have used (1.23) and (1.33). The operators $\tilde{W}(t, t_0; \tilde{a}_0)$ from a unitary group satisfying $\tilde{W}(t_0, t_0; \tilde{a}_0) = \mathbb{1}$ and the equation

$$i\frac{d}{dt}\,\tilde{W}(t, t_0; \tilde{a}_0) = \{\tilde{H}_2(\tilde{a}_0) - H_0(\tilde{a}_0) + \hbar^{-2\mu}\tilde{H}_{\geqq 3}(\hbar^\mu \tilde{a})\}\,\tilde{W}(t, t_0; \tilde{a}_0). \tag{1.46}$$

In order to study the $\hbar \to 0$ limit uniformly in time we need more information on the asymptotic behaviour of the classical solution $\varphi(t)$. For this purpose we introduce the classical free evolution which is the evolution associated with H_0 in the same way as the total classical evolution was associated with H through Eq. (1.11). Since H_0 is quadratic the classical free evolution is represented by a group of real-linear operators $u_0(t)$ and one easily checks the identity

$$U_0(t; a)C(a, \alpha)U_0(t; a)^* = C(a, u_0(t)\alpha). \tag{1.47}$$

We introduce the asymptotic picture for the classical evolution by

$$\begin{cases} \tilde{\varphi}(t) = u_0(-t)\varphi(t) \\ \tilde{\varphi}_\hbar(t) = u_0(-t)\varphi_\hbar(t), \end{cases} \tag{1.48}$$

so that (1.47) implies

$$C(a(t), \varphi_\hbar(t)) = C(\tilde{a}(t), \tilde{\varphi}_\hbar(t)). \tag{1.49}$$

We now assume that the classical solution is asymptotically free, namely that the limits

$$\lim_{t \to \pm\infty} \tilde{\varphi}(t) = \varphi_\pm \tag{1.50}$$

exist in a suitable sense. Under this assumption and in favorable circumstances one expects the following results:

1) The limit

$$\operatorname*{s-lim}_{\hbar \to 0} \tilde{W}(t, t_0; \tilde{a}_0) = \tilde{U}_2(t, t_0; \tilde{a}_0) \tag{1.51}$$

should exist uniformly in t and t_0 for fixed $\tilde{a}_0$. This implies limiting properties of the correlation functions similar to those expressed by (1.30) namely:

$$\underset{\hbar\to 0}{s\text{-lim}}\, C(\tilde{a}_0, \tilde{\varphi}_\hbar(t_0))^* \prod_{i=1}^{m} R_i(\tilde{a}(t_1) - \tilde{\varphi}_\hbar(t_i)) C(\tilde{a}_0, \tilde{\varphi}_\hbar(t_0))$$
$$= \prod_{i=1}^{m} R_i(\tilde{a}_2(t_i)) \tag{1.52}$$

uniformly in t_0 and the t_i's for fixed $\tilde{a}_0$.

2) If in addition the quantum theory is asymptotically complete, then the operators $\tilde{W}(t, t_0; \tilde{a}_0)$ should have strong limits as t and/or t_0 tend to $\pm\infty$ at fixed $\tilde{a}_0$. Similarly $\tilde{U}_2(t, t_0; \tilde{a}_0)$ should have strong limits in the same circumstances. These limits are simply related to the corresponding S-matrices in the Heisenberg picture ($\tilde{a}_0$ being the incoming field)

$$S(\tilde{a}_0) = \underset{\substack{t\to+\infty \\ t_0\to-\infty}}{s\text{-lim}}\, U_0(t; \tilde{a}_0)^* U(t, t_0; \tilde{a}_0) U_0(t_0; \tilde{a}_0) \tag{1.53}$$

and

$$S_2(\tilde{a}_0) = \underset{\substack{t\to+\infty \\ t_0\to-\infty}}{s\text{-lim}}\, \tilde{U}_2(t, t_0; \tilde{a}_0). \tag{1.54}$$

Indeed it follows from (1.25), (1.45), and (1.49) that

$$\tilde{W}(t, t_0; \tilde{a}_0) = C(\tilde{a}_0, \tilde{\varphi}_\hbar(t))^* U_0(t; \tilde{a}_0)^* U(t, t_0; \tilde{a}_0) U_0(t_0; \tilde{a}_0)$$
$$\cdot C(\tilde{a}_0, \tilde{\varphi}_\hbar(t_0)) \exp[i\omega_\hbar(t, t_0)] \tag{1.55}$$

and therefore that

$$\underset{\substack{t\to+\infty \\ t_0\to-\infty}}{s\text{-lim}}\, \tilde{W}(t, t_0; \tilde{a}_0) = C(\tilde{a}_0, \varphi_{\hbar+})^* S(\tilde{a}_0) C(\tilde{a}_0 . \varphi_{\hbar-})$$
$$\cdot \exp[i\omega_\hbar(+\infty, -\infty)]. \tag{1.56}$$

The phase $\omega_\hbar(+\infty, -\infty)$ should be finite for asymptotically free classical solutions. Furthermore the limits $\hbar\to 0$ and $t, t_0 \to \pm\infty$ should commute. In particular

$$\underset{\hbar\to 0}{s\text{-lim}}\, C(\tilde{a}_0, \varphi_{\hbar+})^* S(\tilde{a}_0) C(\tilde{a}_0, \varphi_{\hbar-}) \exp[i\omega_\hbar(+\infty, -\infty)] = S_2(\tilde{a}_0). \tag{1.57}$$

In the absence of asymptotic completeness, one nevertheless expects similar results to hold for the wave operators, provided they exist [see (5.97)].

We conclude this preliminary discussion with a comment on the role of the parameter μ [see Eq. (1.2)] which was left unspecified until now. The parameters $\hbar$ and μ enter into the basic Eqs. (1.19) and (1.37) only through the combination $\hbar^\mu$ so that the entire analysis is insensitive to the choice of μ. The requirement that the quantum mechanical energy tend in some sense to the classical one imposes $\delta = 0$ or equivalently $\mu = \frac{1}{2}$ [see Eq. (1.4)], but this choice is extraneous to the dynamical problem. From now on, we choose $\mu = \frac{1}{2}$, in keeping with common use.

After this discussion, we are in a position to state our results more precisely. They consist in proving (1.51) with the uniformity in time there stated, and (1.57) and its analogues for the wave operators, in the case of a non-relativistic many-boson system with two-body interactions, in space dimension $n \geqq 3$. From the convergence (1.51) with uniformity in time, the convergence (1.52), which is perhaps of more direct physical significance, follows with similar uniformity in time. Since however the derivation of (1.52) from (1.51) is completely straightforward, we shall from now on concentrate on (1.51) and not mention (1.52) any more in the rest of the paper.

We now introduce the notation appropriate to describe the system mentioned above. Let $a(x)$ and $a^*(x)$ satisfy the CCR

$$[a(x), a(y)] = 0, [a(x)a^*(y)] = \delta(x-y). \tag{1.58}$$

The $a(x)$'s play the role of the a_i's. We recall that $a_\hbar(x) = \hbar^{1/2}a(x)$. The total hamiltonian of the system is

$$H_\hbar(a_\hbar) = \frac{\hbar^2}{2m_\hbar}\int dx \nabla a^*(x)\cdot\nabla a(x) + \tfrac{1}{2}\int dx dy V_\hbar(x-y)a^*(x)a^*(y)a(x)a(y), \tag{1.59}$$

where $V_\hbar$ is a real even function. In order that $H_\hbar$ have the form (1.4) with $\delta = 0$ (i.e. $\mu = \frac{1}{2}$) we must impose that

$$m_\hbar = \hbar m, \qquad V_\hbar(x) = \hbar^2 V(x), \tag{1.60}$$

with m and V $\hbar$-independent. From now on, we take in addition $m=1$. The total Hamiltonian can then be rewritten as

$$H_\hbar \equiv H_\hbar(a_\hbar) = \hbar H_0(a) + \hbar^2 H_4(a), \tag{1.61}$$

where

$$H_0(a) = \tfrac{1}{2}\int dx \nabla a^*(x)\cdot\nabla a(x), \tag{1.62}$$

$$H_4(a) = \tfrac{1}{2}\int dx dy V(x-y)a^*(x)a^*(y)a(x)a(y), \tag{1.63}$$

and the operator $H(a)$ defined in (1.4) becomes

$$H(a) = H_0(a) + H_4(a). \tag{1.64}$$

In the classical limit that we are considering, the particle number

$$N = \int dx a^*(x)a(x) \tag{1.65}$$

is $O(\hbar^{-1})$, the total energy is $O(1)$ and therefore the energy per particle is $O(\hbar)$. Since the mass itself is $O(\hbar)$ the De Broglie wavelength is $O(1)$. In the limit $\hbar \to 0$, the particle structure disappears and the system becomes a classical field or a fluid described by the classical variable φ which is now a complex function of x. The quantum mechanical free evolution is generated through (1.32) by the free Hamiltonian $H_0(a)$ defined by (1.62). The operator $H_1(a)$ defined by (1.8) becomes

$$\begin{aligned} H_1(a) = &-\tfrac{1}{2}\int dx(\Delta\varphi(x)a^*(x) + \Delta\bar{\varphi}(x)a(x)) \\ &+ \int dx dy V(x-y)|\varphi(y)|^2(\varphi(x)a^*(x) + \bar{\varphi}(x)a(x)), \end{aligned} \tag{1.66}$$

and therefore the classical evolution Eq. (1.11) obtained through (1.9) becomes

$$i\dot{\varphi} = -\tfrac{1}{2}\Delta\varphi + \varphi(V*|\varphi|^2). \tag{1.67}$$

Similarly, the classical free equation is

$$i\dot{\varphi} = -\tfrac{1}{2}\Delta\varphi, \tag{1.68}$$

so that the free evolution operator $u_0(t)$ is given by

$$u_0(t) = \exp\left(i\frac{t}{2}\Delta\right). \tag{1.69}$$

The remaining terms of the decomposition (1.8) will be given where needed, namely in Sect. 3.

The implementation of our program requires some information on the evolution at finite and infinite times of both the classical and quantum theories. The classical theory will be treated in a companion paper [4], where we study the Cauchy problem for the Eq. (1.67) and the scattering problem for the Eqs. (1.67) and (1.68). The results relevant for the present analysis are collected in Sect. 2. As regards the quantum theory, we need to define and study both the total evolution associated with $U(t, t_0; \tilde{a}_0)$, $\tilde{W}(t, t_0; \tilde{a}_0)$ and the evolution of the fluctuations associated with $\tilde{U}_2(t, t_0; \tilde{a}_0)$. Various degrees of difficulty arise in the analysis of these operators, depending on whether the potential V is locally square integrable ($V \in L^2_{\mathrm{loc}}$) or not. In order to keep this paper to a reasonable length, we treat here only the case where $V \in L^2_{\mathrm{loc}}$. In this situation the evolutions given by the operators $U(t, t_0; \tilde{a}_0)$, $\tilde{W}(t, t_0; \tilde{a}_0)$, and $\tilde{U}_2(t, t_0; \tilde{a}_0)$ are studied in Sects. 3 and 4 respectively. The more singular case is left for a subsequent paper. Having all the necessary ingredients we can then proceed to the proof of the announced convergences when $\hbar \to 0$. This is done in Sect. 5 where the results are stated in a precise form in Proposition 5.1 and Theorem 5.1. The assumptions on the potential vary from section to section and will be given whenever needed.

2. The Classical Theory

In this section we describe some results concerning the Cauchy problem and the scattering theory for the classical equations (1.67) and (1.68) in $\mathbb{R}^n$ with $n \geqq 3$. A more detailed exposition, closely following [3], as well as the proofs will be given in [4]. Here we present the main results in a simplified form which is sufficient for the applications in this paper.

We denote by $\|\cdot\|_q$ the norm in $L^q \equiv L^q(\mathbb{R}^n)$ $(1 \leqq q \leqq \infty)$, except for $q=2$ where the subscript 2 will be omitted, and by $H^k \equiv H^k(\mathbb{R}^n)$ the usual Sobolev spaces. For any interval I (possibly unbounded) of the real line $\mathbb{R}$, for any Banach space $\mathscr{B}$, we denote by $\mathscr{C}(I, \mathscr{B})$ [respectively $\mathscr{C}_b(I, \mathscr{B})$] the space of continuous (respectively bounded continuous) functions from I to $\mathscr{B}$. Let $n \geqq 3$, let $2 \leqq r < 2n/(n-2)$, let k be an integer, $k \geqq 1$; we define:

$$X_{kr} = \{\psi : \psi \in H^k \quad \text{and} \quad D^\alpha \psi \in L^r \quad \text{for all } \alpha, \quad |\alpha| = k\} \tag{2.1}$$

and

$$\mathscr{X}_a(I) = \mathscr{C}_b(I, H^1) \cap \mathscr{C}(I, X_{kr}) . \tag{2.2}$$

It is convenient to rewrite the Cauchy problem for the Eq. (1.67) with the initial condition $\varphi(t_0) = \varphi_0$ in the form of the following integral equation

$$\varphi(t) = u_0(t - t_0)\varphi_0 - i \int_{t_0}^{t} d\tau u_0(t - \tau) \{\varphi(\tau)(V * |\varphi(\tau)|^2\} , \tag{2.3}$$

where $u_0(t)$ is defined by (1.69). We first state a global existence and uniqueness result.

Proposition 2.1. *Let $n \geqq 3$, let X_{kr} and $\mathscr{X}_a(\cdot)$ be defined by* (2.1) *and* (2.2) *respectively. Let V satisfy the following assumptions:*

$$V \in L^{p_1} + L^{p_2} , \tag{2.4}$$

where

$$\begin{cases} 1/2 - 1/r \leqq 1/p_1 \leqq 1/p_2 \leqq \mathrm{Min}(1, 1 - 2/r + 2/n) \\ p_2 > 1 \quad \text{if} \quad k > 1 \quad \text{and} \quad r = n = 3 , \end{cases} \tag{2.5}$$

and

$$V_- \in L^{\mathrm{Max}(p_1, n/2)} + L^{n/2} , \tag{2.6}$$

where $V_\pm = \mathrm{Max}(\pm V, 0)$. Let $t_0 \in \mathbb{R}$, let $\varphi_0 \in X_{kr}$ be such that $u_0(t - t_0)\varphi_0 \in \mathscr{X}_a(\mathbb{R})$. Then the Eq. (2.3) has a unique solution $\varphi(t)$ in $\mathscr{X}_a(\mathbb{R})$. This solution satisfies

$$\|\varphi(t)\| = \|\varphi_0\| \tag{2.7}$$

and

$$H(\varphi(t)) = H(\varphi_0) , \tag{2.8}$$

where $H(\cdot)$ is defined by (1.64)

Remark 2.1. The parameters p_1 and p_2 control the decay of the potential at infinity and its local singularities respectively. They are coupled through (2.5); the worst singularities allowed for V_+ correspond to $p_2 = 1$ for $n = 3$ (with $r > 3$) and to $p_2 > n/4$ for $n \geqq 4$ [with r close to $2n/(n-2)$].

As mentioned in the introduction [see especially the Eq. (1.50)] we need some information on the asymptotic behaviour in time of the solutions of (2.3). In a first step we can construct solutions asymptotically free at $+\infty$ or at $-\infty$ by solving the integral equation

$$\varphi(t) = u_0(t)\tilde{\varphi}_0 - i \int_{t_0}^{t} d\tau u_0(t - \tau) \{\varphi(\tau)(V * |\varphi(\tau)|^2)\} , \tag{2.9}$$

where t_0 lies in a neighborhood of $+\infty$ or of $-\infty$. For this purpose we introduce the following spaces. Let r' satisfy

$$1/2 - 1/n < 1/r \leqq 1/r' < 1/2 - 1/3n . \tag{2.10}$$

For any interval I (possibly unbounded) of $\mathbb{R}$, we define

$$\mathscr{X}_0(I)=\{\psi:\psi\in\mathscr{X}_a(I), |\psi|_{0I}<\infty\}, \tag{2.11}$$

where

$$|\psi|_{0I}=\operatorname*{Sup}_{t\in I}\operatorname*{Max}_{\substack{|\beta|\leq k\\|\alpha|=k}}\{\|D^\beta\psi\|, (1+|t|)^{n/2-n/r'}\|D^\beta\psi\|_{r'}, (1+|t|)^{n/2-n/r'}\|D^\alpha\psi\|_r\}. \tag{2.12}$$

We shall use freely the notation $\psi\in\mathscr{X}_a(\mathbb{R})\cap\mathscr{X}_0(I)$ to denote a function $\psi\in\mathscr{X}_a(\mathbb{R})$, the restriction of which to I lies in $\mathscr{X}_0(I)$.

