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On the Asymptotic Behaviour of Infinite Gradient Systems

R. Lang

Institut für Angewandte Mathematik der Universität, D-6900 Heidelberg,
Federal Republic of Germany

Abstract. We consider gradient systems of infinitely many particles in one-dimensional space interacting via a positive invariant pair potential Φ with a hard core. The main assumption is that Φ is strictly convex within the range R of Φ (where R is a fixed number $\leqq \infty$). Under some technical conditions we prove the following theorems: Let the initial distribution be given by a translation invariant point process ϱ on $\mathbb{R}^1$. Then there exists only one extreme equilibrium state ϱ with a given intensity $I(\varrho)$ satisfying $I(\varrho) \geqq R^{-1}$, and all ergodic initial distributions ϱ with an intensity $I(\varrho) \geqq R^{-1}$ converge weakly as $t \to \infty$ to the extreme equilibrium state with the same intensity.

1. Introduction

In classical statistical mechanics one considers configurations of many particles, which in the mathematical idealization means infinitely many particles, moving according to Newton's equations

$$(1.1) \quad \ddot{x}_i(t) = -\sum_{j \neq i} \operatorname{grad} \Phi(x_i(t) - x_j(t))$$

with $i \in \mathbb{N}$, $x_i \in \mathbb{R}^d (d \in \mathbb{N})$ and a pair potential Φ. Only recently [1] has a non-equilibrium existence proof for (1.1) in the case $d = 1, 2$ been found, and in the series of papers [4] the equilibrium states are characterized as Gibbs measures corresponding to the potential Φ. However, the problem of the asymptotic behaviour of the system (1.1) is not understood as yet (apart from some cases with a degenerate potential Φ there are no results as yet). For a survey on the present state see [2].

Related to (1.1) is the system of stochastic equations

$$(1.2) \quad dx_i(t) = \left[-\sum_{j \neq i} \operatorname{grad} \Phi(x_i(t) - x_j(t)) \right] dt + \beta^{-1/2} d\omega_i(t) \quad (i \in \mathbb{N})$$

with independent Wiener processes $\omega_i(t)$ and the inverse temperature $\beta (\beta > 0)$. As follows from [9] the Gibbs measures for the potential $2\beta\Phi$ are equilibrium states for (1.2). In the limit $\beta\to\infty$ we get from (1.2) the following system of first order differential equations *(gradient system)*:

$$\text{(1.3)}\quad \dot{x}_i(t)=-\sum_{j\neq i}\operatorname{grad}\Phi(x_i(t)-x_j(t)),\ i\in\mathbb{N},\ x_i\in\mathbb{R}^d\qquad(d\in\mathbb{N}).$$

Our aim is to study this system. The first order Eqs. (1.3) are more accessible to mathematical treatment, firstly because one knows Liapunov functions, and secondly because it is possible to exhibit equilibrium states explicitly. If, for example, $d=3$ and Φ is rotation-invariant and convex, there exist several entirely different candidates for ground states, i.e. equilibrium states with minimal energy: Since there are different possibilities to superpose plane layers of closely packed balls (see e.g. [6], p. 41) there exist different close packings of balls in space such that the centers x_i of the balls satisfy the equilibrium condition

$$\text{(1.4)}\quad \sum_{j:j\neq i}\operatorname{grad}\Phi(x_i-x_j)=0\quad\text{for all}\quad i\in\mathbb{N}.$$

In statistical mechanics the term equilibrium state has a somewhat different sense, so in the following we will use the term *rigid state*, when (1.4) is statisfied. Since (1.3) results (formally) from (1.2) in the limit $\beta\to\infty$, one can hope to find connections between the problem of phase transitions for Gibbs measures and the existence of different ground states in the system (1.3).

In this paper however we will confine ourselves on the study of the asymptotic behaviour of (1.3). Since we will get complete results in the case of

$$\text{(1.5)}\quad \text{dimension } d=1,$$

we will consider in the following only this case.

As regards asymptotic behaviour, (1.3) is related to the following system of deterministic equations for infinitely many particles $x_i\in\mathbb{R}^1(i\in\mathbb{Z}, x_i<x_{i+1})$, which was introduced in [13]:

$$\text{(1.6)}\quad \dot{x}_i=f(x_{i+1}-x_i)-f(x_i-x_{i-1})\qquad(i\in\mathbb{Z}),$$

where f is a function $\mathbb{R}_+\to\mathbb{R}_+$ such that there exist numbers m, M with

$$\text{(1.7)}\quad 0<m\leqq\frac{f(x)-f(y)}{x-y}\leqq M<\infty\qquad(x,y\in\mathbb{R}_+, x\neq y).$$

In the special case $f(x)=x$ (1.6) reduces to a system of linear equations which is solvable explicitly; for a generalization of the system (1.6), but which is still linear, see [3]. For general f satisfying (1.7) Spitzer has proved the following ergodic theorem ([13], pp. 217–222 including the footnote on p. 221): Given an ergodic translation invariant point process ϱ on $\mathbb{R}^1$ satisfying some unessential technical conditions, and with intensity $\frac{1}{r}(r>0)$, let ϱ_t be the image of ϱ under the flow defined by (1.6). Then ϱ_t converges weakly as $t\to\infty$ to that translation invariant point process for which adjacent points have equal spacings r. The connection between (1.6) and (1.3) is the following: Given the function f in (1.6) define the potential Φ as $\Phi(x) := \int_0^{|x|} dy f(y)$ – for example in the linear case we get $\Phi(x) = \frac{1}{2}|x|^2$ – then (1.6) can be interpreted as a special case of the gradient system (1.3), namely the case of dimension $d=1$ and of an attractive nearest neighbor potential. Dropping the restriction that only nearest neighbor points can interact, it is more natural to assume the potential in (1.3) is repulsive. Furthermore as an analogue to (1.7) we assume that Φ is strictly convex in the domain $\{x \in \mathbb{R}^1 : 0 < \Phi(x) < \infty\}$, so we have in addition to (1.5) the second assumption on which the study of the asymptotic behaviour of (1.3) is based in this paper:

(1.8) Φ is positive, (has a hard core of radius δ because of technical reasons) and there exists a positive number $R \leqq \infty$ (the range of Φ) such that Φ is strictly convex on the domain $\{x \in \mathbb{R}^1, \delta < |x| < R\}$ and identically zero for $|x| \geqq R$.
The third basic assumption is the following

(1.9) the initial distribution ϱ is translation invariant.

In the next section we consider finite gradient systems on the torus. This simple case illustrates the questions of this paper, in particular it explains the use of the convexity of Φ. With the help of the latter we give in the third section a simple proof of the existence of a solution of (1.3) (Theorem 1). In Sect. 4 we prove that there exists only one translation invariant extreme rigid state with given intensity $I \geqq R^{-1}$ (Theorem 2). It is characterized as that translation invariant state with minimal energy (Theorem 3). Under the assumption that the intensity I of the ergodic initial distribution satisfies $I \geqq R^{-1}$, we deduce in Sect. 5 an ergodic theorem (Theorem 4) analogous to Spitzer's theorem for (1.6) described above. The proof is based on the fact, that the average energy is a *Liapunov function*, even a convex function (in the variable t) and furthermore, that taking the average of the square of the spacing between two particles gives a Liapunov function as well. Combining this last property, which is typically one-dimensional, with an idea of Kesten, and in the case $R < \infty$ using in addition an idea of Nguyen Xuan Xanh (see the proof of Lemma 5), we deduce the ergodic theorem.

The discussion above about the ground states in dimension $d=3$ shows that such a result does not hold in higher dimensions. However we can still deduce for any d that every ω-limit point of $\{\varrho_t : t \geqq 0\}$ is a rigid state (the proof is based on the notion of Palm measure from the theory of point processes). Results for $d \geqq 2$ will be communicated later.

The starting point for this paper was the ideas of Spitzer in [13, pp. 217–222], as well as some conversations with J. Fritz.

To both I am very grateful, to F. Spitzer also for his communication of Kesten's proof (the proof of Theorem 4). My thanks go to H. O. Georgii and Nguyen Xuan Xanh for encouragement and substantial contributions, to Ch. Preston for his help with English as well as to H. Rost and H. Zessin.

2. Preparations: Finite Gradient Systems and Notations

Before we study the infinite gradient system, we first consider in this section finite systems. The use of the convexity of the potential will be explained in this simple case, so preparing for the following sections.

Let $r>0$, $n\in\mathbb{N}$, $\mathbb{D}=\{x=(x_1,\dots,x_{n-1})\in\mathbb{R}^{n-1}:0<x_1<\dots<x_{n-1}<nr\}$; given $(x_1,\dots,x_{n-1})\in\mathbb{D}$, define the infinite sequence $\{x_i\}$ by periodic continuation: $x_0:=0$, $x_{i+n}:=x_i+nr\,(i\in\mathbb{Z})$ and let $\bar{\mathbb{D}}$ be the set of all such periodic sequences; in the following identify $\mathbb{D}$ and $\bar{\mathbb{D}}$.

Let $\Phi:\mathbb{R}\backslash\{0\}\to\mathbb{R}_+$ be a function with the following properties:

(i) Φ is symmetric,

(ii) Φ is twice continuously differentiable,

(iii) there exists a number R (the so called range of Φ) with $0<R\leqq nr$, such that Φ is positive and strictly convex on $(0,R)$ and identically zero on $[R,\infty)$.

(iv) $\lim\limits_{x\downarrow 0}\Phi(x)=\infty$.