We are now in a position to state the following result.

Proposition 2.2. *Let $n\geq 3$, let X_{kr}, $\mathscr{X}_a$ and $\mathscr{X}_0$ be defined by* (2.1), (2.2), *and* (2.11) *respectively. Let V satisfy* (2.4)–(2.6) *and in addition*

$$1/2-1/r'+1/n<1/p_1. \tag{2.13}$$

Let $\tilde\varphi_0\in X_{kr}$ be such that $u_0(t)\tilde\varphi_0\in\mathscr{X}_a(\mathbb{R})\cap\mathscr{X}_0(\mathbb{R}^+)$. Then

1) For t_0 sufficiently large (depending on $\tilde\varphi_0$) and in particular for $t_0=\infty$, the Eq. (2.9) *has a unique solution φ in $\mathscr{X}_a(\mathbb{R})\cap\mathscr{X}_0(\mathbb{R}^+)$.*

2) Let $t_0\in\mathbb{R}$ or $t_0=+\infty$ and let $\varphi\in\mathscr{X}_a(\mathbb{R})\cap\mathscr{X}_0(\mathbb{R}^+)$ be solution of the Eq. (2.9). *Then $u_0(\cdot)\tilde\varphi(s)\in\mathscr{X}_a(\mathbb{R})\cap\mathscr{X}_0(\mathbb{R}^+)$ for all $s\in\mathbb{R}$. Furthermore, there exists $\varphi_+\in X_{kr}$ such that $u_0(t)\varphi_+\in\mathscr{X}_a(\mathbb{R})\cap\mathscr{X}_0(\mathbb{R}^+)$, $u_0(\cdot)\tilde\varphi(s)-u_0(\cdot)\varphi_+\in\mathscr{X}_0(\mathbb{R})$, and*

$$\lim_{s\to\infty} u_0(\cdot)\tilde\varphi(s)-u_0(\cdot)\varphi_+=0 \quad \text{in} \quad \mathscr{X}_0(\mathbb{R}), \tag{2.14}$$

where $\tilde\varphi(s)$ is defined by (1.48). *In particular*

$$\lim_{s\to\infty}\tilde\varphi(s)=\varphi_+ \quad \text{in} \quad X_{kr}. \tag{2.15}$$

Furthermore, for all $t\in\mathbb{R}$,

$$\|\varphi(t)\|=\|\varphi_+\|, \tag{2.16}$$

$$H(\varphi(t))=\tfrac{1}{2}\|\nabla\varphi_+\|^2. \tag{2.17}$$

If $t_0=+\infty$, $\varphi_+=\tilde\varphi_0$.

Similar results hold for t_0 in a neighborhood of $-\infty$ and $u_0(t)\tilde\varphi_0$ or φ in $\mathscr{X}_a(\mathbb{R})\cap\mathscr{X}_0(\mathbb{R}^-)$.

Remark 2.2. The condition (2.13) imposes an additional restriction on the decay of the potential at infinity. It follows from (2.10) and (2.13) that $p_1<3n/4$. One checks easily that for any allowed choice of r and r', (2.13) is stronger than the upper limitation on p_1 and compatible with the lower limitation on p_1 that come from (2.5).

Remark 2.3. Under the assumptions of the Proposition 2.2, the wave operators formally given by

$$\Omega_{c\pm}\varphi_\pm=\varphi(0)=\varphi_\pm-i\int_{\pm\infty}^{0} d\tau\, u_0(-\tau)\{\varphi(\tau)(V*|\varphi(\tau)|^2)\} \tag{2.18}$$

are well defined as (non linear) maps from $Y_\pm$ to itself, where

$$Y_\pm = \{\psi : u_0(t)\psi \in \mathscr{X}_a(\mathbb{R}) \cap \mathscr{X}_0(\mathbb{R}^\pm)\}. \tag{2.19}$$

A solution of (2.9) in $\mathscr{X}_a(\mathbb{R})$ will be called dispersive (respectively dispersive in the past or in the future) if it belongs to $\mathscr{X}_0(\mathbb{R})$ [respectively to $\mathscr{X}_0(\mathbb{R}^-)$ or $\mathscr{X}_0(\mathbb{R}^+)$]. Proposition 2.2 provides us with solutions that are dispersive in the past *or* in the future but not necessarily in both directions. For such solutions the uniformly of the $\hbar \to 0$ limit announced after (1.51) holds only when t or t_0 tends to $-\infty$ *or* to $+\infty$. As a consequence, for such solutions one cannot obtain in general the limiting property (1.57) for the S-matrix but only the corresponding property for the wave operators.

In order to obtain dispersive solutions, more information is necessary. In particular, we shall see that under an additional restriction on the initial data φ_0, all solutions of the Eq. (2.3) obtained from Proposition 2.1 are dispersive if the potential V is repulsive in a suitable sense. For this purpose we define a new space Σ as the Hilbert space

$$\Sigma = \{\psi : \psi \in H^1 \text{ and } x\psi \in L^2\}, \tag{2.20}$$

with the norm defined by

$$\|\psi\|_\Sigma^2 = \|\psi\|^2 + \|\nabla\psi\|^2 + \|x\psi\|^2. \tag{2.21}$$

The relevance of Σ is best seen from the following lemma.

Lemma 2.1. *Let $\psi \in \Sigma$, let $2 \leqq q \leqq 2n/(n-2)$. Then*

$$\|u_0(t)\psi\|_q \leqq \tilde{a}_q (1+|t|)^{n/q - n/2} \|\psi\|_\Sigma, \tag{2.22}$$

where the constant $\tilde{a}_q$ depends only on n and q.

We can now state the following result.

Proposition 2.3. *Let $n \geqq 3$, let X_{kr}, $\mathscr{X}_a(\cdot)$, $\mathscr{X}_0(\cdot)$ and Σ be defined by* (2.1), (2.2), (2.11), *and* (2.20). *Let V satisfy* (2.4)–(2.6) *and in addition*

$$x \cdot \nabla V \in L^\infty + L^{p_4}, \tag{2.23}$$

where

$$1/p_4 = \mathrm{Min}(1, 2 + 4k/n - 4/r) \tag{2.24}$$

and the derivative of V is taken in the distribution sense. Let $\tilde{\varphi}_0 \in X_{kr} \cap \Sigma$ be such that $u_0(t)\tilde{\varphi}_0 \in \mathscr{X}_a(\mathbb{R})$.

1) *Let t_0 be finite and let φ be the solution of* (2.9) *in $\mathscr{X}_a(\mathbb{R})$ as obtained from Proposition 2.1. Then $\varphi \in \mathscr{C}(\mathbb{R}, \Sigma)$.*

Assume in addition that V is repulsive in the following sense:

V is non negative and $\lambda^2 V(\lambda x)$ is decreasing in λ for all $x \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}^+$.

Then $\tilde{\varphi} \in \mathscr{C}_b(\mathbb{R}, \Sigma)$.

Assume in addition that V satisfies (2.13) *and that $u_0(\cdot)\tilde{\varphi}_0 \in \mathscr{X}_0(\mathbb{R})$. Then $\varphi \in \mathscr{X}_0(\mathbb{R})$.*

Let $\varphi_\pm$ be defined as in Proposition 2.2; then $\varphi_\pm \in \Sigma$ and

$$\lim_{s\to\pm\infty} \tilde{\varphi}(s) = \varphi_\pm \quad in \quad \Sigma. \tag{2.25}$$

2) *Let r' be such that*

$$1/2 - 1/n < 1/r \leqq 1/r' < 1/2 - 1/2n, \tag{2.26}$$

let

$$p_1 < n/2, \tag{2.27}$$

and let V satisfy in addition

$$x \cdot \nabla V \in L^{p_3} + L^{p_4} \tag{2.28}$$

with

$$p_3 < n/2. \tag{2.29}$$

Let $t_0 \in \mathbb{R}$ or $t_0 = +\infty$ and let $\varphi \in \mathscr{X}_a(\mathbb{R}) \cap \mathscr{X}_0(\mathbb{R}^+)$ be solution of the Eq. (2.9). *Then $\varphi \in \mathscr{C}(\mathbb{R}, \Sigma)$. Let φ_+ be defined as in Proposition 2.2, then $\varphi_+ \in \Sigma$ and*

$$\lim_{s\to+\infty} \tilde{\varphi}(s) = \varphi_+ \quad in \quad \Sigma. \tag{2.30}$$

For repulsive interactions in the previous sense, Proposition 2.3 provides us with a large class of dispersive solutions. Actually it implies asymptotic completeness in the space Z:

$$Z = \{\psi : \psi \in \Sigma \text{ and } u_0(\cdot)\psi \in \mathscr{X}_0(\mathbb{R})\}. \tag{2.31}$$

Corollary 2.1. *Let r and r' satisfy* (2.26). *Let V satisfy* (2.4), (2.5), (2.27), *the repulsivity condition of Proposition 2.3,* (2.28), (2.29), *and* (2.24). *Then the (non linear) mapping*

$$\varphi_- \to \varphi_+ = S_c \varphi_- \tag{2.32}$$

defined formally by

$$\varphi_+ = \varphi_- - i \int_{-\infty}^{\infty} d\tau\, u_0(-\tau) \{\varphi(\tau)(V * |\varphi(\tau)|^2)\} \tag{2.33}$$

and, more precisely, by a combined use of Proposition 2.2 and 2.3, is a bijection of Z onto Z.

Remark 2.4. In each of the situations covered by the previous propositions one can also obtain boundedness properties of the solutions and continuity properties with respect to the initial data.

3. The Quantum Theory

In this section, we define the quantum theory formally described in Sect. 1, and in particular we derive the main properties of the operators $\tilde{W}(t, s; a)$ defined by (1.45) and (1.55) that will be used later.

The basic space of the theory is the boson Fock space

$$\mathscr{H} = \bigoplus_{N=0}^{\infty} \mathscr{H}_N, \tag{3.1}$$

where $\mathscr{H}_N$ is the space of totally symmetric square integrable functions of N variables in $\mathbb{R}^n$. The Fock vacuum is denoted by Ψ_0. The scalar product in $\mathscr{H}$ is denoted by

$$\langle \Phi, \Psi \rangle = ⨍ dX\, \bar{\Phi}(X)\, \Psi(X), \tag{3.2}$$

where $X = (x_1, \ldots, x_N)$ and

$$⨍ dX = \sum_{N=0}^{\infty} (N!)^{-1} \int dx_1 \ldots dx_N. \tag{3.3}$$

The norm in $\mathscr{H}$ is denoted by $\|\cdot\|$. No confusion should arise with the scalar product and the norm in $L^2 \equiv \mathscr{H}_1$ for which we use the same notation. The norm of a bounded operator A in $\mathscr{H}$ is denoted by $|||A|||$. The creation and annihilation operators are defined, for any $\alpha \in L^2$, by

$$\begin{cases} (a(\bar{\alpha})\Psi)(X) = \int dx\, \bar{\alpha}(x)\, \Psi(X, x), \\ (a^*(\alpha)\Psi)(X) = \sum_{i=1}^{N} \alpha(x_i)\, \Psi(X \backslash x_i). \end{cases} \tag{3.4}$$

In (3.2)–(3.4), we follow the convention of Friedrichs [2]. The particle number operator N is defined by (1.65).

For any self-adjoint semi-bounded operator $A \geqq c\mathbb{1}$ in $\mathscr{H}$, we denote by $Q(A)$ the form domain of A, namely $Q(A) = \mathscr{D}((A - c\mathbb{1})^{1/2})$. $Q(A)$ is a Hilbert space with norm $\|((1-c)\mathbb{1} + A)^{1/2}\Phi\|$. We denote by $Q^*(A)$ the completion of $\mathscr{H}$ in the norm $\|((1-c)\mathbb{1} + A)^{-1/2}\Phi\|$. We shall also use the space $\mathscr{C}_0(N)$ of vectors in $\mathscr{H}$ with finitely many particles.

We now begin the study of $\tilde{W}(t, s; a)$. For this purpose we need some properties of the Weyl operators defined [cf. Eq. (1.15)] by

$$C(a, \alpha) = \exp[a^*(\alpha) - a(\bar{\alpha})] \tag{3.5}$$

for any $\alpha \in L^2$.

Lemma 3.1.

1) *$C(a, \alpha)$ is unitary and strongly continuous as a function of $\alpha \in L^2$.*

2) *Let $\alpha \in H^1$. Then $C(a, \alpha)$ is bounded in $Q(H_0)$ and in $Q^*(H_0)$ uniformly for α in a bounded set of H^1. (H_0 is defined by (1.62)).*

3) *Let $\alpha : t \to \alpha(t) \in \mathscr{C}^1(\mathbb{R}, L^2)$. Then $C(a, \alpha(t))$ is strongly differentiable in t from $Q(N)$ to $\mathscr{H}$. The derivative is given by*

$$\dot{C}(a, \alpha(t)) = C(a, \alpha(t))\,(a^*(\dot{\alpha}) - a(\dot{\bar{\alpha}}) + i\,\mathrm{Im}\langle \alpha, \dot{\alpha} \rangle), \tag{3.6}$$

where $\dot{\alpha} = d\alpha/dt$.

Proof.

1) Unitarity is obvious. Strong continuity then follows from weak continuity on $\mathscr{C}_0(N)$.

2) We have

$$C(a,\alpha)^* H_0 C(a,\alpha) = \tfrac{1}{2}\int dx(\nabla a^*(x) + \nabla\bar{\alpha}(x))\cdot(\nabla a(x) + \nabla a(x)) \tag{3.7}$$

$$\leqq 2H_0 + \|\nabla\alpha\|^2 . \tag{3.8}$$

This proves 2) for $Q(H_0)$ and by duality for $Q^*(H_0)$.

3) Let Ψ_1, $\Psi_2 \in \mathscr{C}_0(N)$. Then by an elementary computation

$$\frac{d}{dt}\langle \Psi_1, C(a,\alpha(t))(N+1)^{-1/2}\Psi_2\rangle$$

$$= \langle \Psi_1, C(a,\alpha(t))D(t)(N+1)^{-1/2}\Psi_2\rangle , \tag{3.9}$$

where

$$D(t) = a^*(\dot{\alpha}(t)) - a(\dot{\bar{\alpha}}(t)) + i\,\mathrm{Im}\langle \alpha(t), \dot{\alpha}(t)\rangle . \tag{3.10}$$

The operator $D(t)(N+1)^{-1/2}$ is bounded and norm continuous in t. Therefore

$$[C(a,\alpha(t)) - C(a,\alpha(s))]\Psi = \int_s^t d\tau\, C(a,\alpha(\tau))D(\tau)\Psi \tag{3.11}$$

as a strong Riemann integral in $\mathscr{H}$ for any Ψ in $Q(N)$. This proves 3). Q.E.D

The main constituent of $\tilde{W}(t,s;a)$ is the total evolution operator $U(t,s;a)$ which we now define. We do not strive for the greatest possible generality, since the strongest assumptions we shall need on V will arise in the study of the limit $\hbar\to 0$ in Sect. 5. Let $n \geqq 3$ and $V_\pm = \mathrm{Max}(\pm V, 0)$. We assume that V satisfies the following conditions

$$\begin{cases} V \in L^\infty + L^p & (p \geqq 1) . \\ V_- \in L^\infty + L^{n/2} & \end{cases} \tag{3.12}$$

The Hamiltonians $H_\hbar$, H_0, and H_4 [see (1.61)] are defined through the direct sum decomposition

$$H_\hbar = \bigoplus_{N=1}^{\infty} H_{\hbar N} = \bigoplus_{N=1}^{\infty} \hbar H_{0N} + \hbar^2 H_{4N} . \tag{3.13}$$

Under the assumptions made on V, $H_{\hbar N}$ is for each N the self-adjoint operator defined as a sum in the sense of quadratic forms, with $Q(H_{\hbar N}) \subset Q(H_{0N})$; it is semi-bounded [10]. Furthermore $H_\hbar$ is essentially self-adjoint on $\mathscr{D}(H_\hbar) \cap \mathscr{C}_0(N)$. If in addition $V_+ \in L^2 + L^\infty$ and $V_- \in L^2 + L^\infty$ (resp. $V_- \in L^{2+\varepsilon} + L^\infty$) for $n=3$ (resp. for $n=4$), then $H_\hbar$ is essentially self-adjoint on the subspace of vectors with finitely many particles and smooth wave functions [8]. The total evolution [see (1.6)] is defined by

$$U(t,s;a) \equiv U(t-s;a) = \exp[-i\hbar^{-1}(t-s)H_\hbar] . \tag{3.14}$$

In order to study $\tilde{W}(t,s;a)$ we need the explicit form of the decomposition (1.8). $H_1(a)$ is given by (1.66). The other terms are given below:

$$H_2(a) = H_0(a) + G(a) + K(a) + L(a) + L^*(a) , \tag{3.15}$$

where $H_0(a)$ is defined by (1.62), and

$$G(a)=\int dx\, g(x)\, a^*(x)\, a(x)\,, \tag{3.16}$$

$$K(a)=\int dx\, dy\, k(x,y)\, a^*(x)\, a(y)\,, \tag{3.17}$$

$$L(a)=\tfrac{1}{2}\int dx\, dy\, \bar{l}(x,y)\, a(x)\, a(y)\,, \tag{3.18}$$

with

$$g(x)=\int dy\, V(x-y)\,|\varphi(y)|^2=(V*|\varphi|^2)(x)\,, \tag{3.19}$$

$$k(x,y)=\varphi(x)\, V(x-y)\,\bar{\varphi}(y)\,, \tag{3.20}$$

$$l(x,y)=\varphi(x)\, V(x-y)\,\varphi(y)\,, \tag{3.21}$$

$$H_{\geqq 3}(a)=H_3(a)+H_4(a)\,, \tag{3.22}$$

with

$$H_3(a)=A_3(a)+A_3^*(a)\,, \tag{3.23}$$

$$A_3(a)=\int dx\, dy\, V(x-y)\,\bar{\varphi}(x)\, a^*(y)\, a(x)\, a(y)\,, \tag{3.24}$$

and $H_4(a)$ is given by (1.63).

We first derive some properties of $H_2(a)$.

Lemma 3.2. *Let $V\in L^\infty+L^p$ i.e. $V=V_1+V_2$ with $V_1\in L^\infty$, $V_2\in L^p$, $p\geqq 2$ and let $\varphi\in L^2\cap L^q$ with $1/p+2/q=1$. Then the operators $N^{-1}G$, $N^{-1}K$ and $(N(N-1))^{-1}L^*L$ are bounded and*

$$|||N^{-1}G|||\leqq \|g\|_\infty\,, \tag{3.25}$$

$$|||N^{-1}K|||\leqq c\,, \tag{3.26}$$

$$|||(N(N-1))^{-1}L^*L|||\leqq \tfrac{1}{4}c^2\,, \tag{3.27}$$

$$|||((N+1)(N+2))^{-1}LL^*|||\leqq \tfrac{1}{4}c^2\,, \tag{3.28}$$

where

$$c=\|k\|_{\mathrm{HS}}=2^{1/2}\|L^*\Psi_0\|=\{\int dx(|\varphi|^2(V^2*|\varphi|^2)\}^{1/2}\,, \tag{3.29}$$

and $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm in $\mathscr{H}_1$. Furthermore

$$\mathrm{Max}(c,\|g\|_\infty)\leqq \|V_1\|_\infty\|\varphi\|^2+\|V_2\|_p\|\varphi\|_q^2\,. \tag{3.30}$$

Proof. (3.25) and (3.26) are obvious. In order to prove (3.27) it suffices to show that

$$L^*L\leqq \tfrac{1}{4}c^2 N(N-1) \tag{3.31}$$

because L^*L commutes with N. Now

$$\begin{aligned}\|L\Psi\|^2&=⨍ dX\,|\tfrac{1}{2}\int dy_1\, dy_2\,\bar{l}(y_1,y_2)\,\Psi(X,y_1,y_2)|^2\\ &\leqq \|L^*\Psi_0\|^2 ⨍ dX\,\tfrac{1}{2}\int dy_1\, dy_2\,|\Psi(X,y_1,y_2)|^2\\ &=\tfrac{1}{2}\|L^*\Psi_0\|^2\langle\Psi,N(N-1)\Psi\rangle\end{aligned} \tag{3.32}$$

by Schwarz's inequality. The proof of (3.28) is similar. Finally (3.30) follows from Hölder's and Young's inequalities. Q.E.D.

Corollary 3.1. *Let the assumptions of Lemma 3.2 be satisfied. Then, for any* $\omega \in \mathbb{C}$, $|\omega|=1$,

$$\omega L+\bar{\omega}L^* \leqq c(N+1) \tag{3.33}$$

and

$$|||(N+1)^{-1/2}L^{(*)}(N+1)^{-1/2}||| \leqq c\,, \tag{3.34}$$

where c is given by (3.29).

Proof.

$$\left[\left(\frac{c}{2}(N+2)\right)^{1/2}-\bar{\omega}\left(\frac{c}{2}N\right)^{-1/2}L^*\right]\left[\left(\frac{c}{2}(N+2)\right)^{1/2}-\omega L\left(\frac{c}{2}N\right)^{-1/2}\right] \geqq 0 \tag{3.35}$$

implies

$$\omega L+\bar{\omega}L^* \leqq \frac{c}{2}(N+2)+\left(\frac{c}{2}N\right)^{-1}L^*L \leqq c(N+1) \tag{3.36}$$

by (3.31). This proves (3.33), from which (3.34) follows by sum and difference. Q.E.D.

In Sect. 5 we shall make essential use of a regularized version of the Eq. (1.46). For this purpose we introduce a regularization operator $P_{\nu\kappa}$ as follows: let $\sigma_1 \in \mathscr{C}^1(\mathbb{R}^+)$ be positive and decreasing, $\sigma_1(s)=1$ if $s \leqq 1$, $\sigma_1(s)=0$ if $s \geqq 2$. We denote by σ_ν the operator $\sigma_1(N/\nu)$ in $\mathscr{H}$. Let $\varrho_1 \in L^1 \cap L^2$ be even, positive, with $\|\varrho_1\|_1=1$. Let $\hat{\varrho}_1$ be its Fourier transform. Assume in addition that $\hat{\varrho}_1 \geqq 0$, that $|k|\hat{\varrho}_1(k)$ is bounded and that $\hat{\varrho}_1(\lambda k)$ is a decreasing function of λ for all $\lambda \in \mathbb{R}^+$ and $k \in \mathbb{R}^n$. For any $\kappa>0$ let $\varrho_\kappa(x)=\kappa^n \varrho_1(\kappa x)$ and let $\varrho_{\kappa *}$ be the operator in $\mathscr{H}_1$ defined by $\varrho_{\kappa *}\psi = \varrho_\kappa * \psi$ for any $\psi \in \mathscr{H}_1$. Let $R_\kappa = \bigoplus_{N=0}^{\infty} \varrho_{\kappa *}^{\otimes_S N}$ and $P_{\nu\kappa}=\sigma_\nu R_\kappa$ where the superscript $\otimes_S N$ denotes the symmetrized N^{th} tensor power. The parameters ν and κ are a particle number cut-off and a momentum cut-off respectively. The operators σ_ν, R_κ and $P_{\nu\kappa}$ commute among themselves and with N and H_0. They satisfy $0 \leqq P_{\nu\kappa} = P_{\nu\kappa}^* \leqq \mathbb{1}$, are increasing in ν and κ and satisfy

$$\underset{\nu,\kappa\to\infty}{s\text{-lim}}\, P_{\nu\kappa}=\mathbb{1}\,. \tag{3.37}$$

Some of the regularizing properties of $P_{\nu\kappa}$ are expressed by the following lemma.

Lemma 3.3. *Let V satisfy the assumptions of Lemma 3.2.*

1) Let $\varphi \in L^2 \cap L^q$ *with* $1/p+2/q=1$. *Then* $\sigma_\nu(H_2-H_0)$ *and therefore* $P_{\nu\kappa}(H_2-H_0)$ *are bounded in* $\mathscr{H}$ *and strongly continuous with respect to* $\varphi \in L^2 \cap L^q$.

2) Let $\varphi \in L^2$. *Then* $P_{\nu\kappa}H_3$ *and* $P_{\nu\kappa}H_4$ *are bounded in* $\mathscr{H}$ *and strongly continuous with respect to* $\varphi \in L^2$ *(*H_3 *and* H_4 *are defined by* (3.23), (3.24), *and* (1.63) *and the a dependence is omitted).*

Proof. Part 1) follows immediately from Lemma 3.2.

2) Because of the particle number cut-off it is enough to prove that $R_\kappa A_3$, $R_\kappa A_3^*$ and $R_\kappa H_4$ are bounded from $\mathscr{H}_2$ to $\mathscr{H}_1$, from $\mathscr{H}_1$ to $\mathscr{H}_2$ and from $\mathscr{H}_2$ to $\mathscr{H}_2$ respectively. Let $\psi \in \mathscr{H}_1$ and Ψ, $\Phi \in \mathscr{H}_2$. Then

$$\begin{aligned}\langle \psi, R_\kappa A_3 \Psi\rangle &= \int dx_1\, dx_2\, dx_1'\, \bar{\psi}(x_1)\bar{\varphi}(x_2)\\ &\quad\cdot \varrho_\kappa(x_1 - x_1')V(x_1' - x_2)\Psi(x_1', x_2),\\ \langle \Psi, R_\kappa A_3^* \psi\rangle &= \int dx_1\, dx_2\, dx_1'\, dx_2'\, \bar{\Psi}(x_1, x_2)\varrho_\kappa(x_1 - x_1')\\ &\quad\cdot \varrho_\kappa(x_2 - x_2')V(x_1' - x_2')\psi(x_1')\varphi(x_2'),\\ \langle \Psi, R_\kappa H_4 \Phi\rangle &= \tfrac{1}{2}\int dx_1\, dx_2\, dx_1'\, dx_2'\, \bar{\Psi}(x_1, x_2)\varrho_\kappa(x_1 - x_1')\\ &\quad\cdot \varrho_\kappa(x_2 - x_2')V(x_1' - x_2')\Phi(x_1', x_2').\end{aligned}$$

It is therefore sufficient to prove that the operator B in $L^2 \otimes L^2$ defined by

$$(B\theta)(x_1, x_2) = \int dx_1'\, \varrho_\kappa(x_1 - x_1')V(x_1' - x_2)\theta(x_1', x_2) \tag{3.38}$$

is bounded. Now it follows from Young's and Schwarz's inequality that $\|B\| \leqq \|V_1\|_\infty + \|\varrho_\kappa\|\, \|V_2\|$. This proves boundedness; continuity is then obvious. Q.E.D.

We are now in a position to study $\tilde{W}(t, s; a)$ defined formally by (1.55) through (1.25), (1.26), (1.45) where $\varphi_\hbar$, $\tilde{\varphi}$, $\tilde{\varphi}_\hbar$ are defined by (1.13) with $\mu = 1/2$ and (1.48). We assume that V satisfies (3.12) and that

$$\varphi \in \mathscr{C}(\mathbb{R}, L^2 \cap L^q) \quad \text{with} \quad 1/p + 2/q = 1\,. \tag{3.39}$$

Then $C(a, \tilde{\varphi}_\hbar(t))$ and $U(t, s; a)$ are well defined through (3.5) and (3.14). The phase $\omega_\hbar(t, s)$ defined by (1.26) now becomes

$$\omega_\hbar(t, s) = -\hbar^{-1}\int_s^t d\tau\, H_4(\varphi(\tau))\,. \tag{3.40}$$

It follows from (3.12) and (3.39), by Hölder's and Young's inequalities, that $\omega_\hbar(t, s)$ is well defined and continuously differentiable in t. This completes the precise definition of $\tilde{W}(t, s; a)$. We now derive the regularized version of (1.46) mentioned above. In order to be precise, we indicate explicitly the time dependence coming from φ in the operators $H_k(a) \equiv H_k(t, a)$, $k = 2, 3, 4$, defined by (3.15), (3.23) and (1.63), and $\tilde{H}_k(a) \equiv \tilde{H}_k(t, a)$ defined by (1.37b), which then becomes

$$\tilde{H}_k(t, a) = U_0(t; a)^* H_k(t, a) U_0(t, a) \qquad (k = 2, 3, 4)\,. \tag{3.41}$$

Proposition 3.1. *Let V satisfy* (3.12) *with $p \geqq 2$. Let φ satisfy*

$$\varphi \in \mathscr{C}(\mathbb{R}, H^1 \cap L^q) \quad \textit{with} \quad 1/p + 2/q = 1 \tag{3.42}$$

and the Eq. (2.9). *Then the operator $P_{\nu\kappa}\tilde{W}(t, s; a)$ defined above is strongly differentiable in $\mathscr{H}$ and its derivative is given by*

$$\begin{aligned} i\frac{d}{dt} P_{\nu\kappa}\tilde{W}(t, s; a) &= P_{\nu\kappa}(\tilde{H}_2(t, a) - H_0(a) + \hbar^{1/2}\tilde{H}_3(t, a)\\ &\qquad + \hbar\tilde{H}_4(t, a))\tilde{W}(t, s; a)\,.\end{aligned} \tag{3.43}$$

Proof. We first remark that under the assumptions made on V and φ, the integrand in the R.H.S. of (2.9) lies in $\mathscr{C}(\mathbb{R}, L^2)$. Therefore the integral is well defined as a strong Riemann integral in L^2 and the Eq. (2.9) makes sense. Furthermore $\tilde{\varphi}$ satisfies the equation

$$\tilde{\varphi}(t)=\tilde{\varphi}(t_0)-i\int_{t_0}^{t} d\tau\, u_0(-\tau)\{\varphi(\tau)(V*|\varphi(\tau)|^2)\}, \tag{3.44}$$

from which it follows that $\tilde{\varphi}\in\mathscr{C}^1(\mathbb{R}, L^2)$ and that

$$i\dot{\tilde{\varphi}}(t)=u_0(-t)\{\varphi(t)(V*|\varphi(t)|^2)\}. \tag{3.45}$$

In the rest of this proof we drop the a dependence in all the operators. It follows from Lemma 3.3 and from (3.42) that the R.H.S. of (3.43) is a bounded operator in $\mathscr{H}$ and is strongly continuous with respect to t. Therefore it is sufficient to prove (3.43) on a dense subset $\mathscr{D}$ of $\mathscr{H}$. We take

$$\mathscr{D}=\{\Psi: U_0(s)C(\tilde{\varphi}_\hbar(s))\Psi\in\mathscr{D}(H_\hbar)\cap\mathscr{C}_0(N)\}. \tag{3.46}$$

Using the facts that $U(t-s)$ is strongly differentiable on $\mathscr{D}(H_\hbar)$, that $\mathscr{D}(H_\hbar)\subset Q(H_\hbar)\subset Q(H_0)$, that $U_0(t)$ is strongly differentiable from $Q(H_0)$ into $Q^*(H_0)$, that $C(\tilde{\varphi}_\hbar(t))$ is bounded on $Q^*(H_0)$ and strongly differentiable from $\mathscr{C}_0(N)$ into $\mathscr{H}$ (by Lemma 3.1), that $P_{\nu\kappa}$ is bounded from $Q^*(H_0)$ to $\mathscr{H}$ [because $|k|\varrho_1(k)$ is bounded], we see that, for any $\Psi\in\mathscr{D}$, $P_{\nu\kappa}\tilde{W}(t,s)\Psi$ is differentiable, and that its time derivative is given by:

$$\begin{aligned} i\frac{d}{dt}P_{\nu\kappa}\tilde{W}(t,s)\Psi=P_{\nu\kappa}C(\tilde{\varphi}_\hbar(t))^*\{&U_0(t)^*(H_0+\hbar H_4)\\ &-H_0U_0(t)^*-i[a^*(\dot{\tilde{\varphi}}_\hbar(t))-a(\dot{\bar{\tilde{\varphi}}}_\hbar(t))+i\,\mathrm{Im}\langle\tilde{\varphi}_\hbar(t),\dot{\tilde{\varphi}}_\hbar(t)\rangle]\\ &\cdot U_0(t)^*+\hbar^{-1}H_4(\varphi(t))U_0(t)^*\}U(t-s)U_0(s)C(\tilde{\varphi}_\hbar(s))\Psi. \end{aligned} \tag{3.47}$$

We are left with the task of writing (3.47) in the form of (3.43). This results from an algebraic computation to be performed below using the Eq. (3.45). This computation amounts to writing the same operator in two ways as the sum of various terms and is justified by the fact that all terms involved are well defined as bounded operators in $\mathscr{H}$. Using (3.45) and (1.49) we obtain in the R.H.S. of (3.47)

$$\begin{aligned} &P_{\nu\kappa}U_0(t)^*\{C(\varphi_\hbar(t))^*\hbar H_4C(\varphi_\hbar(t))-\hbar^{-1/2}a^*(\varphi(t)(V*|\varphi(t)|^2))\\ &-\hbar^{-1/2}a(\bar{\varphi}(t)(V*|\varphi(t)|^2))-\hbar^{-1}\langle\varphi(t),\varphi(t)(V*|\varphi(t)|^2)\rangle\\ &+\hbar^{-1}H_4(\varphi(t))\}U_0(t)\tilde{W}(t,s)\Psi. \end{aligned}$$

(3.43) then follows from (1.16), (1.8), (1.63), (1.66), (3.15)–(3.24), the fact that $\langle\varphi,\varphi(V*|\varphi|^2)\rangle=2H_4(\varphi)$ and the definition (3.41). Q.E.D.