The equations of motion are given by

$$(2.1)\quad \dot{x}_i(t)=-\sum_{j\neq i}\Phi'(x_i(t)-x_j(t))\qquad (i\in\mathbb{Z})$$

with the initial condition

$$\{x_i(0)\}=\{x_i\}\quad\text{for some}\quad \{x_i\}\in\bar{\mathbb{D}}.$$

To understand the asymptotic behaviour of the solution $x(t)$ of (2.1) it is necessary to know the rigid points, i.e. the points $\{x_i\}\in\bar{\mathbb{D}}$ with

$$(2.2)\quad \sum_{j:j\neq i}\Phi'(x_i-x_j)=0\quad\text{for all}\quad i\in\mathbb{Z}.$$

From the convexity of Φ follows:

Proposition 1. *If $R\geqq r$, then there exists only one rigid point $\bar{x}$, namely $\{\bar{x}_i\}=\{ir\}$.*

Proof. Define the energy function W on (the convex domain) $\mathbb{D}$ by

$$(2.3)\quad W(x)=\tfrac{1}{2}\sum_{|k|\leqq 1}\sum_{i=0}^{n-1}\sum_{\substack{0\leqq j\leqq n-1\\ j\neq i}}\Phi(x_i-x_j+k\cdot nr)=\tfrac{1}{2}\sum_{i=0}^{n-1}\sum_{j\neq i}\Phi(x_i-x_j);$$

then we have for $1\leqq i,j\leqq n-1$:

$$(2.4)\quad \frac{\partial W}{\partial x_i}(x)=\sum_{|k|\leqq 1}\sum_{\substack{0\leqq j\leqq n-1\\ j\neq i}}\Phi'(x_i-x_j+k\cdot nr)=\sum_{j\neq i}\Phi'(x_i-x_j).$$

$$(2.5)\quad \frac{\partial^2 W}{\partial x_i\partial x_j}(x)=\begin{cases}\sum\limits_{|k|\leqq 1}\sum\limits_{\substack{0\leqq j\leqq n-1\\ j\neq i}}\Phi''(x_i-x_j+k\cdot nr), & \text{if}\quad i=j\\ \sum\limits_{|k|\leqq 1}\Phi''(x_i-x_j+k\cdot nr), & \text{if}\quad i\neq j.\end{cases}$$

Denoting the gradient on $\mathbb{R}^{n-1}$ by D, from (2.2) and (2.4) we get

$$(2.6)\quad x\in\mathbb{D}\quad\text{is a rigid point if and only if}\quad DW(x)=0.$$

By virtue of (2.5) it follows for all $\alpha=(\alpha_1,\dots,\alpha_{n-1})\in\mathbb{R}^{n-1}$ and $x\in\mathbb{D}$ that

$$
\begin{aligned}
(2.7)\qquad \sum_{1\leqq i,j\leqq n-1}\alpha_i\alpha_j\frac{\partial^2 W}{\partial x_i\partial x_j}(x) &= \sum_{|k|\leqq 1}\sum_{i=1}^{n-1}\alpha_i^2\sum_{\substack{0\leqq j\leqq n-1\\ j\neq i}}\Phi''(x_i-x_j+k\cdot nr)\\
&\quad-\sum_{|k|\leqq 1}\sum_{i=1}^{n-1}\sum_{\substack{1\leqq j\leqq n-1\\ j\neq i}}\alpha_i\alpha_j\Phi''(x_i-x_j+k\cdot nr)\\
&=\sum_{|k|\leqq 1}\sum_{i=1}^{n-1}\alpha_i^2\Phi''(x_i+k\cdot nr)\\
&\quad+\sum_{|k|\leqq 1}\sum_{i=1}^{n-1}\alpha_i^2\sum_{\substack{1\leqq j\leqq n-1\\ j\neq i}}\Phi''(x_i-x_j+k\cdot nr)\\
&\quad-\sum_{|k|\leqq 1}\sum_{i=1}^{n-1}\sum_{\substack{1\leqq j\leqq n-1\\ j\neq i}}\alpha_i\alpha_j\Phi''(x_i-x_j+k\cdot nr),
\end{aligned}
$$

and using $\Phi''(x)=\Phi''(-x)$, the right-hand side of (2.7) can be written

$$
\begin{aligned}
&=\sum_{|k|\leqq 1}\sum_{i=1}^{n-1}\alpha_i^2\Phi''(x_i+k\cdot nr)+\tfrac12\sum_{|k|\leqq 1}\sum_{i=1}^{n-1}\sum_{\substack{1\leqq j\leqq n-1\\ j\neq i}}\alpha_i^2\Phi''(x_i-x_j+k\cdot nr)\\
&\quad+\tfrac12\sum_{|k|\leqq 1}\sum_{i=1}^{n-1}\sum_{\substack{1\leqq j\leqq n-1\\ j\neq i}}\alpha_j^2\,\Phi''(x_i-x_j+k\cdot nr)\\
&\quad-\tfrac12\sum_{|k|\leqq 1}\sum_{i=1}^{n-1}\sum_{\substack{1\leqq j\leqq n-1\\ j\neq i}}2\alpha_i\alpha_j\Phi''(x_i-x_j+k\cdot nr)\\
&=\sum_{|k|\leqq 1}\sum_{i=1}^{n-1}\alpha_i^2\Phi''(x_i+k\cdot nr)+\tfrac12\sum_{|k|\leqq 1}\sum_{i=1}^{n-1}\sum_{\substack{1\leqq j\leqq n-1\\ j\neq i}}(\alpha_i-\alpha_j)^2\Phi''(x_i-x_j+k\cdot nr)\\
&\geqq 0.
\end{aligned}
$$

Therefore W is convex on $\mathbb{D}$.