We conclude this section with some comments on the limit of $\tilde{W}(t,s;a)$ when t and/or s tend to $\pm\infty$. We first consider the operators $U_0(t;a)^*U(t-s;a)U_0(s;a)$. The existence of their strong limits as $s\to\pm\infty$ is equivalent to the existence of the wave operators

$$\Omega_\pm(a)=\underset{s\to\pm\infty}{s\text{-lim}}\;U(-s;a)U_0(s;a). \tag{3.48}$$

By an easy extension of standard arguments one proves that the latter exist if

$$\begin{cases} V\in L^{p_1}+L^{p_2} \quad \text{with} \quad 1\leqq p_2\leqq p_1<n\,, \\ V_-\in L^{\mathrm{Max}(p_1,n/2)}+L^{n/2}\,. \end{cases} \tag{3.49}$$

On the other hand the existence of the limit as $t\to\pm\infty$ is a much harder problem since it is equivalent to asymptotic completeness. It can be expected to hold only if, for all N, the N-particle system behaves as a free system for large time, for instance if the interaction is repulsive in some sense. If V is positive and $V(\lambda x)$ is decreasing in $\lambda\in\mathbb{R}^+$ for all x, it can be proved [9] that for each N the spectrum of $H_{\hbar N}$ is absolutely continuous and coincides with $[0,\infty)$. Asymptotic completeness follows from similar but stronger assumptions [9]. It is likely and it would be desirable to prove that asymptotic completeness holds under the repulsivity condition of Sect. 2.

Finally the operator $C(a,\tilde{\varphi}_\hbar(t))$ converges strongly to $C(a,\varphi_{\hbar\pm})$ when $t\to\pm\infty$ provided $\tilde{\varphi}_\hbar(t)$ converges to $\varphi_{\hbar\pm}$ in L^2. This convergence as well as the existence of the limit for the phase $\omega_\hbar(t,s)$ as t and/or s tend to $\pm\infty$ hold for all dispersive solutions of (2.9) in the sense of Sect. 2 for potentials V satisfying the assumptions of Proposition 2.2.

4. The Quantum Fluctuations

In this section we give a precise definition and study the properties of the unitary group of operators $\tilde{U}_2(t,s;a)$ formally defined by (1.41) and (1.22). As explained in Sect. 1, this group describes the evolution of the quantum fluctuations around the classical solution. The main problem consists in solving the Eq. (1.38), namely an evolution equation with time-dependent generator. There is an important literature on this subject (see for instance [7]), but it is simpler to give a direct treatment taking full advantage of the special features of the present case. We follow the usual method, namely we first define a truncated unitary group and then we obtain $\tilde{U}_2(t,s;a)$ from it by a limiting procedure. In all this section we shall drop the dependence of the various operators on a. We shall need the spaces $\mathscr{H}^\delta$, $\delta\in\mathbb{R}$, defined by $\mathscr{H}^\delta=Q(N^\delta)$ for $\delta\geqq 0$ and by $\mathscr{H}^\delta=Q^*(N^{|\delta|})$ for $\delta\leqq 0$, with norms

$$\|\Phi\|_\delta=\|(1+N)^{\delta/2}\Phi\|\,. \tag{4.1}$$

We denote by $\mathscr{B}(\delta,\delta')$ the space of bounded operators from $\mathscr{H}^\delta$ to $\mathscr{H}^{\delta'}$ and by $\|\cdot\|_{\delta,\delta'}$ the norm in $\mathscr{B}(\delta,\delta')$.

We first derive some properties of the generator of $\tilde{U}_2(t,s)$

$$A(t)\equiv\tilde{H}_2(t)-H_0=\tilde{G}(t)+\tilde{K}(t)+\tilde{L}(t)+\tilde{L}^*(t)\,, \tag{4.2}$$

where $\tilde{H}_2(t)$ is defined by (3.41) and $\tilde{G}(t)$, $\tilde{K}(t)$, $\tilde{L}(t)$ and $\tilde{L}^*(t)$ are defined similarly. We introduce the self-adjoint approximants

$$A_\nu(t)=\sigma_\nu A(t)\sigma_\nu \tag{4.3}$$

for any integer $\nu\geqq 1$, where σ_ν is the particle number cut-off operator defined in Sect. 3.

Lemma 4.1. *Let $V \in L^\infty + L^p$ with $p \geqq 2$ and let φ satisfy* (3.39). *Then:*

1) For any $\delta \in \mathbb{R}$, $A(t)$ belongs to $\mathscr{B}(\delta+2, \delta)$ and is norm continuous as a function of t.

2) $A_\nu(t)$ satisfies 1) with bounds uniform in ν. Furthermore, for any $\delta \in \mathbb{R}$, $A_\nu(t)$ belongs to $\mathscr{B}(\delta, \delta)$ and is norm continuous as a function of t.

3) For any $\delta \in \mathbb{R}$, $A_\nu(t)$ tends to $A(t)$ as $\nu \to \infty$, in norm in $\mathscr{B}(\delta+2+\varepsilon, \delta)$, $\varepsilon > 0$, and strongly in $\mathscr{B}(\delta+2, \delta)$, uniformly for t in a compact interval.

Proof.

1) It follows from Lemma 3.2 and an argument similar to the proof of Corollary 3.1 that

$$\|\tilde{G}(t) + \tilde{K}(t)\|_{\delta+2,\delta} \leqq \|g(t)\|_\infty + c(t), \tag{4.4}$$

$$\|\tilde{L}(t) + \tilde{L}^*(t)\|_{\delta+2,\delta} \leqq (1 + 3^{|\delta+1|/2}) c(t), \tag{4.5}$$

where

$$g(t) = V * |\varphi(t)|^2, \tag{4.6}$$

and $c(t)$ is the continuous function defined by [see (3.29) and (3.30)]:

$$c(t) = \{\int dx |\varphi(t)|^2 (V^2 * |\varphi(t)|^2\}^{1/2}. \tag{4.7}$$

This proves boundedness. Continuity is proved in the same way.

2) Follows from 1) and the fact that $\sigma_\nu \in \mathscr{B}(\delta, \delta')$ for any δ and δ'.

3) Strong convergence of $A_\nu(t)$ to $A(t)$ in $\mathscr{B}(\delta+2, \delta)$ follows from the obvious strong convergence on $\mathscr{C}_0(N)$ and the fact that $A_\nu(t)$ is bounded in $\mathscr{B}(\delta+2, \delta)$ uniformly in ν. Norm convergence in $\mathscr{B}(\delta+2+\varepsilon, \delta)$ follows from the fact that $(\mathbb{1} - \sigma_\nu)(1+N)^{-\varepsilon}$ tends to zero in norm as an operator in $\mathscr{H}$. Q.E.D.

We can now define the unitary group $\tilde{U}_{2,\nu}(t,s)$ by the series

$$\tilde{U}_{2,\nu}(t,s) = \sum_{m=0}^{\infty} (-i)^m \int_s^t dt_1 \int_s^{t_1} dt_2 \dots \int_s^{t_{m-1}} dt_m A_\nu(t_1) \dots A_\nu(t_m). \tag{4.8}$$

By Lemma 4.1 the series (4.7) converges in norm in $\mathscr{B}(\delta, \delta)$ and $\tilde{U}_{2,\nu}(t,s)$ is norm continuous and norm differentiable with respect to t in $\mathscr{B}(\delta, \delta)$ for all $\delta \in \mathbb{R}$. The operators $\tilde{U}_{2,\nu}(t,s)$ satisfy the following crucial boundedness property.

Lemma 4.2. *Let V and φ satisfy the assumptions of Lemma* 4.1. *Then $\tilde{U}_{2,\nu}(t,s)$ is bounded in $\mathscr{H}^\delta$ uniformly in ν for all $\delta \in \mathbb{R}$. More precisely.*

$$\|\tilde{U}_{2,\nu}(t,s)\|_{\delta,\delta} \leqq \exp\left(\left|\delta 2^{|\delta|} \int_s^t d\tau c(\tau)\right|\right), \tag{4.9}$$

where $c(t)$ is given by (4.7).

Proof. It is sufficient to consider the case $\delta \geqq 0$. The case $\delta < 0$ will be obtained by duality. Let $\Psi \in \mathscr{H}$. We want to estimate

$$M(t,s) \equiv \|(N+1)^{\delta/2} \tilde{U}_{2,\nu}(t,s)(N+1)^{-\delta/2} \Psi\|^2. \tag{4.10}$$

Now $M(s,s)=\|\Psi\|^2$ and

$$i\frac{d}{dt}M(t,s)=\langle\tilde{U}_{2,\nu}(t,s)(N+1)^{-\delta/2}\Psi,$$
$$\sigma_\nu[(N+1)^\delta,\tilde{L}(t)+\tilde{L}^*(t)]\sigma_\nu\tilde{U}_{2,\nu}(t,s)(N+1)^{-\delta/2}\Psi\rangle. \tag{4.11}$$

By an argument similar to that in the proof of Corollary 3.1 one obtains

$$\pm i[(N+1)^\delta,\tilde{L}(t)+\tilde{L}^*(t)]\leqq\delta c(t)(N+2)^\delta$$
$$+(\delta c(t)N^\delta)^{-1}((N+1)^\delta-(N-1)^\delta)^2\tilde{L}^*(t)\tilde{L}(t) \tag{4.12}$$
$$\leqq\delta c(t)\{(N+2)^\delta+(4\delta^2N^\delta)^{-1}N(N-1)((N+1)^\delta-(N-1)^\delta)^2\} \tag{4.13}$$

by Lemma 3.2,

$$\ldots\leqq 2\delta 2^\delta c(t)(N+1)^\delta$$

by elementary estimates.

Therefore

$$\left|\frac{d}{dt}M(t,s)\right|\leqq 2\delta 2^\delta c(t)M(t,s) \tag{4.14}$$

which yields

$$M(t,s)\leqq\|\Psi\|^2\exp\left(2\delta 2^\delta\left|\int_s^t d\tau c(\tau)\right|\right).\quad \text{Q.E.D.} \tag{4.15}$$

We are now in a condition to define the unitary group $\tilde{U}_2(t,s)$ and to derive its main properties.

Proposition 4.1. *Let V and φ satisfy the assumptions of Lemma 4.1. Then there is a unique group of operators $\tilde{U}_2(t,s)$ satisfying the following properties:*

1) For any $\delta\in\mathbb{R}$, $\tilde{U}_2(t,s)$ is bounded and strongly continuous with respect to t,s in $\mathscr{H}^\delta$ and satisfies

$$\|\tilde{U}_2(t,s)\|_{\delta,\delta}\leqq\exp\left(\left|\delta 2^\delta\int_s^t d\tau c(\tau)\right|\right). \tag{4.16}$$

2) $\tilde{U}_2(t,s)$ is unitary in $\mathscr{H}$.

3) For any $\delta\in\mathbb{R}$, $\tilde{U}_2(t,s)$ is strongly differentiable from $\mathscr{H}^{\delta+2}$ to $\mathscr{H}^\delta$ and

$$i\frac{d}{dt}\tilde{U}_2(t,s)=(\tilde{H}_2(t)-H_0)\tilde{U}_2(t,s). \tag{4.17}$$

In particular it is strongly differentiable from $\mathscr{D}(N)$ to $\mathscr{H}$.

Proof.

1) For any positive integers μ and ν,

$$\tilde{U}_{2,\mu}(t,s)-\tilde{U}_{2,\nu}(t,s)=-i\int_s^t d\tau\tilde{U}_{2,\nu}(t,\tau)(A_\mu(\tau)-A_\nu(\tau))\tilde{U}_{2,\mu}(\tau,s) \tag{4.18}$$

as a Riemann integral in norm in $\mathscr{B}(\delta, \delta)$ for any δ. It follows then from part 2) of Lemma 4.1 and from (4.9) that

$$\begin{aligned} &\|\tilde{U}_{2,\mu}(t,s) - \tilde{U}_{2,\nu}(t,s)\|_{\delta+2+\varepsilon,\delta} \\ &\leqq |t-s| \exp\left(\gamma \left|\int_s^t d\tau c(\tau)\right|\right) \sup_{\tau\in[s,t]} \|A_\mu(\tau) - A_\nu(\tau)\|_{\delta+2+\varepsilon,\delta}, \end{aligned} \tag{4.19}$$

where γ is a constant related to δ. Therefore by part 3) of Lemma 4.1, for any $\delta \in \mathbb{R}$, $\tilde{U}_{2,\nu}(t,s)$ converges in norm in $\mathscr{B}(\delta+2+\varepsilon, \delta)$ as $\nu \to \infty$ uniformly for t, s in a compact set. The limit $\tilde{U}_2(t,s)$ is clearly norm continuous in $\mathscr{B}(\delta+2+\varepsilon, \delta)$ with respect to t, s. The previous convergence and the uniform bound (4.9) imply strong convergence of $\tilde{U}_{2,\nu}(t,s)$ to $\tilde{U}_2(t,s)$ in $\mathscr{B}(\delta, \delta)$ uniformly for t, s in a compact set. As a consequence $\tilde{U}_2(t,s)$ satisfies the bound (4.16) and is strongly continuous in t, s. This proves part 1).

2) Part 2) follows from the unitarity of $\tilde{U}_{2,\nu}(t,s)$ in $\mathscr{H}$ and from the strong convergence of $\tilde{U}_{2,\nu}(t,s)$ and of $\tilde{U}_{2,\nu}(t,s)^* = \tilde{U}_{2,\nu}(s,t)$.