Let x be a rigid point, i.e.

$$(2.8)\qquad DW(x)=0.$$

Assume $x\neq\bar{x}$. To deduce a contradiction we consider two cases:

Case I: $r=R$.

Then one could find an index $i(1\leqq i\leqq n-1)$ with $x_i-x_{i-1}<r=R\leqq x_{i-1}-x_i$ contradicting the condition $\sum_{j<i}\Phi'(x_i-x_j)=\sum_{j>i}\Phi'(x_i-x_j)$.

Case II: $r<R$.

Developing W in a Taylor series first about $\bar{x}$, we get from (2.7) and from $DW(\bar{x})=0$

$$(2.9)\qquad W(x)\geqq W(\bar{x}).$$

By assumption there exist an index $i(1 \leqq i \leqq n-1)$ and an $\varepsilon > 0$ such that

(2.10) $x_{i+1} - x_i \leqq r - \varepsilon$.

Developing W in a Taylor series, this time about x, we obtain from (2.7) and (2.8)

$$(2.11) \quad W(\bar{x}) \geqq W(x) + \inf_{0<\vartheta<1} \Phi''(\vartheta r + (1-\vartheta)(x_{i+1} - x_i))(x_{i+1} - x_i - r)^2$$

$$\geqq W(x) + \varepsilon^2 \inf_{0<x\leqq r} \Phi''(x) > W(x),$$

because $r < R$ and Φ is strictly convex on $(0, R)$.

Remark 2.1. Let $R \geqq r$. If $x \in \mathbb{D}$ such that $W(x) = \min_{y \in \mathbb{D}} W(y)$, then $x = \bar{x}$.

The question of the asymptotic behaviour of the solution $x(t)$ of (2.1) is answered by

Proposition 2. *Let $R \geqq r$. Then $\lim_{t\to\infty} (x_{i+1}(t) - x_i(t)) = r$ for all $x \in \bar{\mathbb{D}}$ and all $i \in \mathbb{Z}$*

Proof. Let $x \in \mathbb{D}$ and $x(t)$ be the solution of (2.1) with initial condition x; write

$$(2.12) \quad v_i(t) = -\sum_{j \neq i} \Phi'(x_i(t) - x_j(t)).$$

The proof is based on the fact that the functions

$$t \to W(x(t)) = \tfrac{1}{2} \sum_{|k|\leqq 1} \sum_{i=0}^{n-1} \sum_{|j| \leqq n-1, j \neq i} \Phi(x_i(t) - x_j(t) + k \cdot nr)$$

and $t \to \sum_{i=0}^{n-1} v_i^2(t)$ are *Liapunov functions* (W is a Liapunov function also for general gradient systems with not necessarily convex Φ, see e.g. [7]), and furthermore that, as a consequence of (2.4) and (2.5), $t \to W(x(t))$ is convex [the following computation for (2.14) is similar to that for (2.7)]:

$$(2.13) \quad \frac{d}{dt} W(x(t)) = \sum_{i=0}^{n-1} \frac{\partial W}{\partial x_i} \dot{x}_i = -\sum_{i=0}^{n-1} v_i^2(t),$$

$$(2.14) \quad \frac{d}{dt} \sum_{i=0}^{n-1} v_i^2(t) = 2 \sum_{i=0}^{n-1} v_i \dot{v}_i$$

$$= -2 \sum_{i=0}^{n-1} v_i \sum_{|k|\leqq 1} \sum_{\substack{0 \leqq j \leqq n-1 \\ j \neq i}} \Phi''(x_i - x_j + k \cdot nr) v_i$$

$$+ 2 \sum_{i=0}^{n-1} v_i \sum_{|k|\leqq 1} \sum_{\substack{0 \leqq j \leqq n-1 \\ j \neq i}} \Phi''(x_i - x_j + k \cdot nr) v_j$$

$$= -\sum_{|k|\leqq 1} \sum_{i=0}^{n-1} \sum_{\substack{0 \leqq j \leqq n-1 \\ j \neq i}} (v_i(t) - v_j(t))^2 \Phi''(x_i(t) - x_j(t) + k \cdot nr)$$

$$\leqq 0.$$

Hence the (positive) function $t \to W(x(t))$ is convex and decreasing, and thus it follows that $\lim_{t\to\infty} \frac{d}{dt} W(x(t)) = 0$, which by (2.13) means:

(2.15) $\lim_{t\to\infty} v_i(t) = 0 \quad \text{for all} \quad i \in \mathbb{Z}$.

Given a sequence t_k of positive numbers such that $t_k \uparrow \infty$ and $\lim_{k\to\infty} (x_i(t_k) - x_{i-1}(t_k)) = : z_i (i \in \mathbb{Z})$, we have $z_i > 0$ for all $i \in \mathbb{Z}$, because otherwise from the singularity of Φ at the origin it would follow that the sequence $W(x(t_k))$ was unbounded, in contradiction to (2.13); by the periodicity of $\mathbb{D}$ we have $x_n(t) - nr \equiv x_0(t)$. Therefore the point y defined by $y_i := z_1 + \ldots + z_i (1 \leqq i \leqq n-1)$ is in $\mathbb{D}$.

From (2.15) we get

(2.16) $0 = \lim_{k\to\infty} \sum_{j\neq i} \Phi'(x_i(t_k) - x_j(t_k)) = \sum_{j\neq i} \Phi'(y_i - y_j)$,

from which we conclude by virtue of Proposition 1 that $y_i = ir (1 \leqq i \leqq n-1)$, hence $z_i = r (i \in \mathbb{Z})$. This proves Proposition 2.

In this note we want to answer the question: given a not necessarily periodic initial condition $x = \{x_i\}$, does a theorem analogous to Proposition 2 still hold? For this purpose it is natural to replace the energy function (2.3) with the averaged energy

(2.17) $\bar{W}(x) := \lim_{n\to\infty} \frac{1}{n} \cdot \frac{1}{2} \sum_{\substack{|i| \leqq n, |j| \leqq n \\ i \neq j}} \Phi(x_i - x_j)$

(for all $x = \{x_i\}$ for which the limit exists).

Similarly to (2.13) it is easy to see that this function is still a Liapunov function[1]. It is not clear however whether Proposition 2 holds pointwise for sufficiently many non-periodic initial conditions; obviously the analogue to Remark 2.1, pointwise formulated with the function $\bar{W}$ defined in (2.17), is not correct. Concerning Proposition 1 for non-periodic $\{x_i\}$ it is as yet impossible to prove it except in some special cases (one case is $\Phi(x) = e^{-x}$, which is solvable by direct computation, another case is when (2.2) is replaced by the condition $\sum_{j: 1 \leqq |i-j| \leqq 2} \Phi'(x_i - x_j) = 0$ for $i \in \mathbb{Z}$). Therefore, it seems reasonable to be content with statistical statements, i.e. to start with an initial distribution ϱ, which means a point process on $\mathbb{R}^1$, and to study the asymptotic behaviour of ϱ_t, where ϱ_t is the image of ϱ under the flow defined by (1.4).

Notations. Let $\{x_i\}$ be a sequence of points (a "configuration") with $x_i \in \mathbb{R}^1$, $x_i \uparrow \infty$ for $i \to \infty$ and $x_i \downarrow -\infty$ for $i \to -\infty$. Denote by

(2.18) $\mathbb{M}$ = set of configurations $\{x_i\}$ such that $x_{i+1} - x_i > \delta$ for all $i \in \mathbb{Z}$, where

(2.19) δ is a fixed positive number denoting the length of the hard core of the potential Φ [cf. (2.31)].

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1 It seems to be possible to conclude from this fact, that $\lim_{t\to\infty} v_i(t) = 0 (i \in \mathbb{Z})$ for sufficiently many initial conditions (cf. [10])

Instead to work with a translation invariant point process, i.e. a translation invariant probability measure on the space $\mathbb{M}$ of configurations, it does not make any essential difference to consider shift invariant probability measures P on the space of sequences

(2.20) $z=(z_i)_{i\in\mathbb{Z}},\ z_i>\delta$

via the transformation

(2.21) $z_i = x_i - x_{i-1}$

(this is a well known fact in the theory of point processes, cf. [8, 11]. But because we do not want to use this theory we work exclusively with measures P on $(\mathbb{R}_+)^{\mathbb{Z}}$ in the following, however it is expedient to use the picturesque language of point process theory sometimes too).

Conversely, given a sequence $z=(z_i)_{i\in\mathbb{Z}}$ we define $\{x_i\}$ by

(2.22) $$x_i=\begin{cases} z_1+\ldots+z_i, & i\geqq 1\\ 0, & i=0\\ -(z_{i+1}+\ldots+z_0), & i\leqq -1.\end{cases}$$

In the following we use both notations $(z_i)_{i\in\mathbb{Z}}$ and $\{x_i\}$ and do not always mention the corresponding transformation (2.21) resp. (2.22) $\left(\text{e.g. we write } P\left\{\lim_{i\to\infty}\frac{x_i}{i}=r\right\}\right.$ for a probability measure P on $\left.(\mathbb{R}_+)^{\mathbb{Z}}\right)$.