3) In order to prove part 3) we write

$$\tilde{U}_{2,\nu}(t,s)\Psi = \Psi - i\int_s^t d\tau A_\nu(\tau)\tilde{U}_{2,\nu}(\tau,s)\Psi \tag{4.20}$$

as a strong Riemann integral in $\mathscr{H}^\delta$ for $\Psi \in \mathscr{H}^{\delta+2}$. By part 3) of Lemma 4.1 and by the previous convergence we can take the limit $\nu \to \infty$ in (4.20). The result then follows from part 1) of Lemma 4.1 and from part 1) of this proposition. Q.E.D.

We finally study the limit of $\tilde{U}_2(t,s)$ as t and/or s tend to $\pm\infty$.

Proposition 4.2. *Let V and φ satisfy the assumptions of Lemma* 4.1 *and let the function $c(\cdot)$ defined by* (4.7) *be integrable at $+\infty$ (resp. at $-\infty$, resp. in $\mathbb{R}$). Then*

1) For all $\delta \in \mathbb{R}$, $\tilde{U}_2(t,s)$ is bounded in $\mathscr{H}^\delta$ uniformly in t, s for t, s in $\mathbb{R}^+$ (resp. $\mathbb{R}^-$, resp. $\mathbb{R}$).

2) Assume in addition that the function $\|g(t)\|_\infty$ defined by (4.6) *is integrable at $+\infty$ (resp. $-\infty$, resp. in $\mathbb{R}$). Then $\tilde{U}_2(t,s)$ has norm limits in $\mathscr{B}(\delta+2, \delta)$ and therefore strong limits in $\mathscr{B}(\delta, \delta)$ for all $\delta \in \mathbb{R}$ when t and/or s tend to $+\infty$ (resp. $-\infty$, resp. $\pm\infty$).*

Proof. Part 1) follows from (4.16). In order to prove part 2) we write

$$\{\tilde{U}_2(t,s) - \tilde{U}_2(t',s)\}\Psi = -i\int_{t'}^t d\tau A(\tau)\tilde{U}_2(\tau,s)\Psi \tag{4.21}$$

as a strong Riemann integral in $\mathscr{H}^\delta$ for $\Psi \in \mathscr{H}^{\delta+2}$. Then by (4.4), (4.5), and (4.16),

$$\begin{aligned} &\|\tilde{U}_2(t,s) - \tilde{U}_2(t',s)\|_{\delta+2,\delta} \\ &\leqq \left|\int_{t'}^t d\tau (\|g(\tau)\|_\infty + \gamma c(\tau)) \exp\left(\gamma \left|\int_s^\tau d\tau' c(\tau')\right|\right)\right|, \end{aligned} \tag{4.22}$$

where γ is a constant related to δ. Part 2) follows immediately. Q.E.D.

In the case where the relevant limits exist, as described in the previous proposition, we define the wave operators and the S matrix for the quantum fluctuations by

$$\Omega_{2\pm} = \operatorname*{s-lim}_{s\to\pm\infty} \tilde{U}_2(0,s), \tag{4.23}$$

$$S_2 = \operatorname*{s-lim}_{\substack{s\to-\infty \\ t\to+\infty}} \tilde{U}_2(t,s). \tag{4.24}$$

In particular, if $c(\cdot)$ and $\|g(\cdot)\|_\infty$ are integrable in $\mathbb{R}$, the quantum fluctuation system is asymptotically complete. The integrability condition holds for dispersive solutions in the sense of Sect. 2 and for suitable potentials.

5. The $\hbar \to 0$ Limit

In this section we prove the main result of this paper, namely the strong convergence of $\tilde{W}(t,t_0;\tilde{a}_0)$ to $\tilde{U}_2(t,t_0;\tilde{a}_0)$ uniformly in t and t_0 for fixed $\tilde{a}_0$ [see (1.51)]. In Sect. 1, $\tilde{a}_0$ was the initial condition at time t_0 common to the Eqs. (1.37) and (1.38). However it is clear from (1.55) that the definition of $\tilde{W}(t,t_0;\tilde{a}_0)$ does not imply any relation between t_0 and $\tilde{a}_0$ and does not contain any reference to the role of t_0 as initial time. We shall therefore consider $\tilde{W}(t,s;\tilde{a}_0)$ as defined by (1.55), for any t and s belonging to $\mathbb{R}$, in the Fock space associated with the $\tilde{a}_0$'s. From now on, we shall replace the notation $\tilde{a}_0$ by the simpler one a. In particular we shall freely use the results of Sects. 3 and 4, already written with this notation. The same remarks apply to $\tilde{U}_2(t,t_0;\tilde{a}_0)$.

The basic tool of the proof is the Duhamel formula between $\tilde{W}(t,s;a)$ and $\tilde{U}_2(t,s;a)$ as in [6]. However two complications arise. 1) For the potentials we want to consider, the derivative in the Duhamel formula is too singular and we must regularize it by the use of the operators $P_{\nu\kappa}$. This leads to strong convergence for finite times, stated in Proposition 5.1. 2) Uniformity in time is not apparent at this stage. In order to make it explicit we need to differentiate a second time. This leaves us with two integrals on time variables. One is a simple integral similar to the one encountered in the usual existence proof of the wave operators in linear scattering and is controlled by the space decay of the wave function and the dispersive properties of the free evolution. The second one is a double integral over two time variables. One integral is controlled by the dispersive properties of the free evolution and the other one by the time decay of the classical solution. The main result is stated in Theorem 5.1 at the end of this section, and most of the latter is devoted to its proof.

Before starting the proof we introduce some more notation. We denote by $|||\cdot|||_1$ and $|||\cdot|||_2$ the norm of bounded operators in the one- and two-particle spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. Let f_1 and f_2 be operators in $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. We define the second quantized operators $\Gamma_1(f_1)$ and $\Gamma_2(f_2)$ by

$$\Gamma_1(f_1) = \bigoplus_{N=1}^{\infty} \sum_{1\leqq i\leqq N} f_1(i), \tag{5.1}$$

$$\Gamma_2(f_2) = \bigoplus_{N=2}^{\infty} \sum_{1\leqq i<j\leqq N} f_2(i,j). \tag{5.2}$$

For instance we shall consider V as the operator in $\mathscr{H}_2$ defined by $(V\psi)(x_1, x_2) = V(x_1 - x_2)\psi(x_1, x_2)$. Then $H_4 = \Gamma_2(V)$. If A is an operator in $\mathscr{H}$ commuting with N we shall use the notation $\Gamma_2(A)$ for $\Gamma_2(A_2)$ where A_2 is the restriction of A to $\mathscr{H}_2$. Let b be the function.

$$b(x) = (1 + x^2)^\beta \tag{5.3}$$

with $\beta > 1$. We denote by b_1 the operator of multiplication by b in $\mathscr{H}_1$ and by b_2 the operator in $\mathscr{H}_2$ defined by

$$(b_2\psi)(x_1, x_2) = b(x_1 - x_2)\psi(x_1, x_2)\,. \tag{5.4}$$

We define $B_1 = \Gamma_1(b_1)$ and $B_2 = \Gamma_2(b_2)$. We shall write explicitly the x dependence coming from φ in $g(t; x)$, $k(t; x, y)$ and $l(t; x, y)$ defined by (3.19)–(3.21). The space variables will be omitted when confusion does not arise. In addition we shall use the notation $g(t)$ and $k(t)$ for the operators in $\mathscr{H}_1$ defined respectively as the multiplication by $g(t)$ and as the integral operator with kernel $k(t)$. For brevity, in all this section we shall omit the dependence on a of the various operators under study, the subscripts ν and κ in σ_ν, ϱ_κ, $\varrho_{\kappa *}$, R_κ and $P_{\nu\kappa}$ and, most of the time, the variable s.

We now begin the proof of (1.51). In all this section we suppose that V and φ satisfy the following conditions:

$$\begin{cases} V \in L^{p_1} + L^{p_2} \quad \text{with} \quad 2 \leqq p_2 \leqq p_1 < \infty\,, \\ V_- \in L^{\mathrm{Max}(p_1, n/2)} + L^{n/2}\,, \end{cases} \tag{5.5}$$

$$\varphi \in \mathscr{C}(\mathbb{R}, H^1 \cap L^{q_2}) \quad \text{with} \quad 1/p_2 + 2/q_2 = 1\,, \tag{5.6}$$

and that φ satisfies the Eq. (2.9). We shall write $V = V_1 + V_2$ with $V_i \in L^{p_i}$, $i = 1, 2$. These assumptions are slightly stronger than those of Proposition 3.1 and will not be repeated in the intermediate lemmas. We take a fixed $\Psi \in Q(NB_1)$ and define

$$\begin{cases} \Psi_1(t) = \tilde{W}(t, s)\Psi \\ \Psi_2(t) = \tilde{U}_2(t, s)\Psi\,, \end{cases} \tag{5.7}$$

where the s dependence is omitted, as said above, in the L.H.S. of (5.7). We start estimating the difference

$$\begin{aligned} \|\Psi_1(t) - \Psi_2(t)\|^2 &= 2\,\mathrm{Re}\langle \Psi, (\mathbb{1} - \tilde{W}(t,s)^* \tilde{U}_2(t,s))\Psi\rangle \\ &= 2\langle \Psi, (\mathbb{1} - P)\Psi\rangle - 2\,\mathrm{Re}\langle \Psi_1(t), (\mathbb{1} - P)\Psi_2(t)\rangle \\ &\quad - 2\,\mathrm{Re}\left\langle \Psi, \int_s^t d\tau \frac{d}{d\tau}(P\tilde{W}(\tau,s))^* \tilde{U}_2(\tau,s)\Psi \right\rangle = \sum_{i=0}^4 J_i\,, \end{aligned} \tag{5.8}$$

where

$$J_0 = 2\langle \Psi, (\mathbb{1} - P)\Psi\rangle\,, \tag{5.9}$$

$$J_1 = -2\,\mathrm{Re}\langle \Psi_1(t), (\mathbb{1} - P)\Psi_2(t)\rangle\,, \tag{5.10}$$

$$J_2 = 2\,\mathrm{Im}\int_s^t d\tau \langle \Psi_1(\tau), [\tilde{H}_2(\tau), P]\Psi_2(\tau)\rangle\,, \tag{5.11}$$

$$J_3 = 2\,\mathrm{Im} \int_s^t d\tau \langle \Psi_1(\tau), \hbar^{1/2} \tilde{H}_3(\tau) P \Psi_2(\tau) \rangle , \tag{5.12}$$

$$J_4 = 2\,\mathrm{Im} \int_s^t d\tau \langle \Psi_1(\tau), \hbar \tilde{H}_4(\tau) P \Psi_2(\tau) \rangle . \tag{5.13}$$

In writing (5.8) we have used Proposition 3.1 and part 3) of Proposition 4.1. We want to prove that the J_i's tend to zero when $\hbar \to 0$ uniformly in t and s. We consider them successively. For definiteness, we assume $s \leqq t$. By (3.37), J_0 tends to zero when $\nu, \kappa \to \infty$; J_0 does not depend on t, s. We next consider J_1.

Lemma 5.1. *J_1 tends to zero as $\nu, \kappa \to \infty$ uniformly for t, s in a compact set. If in addition $c(\cdot)$ (defined by (4.7)) is integrable at $+\infty$ (resp. $-\infty$, resp. in $\mathbb{R}$), then the limit is uniform with respect to t in $\mathbb{R}^+$ (resp. $\mathbb{R}^-$, resp. $\mathbb{R}$).*

Proof. We write

$$|J_1| \leqq 2 \|\Psi\| \, \|(\mathbb{1} - P) \tilde{U}_2(t,s) \Psi\| . \tag{5.14}$$

The last norm in the R.H.S. of (5.14) is a decreasing function of ν, κ and tends to zero when $\nu, \kappa \to \infty$ for fixed t, s. It is continuous in t, s for $t, s \in \mathbb{R}$ and, under the integrability condition on $c(\cdot)$ for $t, s \in \mathbb{R} \cup \{+\infty\}$ (resp. $\mathbb{R} \cup \{-\infty\}$, resp. $\mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$) by Proposition 4.2. The result then follows from Dini's theorem. Q.E.D.

We now turn to J_2.

Lemma 5.2. *J_2 satisfies the estimate*

$$|J_2| \leqq 2 \|\Psi\| \int_s^t d\tau \{ 2\nu^{-1} \|\sigma_1'\|_\infty c(\tau) + |||[g(\tau), \varrho_*]|||_1 + 4 M_2(\tau) \} \, \|(N+1) \Psi_2(\tau)\| , \tag{5.15}$$

where σ_1 is the particle number cut-off function, σ_1' its derivative, $c(\cdot)$ is defined by (4.7) and $M_2(\tau)$ by

$$M_2(\tau)^2 \equiv \int dx\, dy \left| \int dx' \varrho(x - x') (k(\tau; x, y) - k(\tau; x', y)) \right|^2 . \tag{5.16}$$

Proof. We write

$$|J_2| \leqq 2 \|\Psi\| \int_s^t d\tau \| [\tilde{H}_2(\tau), P] \Psi_2(\tau) \| . \tag{5.17}$$

Now

$$[\tilde{H}_2(\tau), P] = [\tilde{H}_2(\tau), \sigma] R + \sigma [\tilde{H}_2(\tau), R] \tag{5.18}$$

and

$$\begin{aligned} [\tilde{H}_2(\tau), \sigma] &= [\tilde{L}(\tau) + \tilde{L}^*(\tau), \sigma] \\ &= (\sigma(N+2) - \sigma(N)) \tilde{L}(\tau) + (\sigma(N-2) - \sigma(N)) \tilde{L}^*(\tau) . \end{aligned}$$

Using the estimate

$$|||\sigma(N \pm 2) - \sigma(N)||| \leqq 2\nu^{-1} \|\sigma_1'\|_\infty ,$$

Lemma 3.2 [see especially (3.27) and (3.28)] and the fact that R, N, and U_0 commute, we obtain easily

$$\|[\tilde{H}_2(\tau), \sigma] R \Psi_2(\tau)\| \\ \leqq 2\nu^{-1}\|\sigma_1'\|_\infty c(\tau)\|(N+1)\Psi_2(\tau)\| . \tag{5.19}$$

We consider next $[\tilde{H}_2(\tau), R]$. Obviously

$$\|[\tilde{G}(\tau)+\tilde{K}(\tau), R]\Psi_2(\tau)\| \\ \leqq |||[g(\tau)+k(\tau), \varrho_*]|||_1 \|N\Psi_2(\tau)\| . \tag{5.20}$$

Furthermore

$$|||[k(\tau), \varrho_*]|||_1 \leqq 2\|k(\tau)(\mathbb{1}-\varrho_*)\|_{\mathrm{HS}} = 2M_2(\tau), \tag{5.21}$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm in $\mathscr{H}_1$ and $M_2(\tau)$ is defined by (5.16).

On the other hand one obtains easily

$$\|[\tilde{L}(\tau)+\tilde{L}^*(\tau), R]\Psi_2(\tau)\| \\ \leqq 2^{1/2}\|(1-R)L^*(\tau)\Psi_0\| \; \|(N+1)\Psi_2(\tau)\|$$

by an argument similar to the proof of Lemma 3.2,

$$\ldots \leqq 2M_2(\tau)\|(N+1)\Psi_2(\tau)\| . \tag{5.22}$$

Inserting the estimates (5.19)–(5.22) into (5.17) and (5.18) yields (5.15). Q.E.D.

We next estimate $|||[g(\tau), \varrho_*]|||_1$ and $M_2(\tau)$ in the following lemma.

Lemma 5.3.

$$|||[g(\tau), \varrho_*]|||_1 \leqq \sum_{i=1,2} \|\varphi(\tau)\|_{q_i}^2 \int dy \varrho(y)\|V_i - V_{iy}\|_{p_i}, \tag{5.23}$$

$$M_2(\tau) \leqq \sum_{i=1,2} \{\|\varphi(\tau) - \varrho * \varphi(\tau)\|_{q_i}\|\varphi(\tau)\|_{q_i}\|V_i\|_{p_i} \\ + \|\varphi(\tau)\|_{q_i}^2 \int dy \varrho(y)\|V_i - V_{iy}\|_{p_i}\}, \tag{5.24}$$

where $V_{iy}(x) = V_i(x-y)$ and $1/p_i + 2/q_i = 1$, $i = 1, 2$.

Proof. In the proof we omit the variable τ. Let $\psi \in \mathscr{H}_1$. Then

$$[g, \varrho_*]\psi(x) = \int dy \varrho(y)\psi(x-y)(g(x)-g(x-y))$$

from which (5.23) follows by Hölder's inequality.

On the other hand

$$M_2 = \|\int dx' \varrho(x-x')(\varphi(x)V(x-y) - \varphi(x')V(x'-y))\bar{\varphi}(y)\|_2 ,$$

where $\|\cdot\|_2$ denotes the norm in $L^2(dx\,dy)$

$$\leqq \|(\varphi - \varrho * \varphi)(x)V(x-y)\bar{\varphi}(y)\|_2 \\ + \int dy' \varrho(y')\|\varphi_{y'}(x)(V - V_{y'})(x-y)\bar{\varphi}(y)\|_2 ,$$

where $\varphi_{y'}(x)=\varphi(x-y')$. From this (5.24) follows by Hölder's and Young's inequalities. Q.E.D.

Corollary 5.1.

1) J_2 *tends to zero when* $v,\kappa\to 0$ *uniformly in* t,s *in a compact set.*

2) *If in addition* $\|\varphi(\tau)\|_{q_i}$ *and* $\|\varphi(\tau)-\varrho*\varphi(\tau)\|_{q_i}$ *belong to* $L^2(\mathbb{R}^+,d\tau)$ *(resp.* $L^2(\mathbb{R}^-,d\tau)$, *resp.* $L^2(\mathbb{R},d\tau)$*) for* $i=1,2$, *and if* $\int_{\mathbb{R}^+(\text{resp. }\mathbb{R}^-,\text{resp.}\mathbb{R})} d\tau\|\varphi(\tau)\|_{q_i}\|\varphi(\tau)-\varrho*\varphi(\tau)\|_{q_i}$ *tends to zero when* $\kappa\to\infty$ *for* $i=1,2$, *then the limit* $J_2\to 0$ *is uniform for* $t,s\in\mathbb{R}^+$ *(resp.* $\mathbb{R}^-$, *resp.* $\mathbb{R}$*).*

Proof. Convergence and uniformity for finite times follow from Lemmas 5.1 and 5.2 by a simple application of Lebesgue's dominated convergence theorem. Uniformity for $t,s\in\mathbb{R}$ is obvious after noticing that

$$c(\tau)\leqq\sum_{i=1,2}\|V_i\|_{p_i}\|\varphi(\tau)\|_{q_i}^2. \quad \text{Q.E.D.} \tag{5.25}$$

We next turn to J_3 and J_4. In a first step we write .

$$J_3\leqq 2\hbar^{1/2}\|\Psi\|\int_s^t d\tau\|\tilde{H}_3(\tau)P\Psi_2(\tau)\|, \tag{5.26}$$

$$J_4\leqq 2\hbar\|\Psi\|\int_s^t d\tau\|\tilde{H}_4(\tau)P\Psi_2(\tau)\|, \tag{5.27}$$

and we note that in (5.26) and (5.27) $\hbar$ appears only in the explicit factors $\hbar^{1/2}$ and $\hbar$ while v and κ occur in P. By part 2) of Lemma 3.3, the integrands in (5.26) and (5.27) are continuous functions of τ. Therefore, in order to prove that $\|\Psi_1(t)-\Psi_2(t)\|$ tends to zero when $\hbar\to 0$, we can proceed as follows. By Lemma 5.1 and Corollary 5.1, we can make J_1 and J_2 arbitrarily small by choosing v and κ large enough. We can then make J_3 and J_4 arbitrarily small by taking $\hbar$ sufficiently small for fixed v and κ. At the present stage, we can already prove the strong convergence (1.51) at finite times.

Proposition 5.1. *Let* V *satisfy* (5.5), *let* φ *satisfy* (5.6) *and the Eq.* (2.9). *Then the following strong limit exists*

$$\operatorname*{s-lim}_{\hbar\to 0}\tilde{W}(t,s)=\tilde{U}_2(t,s) \tag{5.28}$$

uniformly for t,s *in compact intervals.*

Proof. The result follows from Lemma 5.1, Corollary 5.1, part 2) of Lemma 3.3 and the preceding remarks. Q.E.D.

We are now left with the task of estimating the integrals in (5.26) and (5.27) uniformly in t,s. We consider first J_3. We define the function $g_1(\tau)$ by

$$g_1(\tau)=V^2*|\varphi(\tau)|^2 \tag{5.29}$$

Lemma 5.4. *The following estimate holds:*

$$H_3(\tau)^2 \leqq 4N^2 G_1(\tau). \tag{5.30}$$

Proof. In this proof we omit the τ dependence. Now by (3.23)

$$H_3^2 = (A_3 + A_3^*)^2 \leqq 2(A_3^* A_3 + A_3 A_3^*),$$

where A_3 and A_3^* are given by (3.24). Let Φ be a vector in $\mathscr{H}$ with finitely many particles and smooth wave functions. Let $X = (x_1, \ldots, x_N)$. Then

$$\begin{aligned} (A_3\Phi)(X) &= \int dx \sum_{1 \leqq i \leqq N} \bar{\varphi}(x) V(x - x_i) \Phi(X, x), \\ (A_3^*\Phi)(X) &= \sum_{1 \leqq i \neq j \leqq N} V(x_i - x_j) \varphi(x_i) \Phi(X \backslash x_i), \end{aligned} \tag{5.31}$$

so that

$$\|A_3 \Phi\|^2 \leqq \int dX \int dx \sum_i |\Phi(X, x)|^2 N \int dy |\varphi(y)|^2 V(x_i - y)^2$$

by Schwarz's inequality,

$$\ldots = \langle \Phi, (N-1)^2 G_1 \Phi \rangle. \tag{5.32}$$

Similarly

$$\begin{aligned} \|A_3^* \Phi\|^2 &\leqq \int dX\, N(N-1) \sum_{i \neq j} |V(x_i - x_j) \varphi(x_i) \Phi(X \backslash x_i)|^2 \\ &= \langle \Phi, N(N+1) G_1 \Phi \rangle. \end{aligned} \tag{5.33}$$

Taking the sum of (5.32) and (5.33), we obtain (5.30). Q.E.D.

It follows from Lemma 5.4. that

$$\begin{aligned} \|\tilde{H}_3(\tau) P \Psi_2(\tau)\| &\leqq \langle \Psi_2(\tau), 4N^2 \sigma^2 R \tilde{G}_1(\tau) R \Psi_2(\tau) \rangle^{1/2} \\ &\leqq 4v \langle \Psi_2(\tau), \tilde{G}_{1R}(\tau) \Psi_2(\tau) \rangle^{1/2}, \end{aligned} \tag{5.34}$$

where

$$\tilde{G}_{1(R)}(\tau) = U_0(\tau)^* G_{1(R)}(\tau) U_0(\tau), \tag{5.35}$$

$$G_{1R}(\tau) = \Gamma_1(g_{1R}(\tau)), \tag{5.36}$$

and

$$g_{1R}(\tau) = \varrho_* g_1(\tau) \varrho_*. \tag{5.37}$$

In order to estimate J_3 we introduce

$$M_3(\tau, \tau') = \langle \Psi_2(\tau'), \tilde{G}_{1R}(\tau) \Psi_2(\tau') \rangle^{1/2}, \tag{5.38}$$

so that

$$M_3(\tau, s) = \langle \Psi, \tilde{G}_{1R}(\tau) \Psi \rangle^{1/2}. \tag{5.39}$$

Then by (5.34)

$$\int_s^t d\tau \|\tilde{H}_3(\tau)P\Psi_2(\tau)\| \leqq 4v \int_s^t d\tau M_3(\tau,\tau)$$
$$\leqq 4v\left\{\int_s^t d\tau M_3(\tau,s) + \int_s^t d\tau \int_s^\tau d\tau' |\dot{M}_3(\tau,\tau')|\right\}, \quad (5.40)$$

where $\dot{M}_3(\tau,\tau') = dM_3(\tau,\tau')/d\tau'$. The quantity $M_3(\tau,s)$ is readily estimated by the following obvious lemma [we recall that $\Psi \in Q(NB_1) \subset Q(B_1)$]:

Lemma 5.5. *$M_3(\tau, s)$ satisfies the estimate*

$$M_3(\tau,s) \leqq \langle \Psi, B_1\Psi\rangle^{1/2} |||g_1(\tau)^{1/2}\varrho_* u_0(\tau) b_1^{-1/2}|||_1 . \quad (5.41)$$

We now estimate $\dot{M}_3(\tau,\tau')$, starting from the identity

$$2iM_3(\tau,\tau')\dot{M}_3(\tau,\tau')$$
$$= \langle \Psi_2(\tau'), [\tilde{G}_{1R}(\tau), \tilde{H}_2(\tau') - H_0]\Psi_2(\tau')\rangle \quad (5.42)$$

which follows from part 3) of Proposition 4.1.

Lemma 5.6. *$\dot{M}_3(\tau,\tau')$ satisfies the estimate*

$$|\dot{M}_3(\tau,\tau')| \leqq \langle \Psi_2(\tau'), (N+1)\Psi_2(\tau')\rangle^{1/2} \{|||g_1(\tau)^{1/2}\varrho_* u_0(\tau-\tau')g(\tau')|||_1$$
$$+ 2\|g_1(\tau)^{1/2}\varrho_* u_0(\tau-\tau')k(\tau')\|_{\mathrm{HS}}\} . \quad (5.43)$$

Proof. We estimate the contributions of the various terms of $\tilde{H}_2(\tau') - H_0$ to the R.H.S. of (5.42). The terms with $\tilde{G}$ and $\tilde{K}$ contribute

$$|\langle \Psi_2(\tau'), [\tilde{G}_{1R}(\tau), \tilde{G}(\tau') + \tilde{K}(\tau')]\Psi_2(\tau')\rangle|$$
$$= |\langle \Psi_2(\tau'), \Gamma_1([\tilde{g}_{1R}(\tau), \tilde{g}(\tau') + \tilde{k}(\tau')])\Psi_2(\tau')\rangle|$$
$$\leqq 2M_3(\tau,\tau')\langle \Psi_2(\tau'), \Gamma_1((\tilde{g}(\tau') + \tilde{k}(\tau'))\tilde{g}_{1R}(\tau)(\tilde{g}(\tau') + \tilde{k}(\tau')))\Psi_2(\tau')\rangle^{1/2}$$

by Schwarz's inequality,

$$\leqq 2M_3(\tau,\tau')\langle \Psi_2(\tau'), N\Psi_2(\tau')\rangle^{1/2}$$
$$\cdot |||g_1(\tau)^{1/2}\varrho_* u_0(\tau-\tau')(g(\tau') + k(\tau'))|||_1 \quad (5.44)$$

by inspection, where $\tilde{g}(\tau) = u_0(\tau)^* g(\tau) u_0(\tau)$ and $\tilde{g}_1(\tau)$ and $\tilde{k}(\tau)$ are defined similarly. We next estimate the contribution of $\tilde{L}^*$. That of $\tilde{L}$ satisfies the same estimate. Now:

$$\langle \Psi_2(\tau'), [\tilde{G}_{1R}(\tau), \tilde{L}^*(\tau')]\Psi_2(\tau')\rangle$$
$$= \sum_N \left\langle \Psi_2(\tau'), \sum_{i<j} (\tilde{g}_{1Ri}(\tau) + \tilde{g}_{1Rj}(\tau))\tilde{l}_{ij}(\tau')\Psi_{2(ij)}(\tau')\right\rangle_N , \quad (5.45)$$

where $\langle \cdot, \cdot \rangle_N$ denotes the N-particle contribution to the scalar product in $\mathscr{H}$,

$$\tilde{l}(\tau') = u_0(\tau') \otimes_S u_0(\tau') l(\tau') \quad (5.46)$$

[so that $\tilde{L}^*(\tau') = \frac{1}{2}\int dx\,dy\,\tilde{l}(\tau'; x, y)a^*(x)a^*(y)$], the subscripts i,j label the variables that occur in $\tilde{l}(\tau')$ and $\Psi_{2(ij)}(\tau')$ denotes the wave function with variables different from x_i, x_j. The previous scalar product is estimated by

$$|\ldots| \leqq \sum_N \left\langle \Psi_2(\tau'), \sum_{i<j} (\tilde{g}_{1Ri}(\tau) + \tilde{g}_{1Rj}(\tau)) \Psi_2(\tau') \right\rangle_N^{1/2}$$
$$\cdot \left\{ \sum_{i<j} \langle \Psi_{2(ij)}(\tau') \tilde{l}_{ij}(\tau'), (\tilde{g}_{1Ri}(\tau) + \tilde{g}_{1Rj}(\tau)) \tilde{l}_{ij}(\tau') \Psi_{2(ij)}(\tau') \rangle_N \right\}^{1/2}$$

by Schwarz's inequality in $\mathscr{H}_N$,

$$\ldots = \sum_N \{ \tfrac{1}{2}(N-1) \langle \Psi_2(\tau'), \tilde{G}_{1R}(\tau) \Psi_2(\tau') \rangle_N$$
$$\cdot \langle \Psi_2(\tau'), \Psi_2(\tau') \rangle_{N-2} \langle \tilde{L}^*(\tau') \Psi_0, \tilde{G}_{1R}(\tau) \tilde{L}^*(\tau') \Psi_0 \rangle \}^{1/2}$$
$$\leqq M_3(\tau, \tau') \langle \Psi_2(\tau'), (N+1) \Psi_2(\tau') \rangle^{1/2}$$
$$\cdot \| g_1(\tau)^{1/2} \varrho_* u_0(\tau - \tau') k(\tau') \|_{\mathrm{HS}} \tag{5.47}$$

by Schwarz's inequality applied to the sum over N. Then (5.43) follows from (5.44), (5.47) and the inequality $|||\cdot|||_1 \leqq \|\cdot\|_{\mathrm{HS}}$. Q.E.D.

The task of estimating J_3 is now reduced to that of estimating the norms in the R.H.S. of (5.41) and (5.43). Before doing this, we perform the same reduction on J_4. We first note that

$$H_4^2 \leqq \tfrac{1}{2} N(N-1) \Gamma_2(V^2). \tag{5.48}$$

Therefore

$$\| \tilde{H}_4(\tau) P \Psi_2(\tau) \|$$
$$\leqq \langle \Psi_2(\tau), \tfrac{1}{2} N(N-1) \sigma^2 R U_0(\tau)^* \Gamma_2(V^2) U_0(\tau) R \Psi_2(\tau) \rangle^{1/2}$$
$$\leqq 2^{1/2} v \langle \Psi_2(\tau), \Gamma_2(\tilde{V}_{2R}(\tau)) \Psi_2(\tau) \rangle^{1/2}, \tag{5.49}$$

where $\tilde{V}_{2R}(\tau)$ is the operator in $\mathscr{H}_2$ defined by

$$\tilde{V}_{2R}(\tau) = U_0(\tau)^* R V^2 R U_0(\tau)|_{\mathscr{H}_2}. \tag{5.50}$$

In order to estimate J_4, we introduce

$$M_4(\tau, \tau') = \langle \Psi_2(\tau'), \Gamma_2(\tilde{V}_{2R}(\tau)) \Psi_2(\tau') \rangle^{1/2}, \tag{5.51}$$

so that

$$M_4(\tau, s) = \langle \Psi, \Gamma_2(\tilde{V}_{2R}(\tau)) \Psi \rangle^{1/2}. \tag{5.52}$$

Then, by (5.49)

$$\int_s^t d\tau \| \tilde{H}_4(\tau) P \Psi_2(\tau) \|$$
$$\leqq 2^{1/2} v \int_s^t d\tau M_4(\tau, \tau)$$
$$\leqq 2^{1/2} v \left\{ \int_s^t d\tau M_4(\tau, s) + \int_s^t d\tau \int_s^\tau d\tau' |\dot{M}_4(\tau, \tau')| \right\}, \tag{5.53}$$

where $\dot{M}_4(\tau, \tau') = dM_4(\tau, \tau')/d\tau'$. The quantity $M_4(\tau, s)$ is readily estimated by the following obvious lemma [we recall that $\Psi \in Q(NB_1) \subset Q(B_2)$]:

Lemma 5.7. *$M_4(\tau, s)$ satisfies the estimate*

$$M_4(\tau,s) \leqq \langle \Psi, B_2 \Psi\rangle^{1/2} |||VRU_0(\tau) b_2^{-1/2}|||_2 . \tag{5.54}$$

We now estimate $\dot{M}_4(\tau, \tau')$, starting from the relation

$$\begin{aligned} &2iM_4(\tau,\tau')\dot{M}_4(\tau,\tau') \\ &\quad = \langle \Psi_2(\tau'), [\Gamma_2(\tilde{V}_{2R}(\tau)), \tilde{H}_2(\tau') - H_0]\,\Psi_2(\tau')\rangle \end{aligned} \tag{5.55}$$

which follows from part 3) of Proposition 4.1.