Endow $(\mathbb{R}_+)^{\mathbb{Z}}$ with the natural topology resp. σ-algebra and define

(2.23) $\mathbb{P}$ = set of probability measures P on $(\mathbb{R}_+)^{\mathbb{Z}}$ satisfying the following properties (2.24), (2.25), and (2.26):

(2.24) $E_P[z_1]<\infty$.

(2.25) P is invariant under the shift $z_i\mapsto z_{i+1}$.

(2.26) $P\{z_1>\delta\}=1$ with δ as in (2.19).

Endow $\mathbb{P}$ with the weak topology.

Define further the intensity $I(P)$ of $P\in\mathbb{P}$ by

(2.27) $I(P)=\{E_P[z_1]\}^{-1}$.

For the special measure $P\in\mathbb{P}$ which is concentrated at the point $z_i=r (i\in\mathbb{Z})$ we will use the notation

(2.28) Q_r, defined by $Q_r\{z_1=r\}=1 \quad (r>0)$.

The equations of motion are given by (2.1). By (2.21) they can be transformed in equations for the z_i, so we can define for $P\in\mathbb{P}$

(2.29) P_t = image of P under the flow given by (2.1).

Suppose further the following properties of the potential Φ:

(2.30) $\Phi:(-\infty,-\delta)\cup(+\delta,+\infty)\to\mathbb{R}_+,\quad \Phi(-x)=\Phi(x)$,

(2.31) there exists a number R with $\delta<R\leqq\infty$ (the range of Φ) such that Φ is strictly convex on (δ,R) and identically zero on $[R,\infty)$,

(2.32) Φ is twice continuously differentiable,

(2.33) $\int\limits_{2\delta}^{\infty} dx\,\Phi(x)<\infty$,

(2.34) behaviour near $\delta:\lim\limits_{x\downarrow\delta}\Phi(x)=\infty$. There exists a positive number α (fixed in the following) with $|\Phi'(x)|\leqq(x-\delta)^{-\alpha}$ and $|\Phi''(x)|\leqq(x-\delta)^{-\alpha-1}$ for all x near $\delta\,(x>\delta)$.

3. Existence

Given a potential Φ satisfying (2.30)–(2.34) we must show: the system of Eqs. (2.1) is solvable for sufficiently many initial conditions $x\in\mathbb{M}$. For this purpose we define for all $\beta>0$

(3.1) $\mathbb{X}_\beta=\left\{x\in\mathbb{M}:\inf\limits_{i\in\mathbb{Z}}(z_i-\delta)\,[\log_+(i)]^{1/\beta}>0\right\}$,

where $\log_+(x)=\log(1+|x|)$.

Theorem 1[2]**.** *Given α as in (2.34). Then for all $\beta>\frac{\alpha+1}{2}$ we have: For every initial condition $x\in\mathbb{X}_\beta$ there exists one and only one solution $x(t)$ of the Eqs. (2.1) satisfying*

(3.2) $\inf\limits_{0\leqq t\leqq T}\inf\limits_{i\in\mathbb{Z}}(z_i(t)-\delta)\,[\log_+(x_i)]^{1/\beta}>0\quad$ for all $\quad T<\infty$.

Example 3.1. Given $\Phi(x)=(|x|-\delta)^{-\lambda}$ with $\lambda>2$. Then we have $\varrho(\mathbb{X}_\lambda)=1$ for all Gibbs measures ϱ w.r.t. the potential Φ ([12], Theorem 3.2).

Proof of the Theorem.

Step 1. Consider a system of n points on $\mathbb{R}^1$ moving according to $\dot{x}_i=-\sum\limits_{\substack{1\leqq j\leqq n\\ j\neq i}}\Phi'(x_i-x_j)=:v_i(1\leqq i\leqq n)$. We formulate mathematically the idea that the particles disperse due to the convexity of Φ. Given $t\geqq 0$ and an index i with $v_i^2(t)=\max\limits_{1\leqq j\leqq n}$ it follows that

(3.3) $\frac{d}{dt}v_i^2(t)\leqq 0,\quad$ because

$$\frac{d}{dt}v_i^2(t)=-2v_i\sum_{j\neq i}\Phi''(x_i-x_j)\,(v_i-v_j)=-2\sum_{j\neq i}\Phi''(x_i-x_j)\,(v_i^2-v_iv_j)\leqq 0,$$

using $\Phi''\geqq 0$ and $|v_i(t)|\geqq|v_j(t)|$ for all j.

2 The aim of this section is only to give a simple existence proof in the case of convex Φ such that the content of this paper is not empty. Certainly one can find an existence proof for other potentials too

Given $t \geqq 0$ and an index i with $z_i(t) = \inf_{2 \leqq j \leqq n} z_j(t)$ it follows that

(3.4) $\frac{d}{dt} z_i(t) \geqq 0$, because

$$\begin{aligned}
\frac{d}{dt} z_i(t) &= \Big(-\sum_{1 \leqq j < i} \Phi'(x_i - x_j) + \sum_{i < j \leqq n} \Phi'(x_j - x_i)\Big) \\
&\quad - \Big(-\sum_{1 \leqq j < i-1} \Phi'(x_{i-1} - x_j) + \sum_{i-1 < j \leqq n} \Phi'(x_j - x_{i-1})\Big) \\
&= \sum_{1 \leqq j < i-1} (-\Phi'(x_i - x_{j+1}) + \Phi'(x_{i-1} - x_j)) - \Phi'(x_i - x_1) \\
&\quad + \sum_{i \leqq j \leqq n-1} (-\Phi'(x_j - x_{i-1}) + \Phi'(x_{j+1} - x_i)) - \Phi'(x_n - x_{i-1}) \\
&\geqq 0,
\end{aligned}$$

Φ' being negative and monotonically increasing on (δ, ∞) and $x_i - x_{j+1} \leqq x_{i-1} - x_j$ respectively $x_j - x_{i-1} \leqq x_{j+1} - x_i$ $(1 \leqq j \leqq n-1)$.

It follows from (3.3) and (3.4) that

(3.5) $t \mapsto \max_{1 \leqq i \leqq n} v_i^2(t)$ is decreasing, $t \mapsto \min_{2 \leqq i \leqq n} z_i(t)$ is increasing:

this holds by virtue of the following elementary fact: Given functions $g_i : \mathbb{R}_+ \to \mathbb{R}$ $(1 \leqq i \leqq n)$, continuously differentiable, $\bar{g}(t) := \max_{1 \leqq i \leqq n} g_i(t)$. If $g_i'(t) \leqq 0$ for all $t \geqq 0$ and all indices i satisfying $g_i(t) = \bar{g}_i(t)$, it follows that

(3.6) $t \mapsto \bar{g}(t)$ is decreasing,

and an analogous statement holds for $\min_i g_i(t)$ and $g_i'(t) \geqq 0$.

Step 2. Given $x = \{x_i\} \in \mathbb{X}_\beta$ and $n \in \mathbb{N}$, consider only the particles x_i with $|i| \leqq n$; these move according to the equations of a gradient system consisting of $2n+1$ particles. Denote the solution of that system by $x^{(n)}(t)$. Step 1 enables us to apply Lanford's iteration method quite analogously to the procedure used in [12] to estimate

(3.7) $$|x_i^{(n+1)}(t) - x_i^{(n)}(t)| \leqq \int_0^t ds \Big| \sum_{j \neq i} \Phi'(x_i^{(n+1)}(s) - x_j^{(n+1)}(s)) - \sum_{j \neq i} \Phi'(x_i^{(n)}(s) - x_j^{(n)}(s)) \Big|,$$

because Step 1 gives estimates (dependent on n) on the Lipschitz constant of Φ' and on the fluctuations of the particles: for $x \in \mathbb{X}_\beta$ we have:

(3.8) $$\begin{aligned}
|v_i| &\leqq \sum_{j \neq i} |\Phi'(x_i - x_j)| \leqq |\Phi'(z_i)| + |\Phi'(z_{i+1})| + 2 \sum_{j \geqq 2} |\Phi'(j\delta)| \\
&\leqq (z_i - \delta)^{-\alpha} + (z_{i+1} - \delta)^{-\alpha} + \text{const} \quad [\text{using (2.34)}] \\
&\leqq \text{const}[\log_+(i)]^{\alpha/\beta} \quad \text{with a constant} < \infty \quad [\text{using (3.1)}].
\end{aligned}$$

According to (3.1), (3.8), and (3.5) there exists a constant $C(x)<\infty$ with

$$(3.9)\quad \inf_{t\geq 0}\inf_{|i|\leq n}(z_i^{(n)}(t)-\delta)\geq \inf_{|i|\leq n}(z_i^{(n)}(0)-\delta)\geq \frac{1}{C(x)}[\log n]^{-1/\beta}$$

for all $n\in\mathbb{N}$,

$$(3.10)\quad \sup_{t\geq 0}\sup_{|i|\leq n}|v_i^{(n)}(t)|\leq \sup_{|i|\leq n}|v_i^{(n)}(0)|\leq C(x)\,|\log n|^{\alpha/\beta}$$

for all $n\in\mathbb{N}$.

So we see that the estimates, which are derived in [12] (Theorems 3.1 and 3.2) for all points in a set of Gibbs measure 1, are satisfied in our case even pointwise for all $x\in\mathbb{X}_\beta$. The hypothesis $(2\lambda_1)^{-1}(2+\lambda_2)<1$, assumed in [12] at the top of p. 83, reads in our notation as $(2\beta)^{-1}(1+\alpha)<1$, and this holds here.