Lemma 5.8. *$\dot{M}_4(\tau, \tau')$ satisfies the estimate*

$$\begin{aligned} |\dot{M}_4(\tau,\tau')| &\leqq 2^{1/2} \|N\Psi_2(\tau')\| \\ &\qquad \cdot \left\{ \operatorname*{Sup}_y |||V_y \varrho_* u_0(\tau-\tau') g(\tau')|||_1 + 2|||w(\tau,\tau')\varrho_*|||_1 \right\} \\ &\quad + \|\Psi\|\ \|VRU_0(\tau-\tau')L^*(\tau')\Psi_0\| , \end{aligned} \tag{5.56}$$

where $w(\tau, \tau')$ is the operator in $\mathscr{H}_1$ of multiplication by the function $w(\tau,\tau') \equiv w(\tau,\tau';x)$ defined by

$$\begin{cases} w^2(\tau,\tau') = V^2 * w_1(\tau,\tau')^2 \\ w_1(\tau,\tau')^2 = \int dy |\varrho_* u_0(\tau-\tau') k(\tau'; \cdot, y)|^2 . \end{cases} \tag{5.57}$$

Proof. We estimate the contribution of the various terms of $\tilde{H}_2(\tau') - H_0$ to the R.H.S. of (5.54). The terms with $\tilde{G}$ and $\tilde{K}$ contribute

$$\begin{aligned} &|\langle \Psi_2(\tau'), [\Gamma_2(\tilde{V}_{2R}(\tau)), \tilde{G}(\tau') + \tilde{K}(\tau')]\,\Psi_2(\tau')\rangle| \\ &\quad = |\langle \Psi_2(\tau'), \Gamma_2([\tilde{V}_{2R}(\tau), \tilde{G}(\tau') + \tilde{K}(\tau')])\,\Psi_2(\tau')\rangle| \\ &\quad \leqq 2^{1/2} M_4(\tau,\tau') \langle \Psi_2(\tau'), N(N-1)\Psi_2(\tau')\rangle^{1/2} \\ &\qquad \cdot |||VRU_0(\tau-\tau')(G(\tau') + K(\tau'))|||_2 \end{aligned} \tag{5.58}$$

by Schwarz's inequality. The contribution of G to the last norm is estimated by

$$|||VRU_0(\tau-\tau')\,G(\tau')|||_2 \leqq 2 \operatorname*{Sup}_y |||V_y \varrho_* u_0(\tau-\tau')\,g(\tau')|||_1 , \tag{5.59}$$

where V_y is the operator in $\mathscr{H}_1$ of multiplication by the function $V_y(x) = V(x-y)$. The contribution of K is estimated as follows. Let $\theta \in \mathscr{H}_2$. Then

$$\begin{aligned} &\|VRU_0(\tau-\tau')K(\tau')\theta\| \\ &\quad \leqq 2\{\textstyle\int dx_1\, dx_2\, V(x_1-x_2)^2 \\ &\qquad \cdot |\textstyle\int dy_2 [\varrho_* u_0(\tau-\tau')k(\tau')]\,(x_2,y_2)\,[\varrho_{*1}\theta]\,(x_1,y_2)|^2\}^{1/2} , \end{aligned}$$

where ϱ_{*1} denotes the operator ϱ_* acting on the variable 1,

$$\leqq 2 \textstyle\int dx_1\, dx_2\, V(x_1-x_2)^2 \int dy_2 |\varrho_{*1}\,\theta(x_1,y_2)|^2\, w_1(\tau,\tau';x_2)^2$$

by Schwarz's inequality applied to the integration over y_2, with w_1 defined by (5.57). From this it follows that

$$|||VRU_0(\tau-\tau')\,K(\tau')|||_2 \leqq 2|||w(\tau,\tau')\varrho_*|||_1 . \tag{5.60}$$

We next consider the contribution of $\tilde{L}^*$ to the R.H.S. of (5.55). That of $\tilde{L}$ satisfies the same estimate.

$$
\begin{aligned}
&|\langle \Psi_2(\tau'), [\Gamma_2(\tilde{V}_{2R}(\tau)), \tilde{L}^*(\tau')]\, \Psi_2(\tau')\rangle| \\
&\quad = \Big| \sum_N \Big\langle \Psi_2(\tau'), \sum_{i<j} (\tilde{V}_{2R}(\tau))_{ij} \tilde{l}_{ij}(\tau')\, \Psi_{2(ij)}(\tau') \Big\rangle_N \\
&\qquad + \sum_N \Big\langle \Psi_2(\tau'), \sum_{ijk\neq} (\tilde{V}_{2R}(\tau))_{ik} \tilde{l}_{ij}(\tau')\, \Psi_{2(ij)}(\tau') \Big\rangle_N \Big| \\
&\quad \leqq \sum_N \Big\langle \Psi_2(\tau'), \sum_{i<j} (\tilde{V}_{2R}(\tau))_{ij}\, \Psi_2(\tau') \Big\rangle_N^{1/2} \\
&\qquad \cdot \Big\{ \sum_{i<j} \langle \Psi_{2(ij)}(\tau') \tilde{l}_{ij}(\tau'), (\tilde{V}_{2R}(\tau))_{ij} \tilde{l}_{ij}(\tau')\, \Psi_{2(ij)}(\tau')\rangle_N \Big\}^{1/2} \\
&\qquad + \sum_N \Big\langle \Psi_2(\tau'), \sum_{ijk\neq} (\tilde{V}_{2R}(\tau))_{ij}\, \Psi_2(\tau') \Big\rangle_N^{1/2} \\
&\qquad \cdot \Big\{ \sum_{ijk\neq} \langle \Psi_{2(ij)}(\tau') \tilde{l}_{ij}(\tau'), (\tilde{V}_{2R}(\tau))_{ik} \tilde{l}_{ij}(\tau')\, \Psi_{2(ij)}(\tau')\rangle_N \Big\}^{1/2}
\end{aligned}
\tag{5.61}
$$

by Schwarz's inequality in $\mathscr{H}_N$,

$$
\begin{aligned}
\ldots &\leqq M_4(\tau, \tau') \Big\{ \|\Psi\|\ \|VRU_0(\tau-\tau') L^*(\tau')\, \Psi_0\| \\
&\quad + \Big[\sum_N 2(N-2) \sum_{ijk\neq} \langle \mathrm{idem} \rangle_N \Big]^{1/2} \Big\}
\end{aligned}
\tag{5.62}
$$

by Schwarz's inequality applied to the sum over N. The last square bracket is estimated by

$$
\begin{aligned}
[\,\cdot\,]^{1/2} &= 2^{1/2} \langle \Psi_2(\tau'), N\Gamma_1(\varrho_* w^2(\tau, \tau') \varrho_*)\, \Psi_2(\tau')\rangle^{1/2} \\
&\leqq 2^{1/2} \|N\Psi_2(\tau')\|\ |||w(\tau, \tau')\varrho_*|||_1 \,.
\end{aligned}
\tag{5.63}
$$

Substituting (5.59) and (5.60) into (5.58), and (5.63) into (5.61) and (5.62), and collecting the various terms, we obtain (5.56). Q.E.D.

In order to complete the estimates of J_3 and J_4, we shall use as an essential ingredient the dispersive properties of the free evolution. For any $q \geqq 2$, we denote by $\bar{q}$ the conjugate index defined by $1/q + 1/\bar{q} = 1$.

Lemma 5.9.

1) *Let* $2 \leqq q \leqq \infty$. *Then for any* $\psi \in L^{\bar{q}}$,

$$
\|u_0(t)\psi\|_q \leqq (2\pi|t|)^{n/q - n/2} \|\psi\|_{\bar{q}} \,.
\tag{5.64}
$$

2) *Let* $2 \leqq q < \infty$, $1/q + 1/l = 1/2$. *Then for any* $\psi \in \mathscr{D}(|x|^{n/l})$,

$$
\|u_0(t)\psi\|_q \leqq a_l |t|^{-n/l} \|\, |x|^{n/l} \psi\| \,,
\tag{5.65}
$$

where a_l *is a constant depending only on* l *and* n.

Proof. 1) The result follows by interpolation between the cases $q = 2$ and $q = \infty$ where it is a consequence of the integral representation

$$
u_0(t; x, y) = (2\pi i t)^{-n/2} \exp[-(x-y)^2/2it] \,.
\tag{5.66}
$$

2) The identity

$$|u_0(1)\psi| = |\mathscr{F}(\exp(-ix^2/2)\psi)| \tag{5.67}$$

(where $\mathscr{F}$ denotes the Fourier transform) and the Sobolev inequality

$$\|\psi\|_q \leqq a_l \|(-\Delta)^{n/2l}\psi\| \tag{5.68}$$

imply

$$\|u_0(1)\psi\|_q \leqq a_l \| \, |x|^{n/l}\psi\| . \tag{5.69}$$

The general case follows by homogeneity. Q.E.D.

We shall need one more technical result, the proof of which is given in the appendix.

Lemma 5.10. *Write $\mathscr{H}_2$ as*

$$\mathscr{H}_2 = L^2(dx_1) \otimes_S L^2(dx_2) = L^2(d\xi) \otimes L_e^2(d\eta), \tag{5.70}$$

where $\xi = (x_1 + x_2)/2$, $\eta = x_1 - x_2$, and the subscript e in L_e^2 means the restriction to even functions. Let f and f' be operators in $\mathscr{H}_1 = L^2$ leaving L_e^2 invariant, and let f_2 and f_2' be the operators in $\mathscr{H}_2$ defined by $f_2 = \mathbb{1}_\xi \otimes f_\eta$ and $f_2' = \mathbb{1}_\xi \otimes f_\eta'$ corresponding to the decomposition (5.70). *Then*

$$|||f_2^* R f_2'|||_2 \leqq \operatorname*{Sup}_{k \in \mathbb{R}^n} |||f^* r_{k*} f'|||_1 , \tag{5.71}$$

where r_{k*} is the operators in $\mathscr{H}_1$ of convolution with the function

$$r_k(x) = \int d\zeta \, e^{-ik\zeta} \varrho\left(\zeta + \frac{x}{2}\right) \varrho\left(\zeta - \frac{x}{2}\right). \tag{5.72}$$

We are now in a condition to estimate the various norms that appear in the R.H.S. of (5.41), (5.43), (5.54), and (5.56). We recall that V and φ satisfy the assumptions (5.5) and (5.6).

Lemma 5.11. *Let $l \geqq p_1$, $l \geqq n/\beta$, $l > 2$ and let*

$$1/l_i = 1/p_i - 1/l \quad \text{for} \quad i = 1, 2 . \tag{5.73}$$

Then

$$|||g_1(\tau)^{1/2} \varrho_* u_0(\tau) b_1^{-1/2}|||_1 \leqq \|\varphi(\tau)\| \lambda_l(\tau), \tag{5.74}$$

$$|||VRU_0(\tau) b_2^{-1/2}|||_2 \leqq 2^{-n/l} \lambda_l(\tau), \tag{5.75}$$

where

$$\lambda_l(\tau) = |\tau|^{-n/l} a_l \sum_{i=1,2} \|V_i\|_{p_i} \|\varrho\|_{\bar{l}_i} . \tag{5.76}$$

Proof. (5.74) follows from (5.29), (5.3), and (5.65) by Hölder's and Young's inequalities. (5.75) follows from Lemma 5.10, from (5.4), (5.65) and the fact that

$$\operatorname*{Sup}_k \|r_k\|_{\bar{l}_i} = \|r_0\|_{\bar{l}_i} \leqq \|\varrho\|_{\bar{l}_i} . \quad \text{Q.E.D.} \tag{5.77}$$

Lemma 5.12. *Let $l \geqq p_1$ and define l_i and q_i' by*

$$1/l_i = 1 - 2/q_i' = 1/p_i - 1/l, \quad \textit{for} \quad i = 1, 2. \tag{5.78}$$

Then

$$|||g_1(\tau)^{1/2} \varrho_* u_0(\tau - \tau') g(\tau')|||_1 \leqq \|\varphi(\tau)\| \mu_l(\tau, \tau'), \tag{5.79}$$

$$\operatorname*{Sup}_y |||V_y \varrho_* u_0(\tau - \tau') g(\tau')|||_1 \leqq \mu_l(\tau, \tau'), \tag{5.80}$$

$$\|g_1(\tau)^{1/2} \varrho_* u_0(\tau - \tau') k(\tau')\|_{\mathrm{HS}} \leqq \|\varphi(\tau)\| \, \mu_l(\tau, \tau'), \tag{5.81}$$

$$|||w(\tau, \tau') \varrho_*|||_1 \leqq \|w(\tau, \tau')\|_\infty \leqq \mu_l(\tau, \tau'), \tag{5.82}$$

$$\|VRU_0(\tau - \tau') L^*(\tau') \Psi_0\| \leqq 2^{-1/2 - n/l} \mu_l(\tau, \tau'), \tag{5.83}$$

where $w(\tau, \tau')$ is defined by (5.57) *and*

$$\mu_l(\tau, \tau') = \left(\sum_{i=1,2} \|V_i\|_{p_i} \|\varrho\|_{\bar{l}_i} \right) (2\pi|\tau - \tau'|)^{-n/l} \cdot \left(\sum_{i=1,2} \|V_i\|_{p_i} \|\varphi(\tau')\|_{q_i'}^2 \right). \tag{5.84}$$

Proof. The estimates (5.79) and (5.80) follow from (5.29), (4.6) and (5.64) by Hölder's and Young's inequalities. We now prove (5.81). By (5.29) and Hölder's and Young's inequalities again,

$$\|g_1(\tau) \varrho_* u_0(\tau - \tau') k(\tau')\|_{\mathrm{HS}} \leqq \|\varphi(\tau)\| \sum_{i=1,2} \|V_i\|_{p_i} \gamma_{s_i}(\tau, \tau'), \tag{5.85}$$

where

$$\gamma_{s_i}(\tau, \tau') = \{\textstyle\int dy \|\varrho_* u_0(\tau - \tau') k(\tau'; \cdot, y)\|_{s_i}^2\}^{1/2} \tag{5.86}$$

and

$$1/s_i + 1/p_i = 1/2 \quad \text{for} \quad i = 1, 2. \tag{5.87}$$

By part 1) of Lemma 5.9,

$$\gamma_{s_i}(\tau, \tau') \leqq \|\varrho\|_{\bar{l}_i} (2\pi|\tau - \tau'|)^{-n/l} \{\textstyle\int dy \|k(\tau'; \cdot, y)\|_{\bar{s}}^2\}^{1/2} \tag{5.88}$$

with $1/s + 1/l = 1/2$. Now

$$\begin{aligned} &\{\textstyle\int dy \|k(\tau'; \cdot, y)\|_{\bar{s}}^2\}^{1/2} \\ &= \{\textstyle\int dy |\varphi(\tau', y)|^2 ((|\varphi(\tau')|^{\bar{s}} * |V|^{\bar{s}})(y))^{2/\bar{s}}\}^{1/2} \\ &\leqq \sum_{i=1,2} \|\varphi(\tau')\|_{q_i'} \, \| \, |\varphi(\tau')|^{\bar{s}} * |V_i|^{\bar{s}} \|^{1/\bar{s}} \\ &\leqq \sum_{i=1,2} \|\varphi(\tau')\|_{q_i'}^2 \, \|V_i\|_{p_i} \end{aligned} \tag{5.89}$$

by Hölder's and Young's inequalities. (5.81) then follows from (5.85)–(5.89).

We now turn to (5.82). The first inequality is obvious. From (5.57) and Hölder's inequality, we obtain

$$\|w(\tau,\tau')\|_\infty \leqq \sum_i \|V_i\|_{p_i} \|w_1(\tau,\tau')\|_{s_i},$$

where s_i is defined by (5.87). Now

$$\|w_1(\tau,\tau')\|_{s_i} \leqq \gamma_{s_i}(\tau,\tau'),$$

where $\gamma_{s_i}(\tau,\tau')$ is defined by (5.86) and we have used the convexity of the norm in $L^{s_i/2}$. The end of the proof is identical with that of (5.81).

Finally we consider (5.83). We first split the L.H.S. as follows

$$\|VRU_0(\tau-\tau')L^*(\tau')\Psi_0\| \leqq \sum_i \||VRU_0(\tau-\tau')|V_i|^{p_i/l}\||_2 \cdot \| |V_i|^{1-p_i/l}\varphi(\tau')\otimes_S\varphi(\tau')\|, \tag{5.90}$$

where the last norm is taken in $\mathscr{H}_2$. By Lemma 5.10, we obtain

$$\begin{aligned}&\||VRU_0(\tau-\tau')|V_i|^{p_i/l}\||_2\\ &\quad\leqq \operatorname*{Sup}_k \|||v|r_{k*}u_0(2(\tau-\tau'))|v_i|^{p_i/l}\||_1,\end{aligned}$$

where v (resp. v_i, $i=1,2$) is the operator of multiplication by V (resp. V_i, $i=1,2$) in $\mathscr{H}_1$,

$$\ldots \leqq \sum_{j=1,2} \|V_j\|_{p_j} \|\varrho\|_{\bar{l}_j}(4\pi|\tau-\tau'|)^{-n/l} \|V_i\|_{p_i}^{p_i/l} \tag{5.91}$$

by part 1) of Lemma 5.9, by (5.77) and Hölder's and Young's inequalities. On the other hand

$$\| |V_i|^{1-p_i/l}\varphi(\tau')\otimes_S\varphi(\tau')\| \leqq 2^{-1/2}\|\varphi(\tau')\|_{q_i'}^2 \|V_i\|_{p_i}^{1-p_i/l}. \tag{5.92}$$

The result now follows from (5.90), (5.91), and (5.92). Q.E.D.

Collecting the estimates contained in Lemmas 5.4–5.12 and using the fact that $\|\varphi(\tau')\| = \|\varphi\|$ is actually independent of τ', we obtain

$$\begin{aligned}J_3 \leqq 8\hbar^{1/2}v\|\Psi\|\,\|\varphi\| \Big\{&\langle\Psi, B_1\Psi\rangle^{1/2}\int_s^t d\tau \operatorname*{Inf}_l \lambda_l(\tau)\\ &+3\int_s^t d\tau \int_s^\tau d\tau' \langle\Psi_2(\tau'), (N+1)\Psi_2(\tau')\rangle^{1/2} \operatorname*{Inf}_l \mu_l(\tau,\tau')\Big\},\end{aligned} \tag{5.93}$$

$$\begin{aligned}J_4 \leqq 2\hbar v\|\Psi\| \Big\{&\langle\Psi, B_2\Psi\rangle^{1/2}\int_s^t d\tau \operatorname*{Inf}_l \lambda_l(\tau)\\ &+\int_s^t d\tau \int_s^\tau d\tau' (6\|N\Psi_2(\tau')\| + \|\Psi\|) \operatorname*{Inf}_l \mu_l(\tau,\tau')\Big\},\end{aligned} \tag{5.94}$$

where $\lambda_l(\tau)$ and $\mu_l(\tau,\tau')$ are defined by (5.76) and (5.84) and the Infimum is taken over the values of l allowed in Lemma 5.11 for λ_l and in Lemma 5.12 for μ_l.

We can now state our main result. We concentrate on situations where we can control the classical theory by the results of Sect. 2, the quantum theory by the results of Sects. 3 and 4, and the limit $\hbar\to 0$ by the estimates of this section.

Theorem 5.1. *Let $n \geqq 3$ and let V satisfy:*

$$\begin{cases} V \in L^{p_1} \cap L^2 & \text{with } p_1 < n/2 \\ \text{for} \quad n = 3, 4, & \\ \begin{cases} V \in L^{p_1} + L^{p_2} \\ V_- \in L^{n/2} \end{cases} & p_2 \geqq 2, n/4 < p_2 \leqq p_1 < n/2, \\ \text{for} \quad n \geqq 5. & \end{cases} \tag{5.95}$$

Let r and r' satisfy the (compatible) conditions

$$1/2 - 1/n < 1/r \leqq 1/r' < 1/2 - 1/2n$$
$$1/r \leqq 1/n + 1/2 - 1/2p_2 .$$

Let X_{kr}, $\mathscr{X}_a(\cdot)$ and $\mathscr{X}_0(\cdot)$ be defined by (2.1), (2.2), *and* (2.11). *Then*

1) The Eq. (2.9) *has global solutions in $\mathscr{X}_a(\mathbb{R})$ in the sense of Proposition* 2.1, *and solutions dispersive in the past (i.e. in $\mathscr{X}_a(\mathbb{R}) \cap \mathscr{X}_0(\mathbb{R}^-)$) or in the future (i.e. in $\mathscr{X}_a(\mathbb{R}) \cap \mathscr{X}_0(\mathbb{R}^+)$) in the sense of Proposition* 2.2.

2) The quantum mechanical evolution operators $U(t-s)$ defined by (3.14) *form a strongly continuous unitary group. The wave operators* (3.48) *exist.*

3) Let φ be a solution of (2.9) *in $\mathscr{X}_a(\mathbb{R})$. Then the operators $\tilde{W}(t,s)$ and $\tilde{U}_2(t,s)$ form strongly continuous unitary groups and satisfy Propositions* 3.1 *and* 4.1 *respectively. If in addition $\varphi \in \mathscr{X}_0(\mathbb{R}^+)$ (resp. $\varphi \in \mathscr{X}_0(\mathbb{R}^-)$, resp. $\varphi \in \mathscr{X}_0(\mathbb{R})$), then $\tilde{W}(t,s)$ has strong limits when s tends to $+\infty$ (resp. $-\infty$, resp. $\pm\infty$) and $\tilde{U}_2(t,s)$ has strong limits when t and/or s tend to $+\infty$ (resp. $-\infty$, resp. $\pm\infty$).*

4) Let φ be a solution of (2.9) *in $\mathscr{X}_a(\mathbb{R}) \cap \mathscr{X}_0(\mathbb{R}^+)$ (resp. $\mathscr{X}_a(\mathbb{R}) \cap \mathscr{X}_0(\mathbb{R}^-)$, resp. $\mathscr{X}_0(\mathbb{R})$). Then for any $\theta \in \mathbb{R}$, for any $\Psi \in \mathscr{H}$,*

$$\underset{\hbar \to 0}{s\text{-lim}}\, \tilde{W}(t,s)\Psi = \tilde{U}_2(t,s)\Psi \tag{5.96}$$

uniformly in t, s for $t, s \geqq \theta$ (resp. $\leqq \theta$, resp. in $\mathbb{R}$). Moreover, if $\varphi \in \mathscr{X}_0(\mathbb{R}^\pm)$, the wave operators (3.48) *converge in the following sense*

$$\underset{\hbar \to 0}{s\text{-lim}}\, C(\varphi_\hbar(0))^* \Omega_\pm C(\varphi_{\hbar\pm}) \exp[i\omega_\hbar(0, \pm\infty)] = \Omega_{2\pm} , \tag{5.97}$$

where $\omega_\hbar$ is defined by (3.40) *and $\Omega_{2\pm}$ by* (4.23).

5) Let φ be a solution of (2.9) *in $\mathscr{X}_0(\mathbb{R})$ and assume that the quantum system is asymptotically complete. Then $\tilde{W}(t,s)$ has strong limits when t and/or s tend to $\pm\infty$. Furthermore*

$$\underset{\hbar \to 0}{s\text{-lim}}\, C(\varphi_{\hbar+})^* S C(\varphi_{\hbar-}) \exp[i\omega_\hbar(+\infty, -\infty)] = S_2 , \tag{5.98}$$

where

$$S = \underset{\substack{t \to +\infty \\ s \to -\infty}}{s\text{-lim}}\, U_0(t)^* U(t-s) U_0(s) \tag{5.99}$$

is the quantum mechanical S-matrix and S_2 is defined by (4.24).

Proof. Parts 1)–3) are restatements of results from Sects. 2–4 in the special case at hand. We turn to part 4). For definiteness we assume that $\varphi \in \mathscr{X}_a(\mathbb{R}) \cap \mathscr{X}_0(\mathbb{R}^+)$; the other cases are similar. It is sufficient to prove (5.96) for Ψ in the dense set $Q(NB_1)$.

From the definition (2.11) of $\mathscr{X}_0(\mathbb{R}^+)$ and the conditions $1/r' < 1/2 - 1/2n$ and $p_1 < n$, it follows that

$$\int_0^\infty d\tau \|\varphi(\tau)\|_{q_i}^2 < \infty, \quad i = 1, 2. \tag{5.100}$$

In particular, by (5.25), $c(\tau)$ is integrable in $\mathbb{R}^+$. By Lemma 5.1, this implies that J_1 tends to zero when $\nu, \kappa \to \infty$ uniformly with respect to t, s in $\mathbb{R}^+$. By an argument similar to the proof of Proposition 2.2, which cannot be given here, one can prove the remaining assumptions of part 2) of Corollary 5.1. Therefore also J_2 tends to zero when $\nu, \kappa \to \infty$ uniformly with respect to t, s in $\mathbb{R}^+$. Furthermore, by part 1) of Proposition 4.2 and by (5.25), it follows from (5.100) that the factors $\langle \Psi_2(\tau'), (N+1)\Psi_2(\tau')\rangle^{1/2}$ and $\|N\Psi_2(\tau')\|$ in (5.93) and (5.94) are bounded uniformly with respect to $\tau' \in \mathbb{R}^+$ for fixed $\Psi \in Q(NB_1) \subset \mathscr{D}(N)$. In order to complete the control of J_3 and J_4 we now pick $\varepsilon > 0$ sufficiently small so that $n - \varepsilon > \mathrm{Max}(p_1, n/\beta, 2)$ and substitute in (5.93) and (5.94) the inequalities

$$\operatorname*{Inf}_l \lambda_l(\tau) \leqq \begin{cases} \lambda_{n-\varepsilon}(\tau) & \text{if} \quad |\tau| \leqq 1 \\ \lambda_{n+\varepsilon}(\tau) & \text{if} \quad |\tau| \geqq 1, \end{cases}$$

$$\operatorname*{Inf}_l \mu_l(\tau, \tau') \leqq \begin{cases} \mu_{n-\varepsilon}(\tau, \tau') & \text{if} \quad |\tau - \tau'| \leqq 1 \\ \mu_{n+\varepsilon}(\tau, \tau') & \text{if} \quad |\tau - \tau'| \geqq 1. \end{cases}$$

Then the integrals over τ for fixed τ' in the R.H.S. of (5.93) and (5.94) are bounded uniformly with respect to t, s, and τ' in $\mathbb{R}^+$. It then follows from the assumptions on V (especially from the condition $p_1 < n/2$) and the definition of $\mathscr{X}_0(\mathbb{R}^+)$ that

$$\int_0^\infty d\tau' \|\varphi(\tau')\|_{q_i(l)}^2 < \infty, \quad i = 1, 2,$$

for both $l = n \pm \varepsilon$ and therefore that the remaining integrals over τ' are bounded uniformly with respect to $t, s \in \mathbb{R}^+$. This completes the proof of (5.95) with the uniformity thereafter stated. Finally, the convergences (5.97) and (5.98) follow immediately from the uniformity of (5.96) in t, s and from the existence of the wave operators and of the S-matrix. Q.E.D.

For the class of potentials considered in Theorem 5.1, Propositions 2.1 and 2.2 provide us with a systematic method of constructing classical solutions satisfying the assumptions of parts 3) and 4) of Theorem 5.1. In order to construct dispersive solutions [i.e. solutions in $\mathscr{X}_0(\mathbb{R})$] as needed in part 5), stronger conditions on the potential are needed. Proposition 2.3 provides such a set of sufficient conditions, which is compatible with (5.95) in dimension $n \geqq 5$. [One can take for instance $V(x) = C|x|^{-\gamma}$ with $C > 0$ and $2 < \gamma < \mathrm{Min}(4, n/2)$.] Under the assumptions of Proposition 2.3, all classical solutions are dispersive and determined by φ_-. If in addition quantum mechanical asymptotic completeness holds, then part 5) of Theorem 5.1 applies and (5.98) can be rewritten as

$$\operatorname*{s-lim}_{\hbar \to 0} C(\hbar^{-1/2} S_c \varphi_-)^* S C(\hbar^{-1/2}\varphi_-) \exp[i\omega_\hbar(+\infty, -\infty)] = S_2, \tag{5.101}$$

where S_c is the classical S-matrix.

Appendix

Proof of Lemma 5.10. Let ψ and $\psi' \in \mathscr{H}_2$, let $\theta = f_2\psi$ and $\theta' = f_2'\psi'$. All these functions are written as functions of variables (x_1, x_2) corresponding to the first decomposition in (5.70). Let $\xi = (x_1 + x_2)/2$ and $\eta = x_1 - x_2$. We define partial Fourier transforms as follows

$$\begin{cases} \hat{\psi}(k,\eta) = \int d\xi e^{-ik\xi}\psi(\xi + \eta/2, \xi - \eta/2) \\ \psi(\xi + \eta/2, \xi - \eta/2) = (2\pi)^{-n}\int dk e^{ik\xi}\hat{\psi}(k,\eta) \end{cases}$$

and similarly for ψ', θ, and θ'. Then

$$\begin{aligned}\langle f_2\psi, Rf_2'\psi'\rangle &= \int dx_1 dx_2 dx_1' dx_2' \bar{\theta}(x_1, x_2)\varrho(x_1 - x_1') \\ &\quad \cdot \varrho(x_2 - x_2')\theta'(x_1', x_2') \\ &= (2\pi)^{-2n}\int dk dk' \int d\xi d\xi' d\eta d\eta' \exp(ik'\xi' - ik\xi)\bar{\hat{\theta}}(k,\eta) \\ &\quad \cdot \varrho(\xi - \xi' + (\eta - \eta')/2)\varrho(\xi - \xi' - (\eta - \eta')/2)\hat{\theta}'(k', \eta').\end{aligned}$$

Let $\xi - \xi' = \zeta$. For fixed k, η, η' and ζ, the integrations over $(\xi + \xi')/2$ and k' are trivial and yield

$$\langle f_2\psi, Rf_2'\psi'\rangle = (2\pi)^{-n}\int dk \int d\eta d\eta' \bar{\hat{\theta}}(k,\eta) r_k(\eta - \eta')\hat{\theta}'(k', \eta'),$$

where we have used the definition (5.72). Taking norms in $L_e^2(d\eta)$, we obtain,

$$\langle f_2\psi, Rf_2'\psi'\rangle \leqq (2\pi)^{-n}\int dk |||f^* r_{k*} f'|||_1 \, \|\hat{\psi}(k, \cdot)\| \, \|\hat{\psi}'(k, \cdot)\|$$

from which (5.71) follows by Schwarz's inequality. Q.E.D.

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