Definition 3.1. Let $\mathscr{L}$ denote the set of all $P\in\mathbb{P}$ such that

(3.11) there exists $\beta>\frac{\alpha+1}{2}$ with $P(\mathbb{X}_\beta)=1$ [$\mathbb{X}_\beta$ as in (3.1), recall (2.21)],

(3.12) there exists $\eta>0$ with

$$E_P[(z_1+\ldots+z_k)^{2+\eta}]<\infty \quad \text{for all} \quad k\in\mathbb{N} \quad \text{and}$$
$$E_P[(z_1-\delta)^{-2\alpha}]<\infty,$$

(3.13) the following functions can be differentiated by first differentiating the integrands and then taking expectations:

$$t\to E_P\Big[\sum_{i\neq 0}\Phi(x_i(t)-x_0(t))\Big],\qquad t\to E_P\Big[\Big|\sum_{i\neq 0}\Phi'(x_i(t)-x_0(t))\Big|^2\Big],$$

$$t\to E_P\Big[\Big(\sum_{i=1}^{k}z_i(t)\Big)^{2+\eta}\Big]\,(k\in\mathbb{N}),\qquad t\to E_P[(z_1(t)-\delta)^{-2\alpha}]$$

[comment to (3.13): These functions are well defined at $t=0$, as follows from (3.12), (2.33), (2.34), (2.21), and the condition about the hard core. The proof of Theorem 1 enables one to find sufficient conditions on the initial distribution P so that the above expectations exist for all $t\geq 0$, and under which it is valid to interchange the operations of differentiation and expectation, but these are unessential details, so the formulation (3.13) is sufficient.]

4. Characterization of the Rigid State

As in (2.31) denote the range of Φ by R ($\delta<R\leq\infty$), let r be a number with $\delta<r<\infty$.

$P\in\mathbb{P}$ is called a *rigid state* [cf. (2.2)] if

$$(4.1)\quad \sum_{j\neq i}\Phi'(x_i-x_j)=0 \quad \text{for all} \quad i\in\mathbb{Z} \quad P\text{-a.e.}$$

In this section we consider $P\in\mathbb{P}$ satisfying the following properties

$$(4.2)\quad P\Big\{\lim_{i\to\pm\infty}\frac{x_i}{i}=r\Big\}=1,$$

$$(4.3)\quad E_P[|\Phi'(z_0)|]<\infty.$$

Furthermore we define

(4.4) $\mathbb{P}(\Phi,r)=\{P\in\mathbb{P}: P$ satisfies (4.2) and the following condition (4.5)$\}$,

(4.5) $V(P)=E_P\left[\sum_{i\neq 0}\Phi(x_i)\right]<\infty$.

With this notation we are ready to formulate the following two theorems about rigid states.

Theorem 2. *If $r\leqq R$, then there exists a unique rigid state $P\in\mathbb{P}$ satisfying* (4.1)–(4.3), *namely $P=Q_r$ (defined by* (2.28)*).*

Proof. We postpone the proof to § 5, p. 145, because then Theorem 2 will follow easily with the same method used to prove Lemma 4. Under the additional assumption of the integrability of the potential at infinity (which is not necessary for Theorem 2 as the proof in § 5 will show) there is an alternative way to prove Theorem 2, namely by translating the idea of the proof of Proposition 1 to the case of infinitely many degrees of freedom. A similar method is used to prove the following

Theorem 3. *Let $r\leqq R$. Then Q_r is characterized as that measure $\bar{P}\in\mathbb{P}(\Phi,r)$ for which*

(4.6) $V(\bar{P})=\min\{V(P): P\in\mathbb{P}(\Phi,r)\}$.

Proof. We will use the following notation:

(4.7) $\bar{x}=\{ir: i\in\mathbb{Z}\}$,

(4.8) $W_n(x)=\frac{1}{n}\cdot\frac{1}{2}\sum_{\substack{|i|\leqq n,|j|\leqq n\\ i\neq j}}\Phi(x_i-x_j)\qquad (x\in\mathbb{M}, n\in\mathbb{N})$,

(4.9) $\mathbb{Y}=\left\{x\in\mathbb{M}: \lim_{i\to\pm\infty}\frac{x_i}{i}=r, \lim_{n\to\infty}W_n(x) \text{ exists}\right\}$,

so that we can define the energy function $\bar{W}$ on $\mathbb{Y}$ by

(4.10) $\bar{W}(x)=\lim_{n\to\infty}W_n(x)\qquad (x\in\mathbb{Y})$.

For $P\in\mathbb{P}(\Phi,r)$ we have $P(\mathbb{Y})=1$ and if in addition P is ergodic, we get by the ergodic theorem and (4.5)

(4.11) $V(P)=\bar{W}(x)\quad P$-a.e.,

(4.12) $V(Q_r)=\bar{W}(\bar{x})$.

The theorem will be proved by the following two steps:

Step 1. Show that

(4.13) $\bar{W}(x)\geqq\bar{W}(\bar{x})$ for all $x\in\mathbb{Y}$

which implies

(4.14) $V(P)\geqq V(Q_r)$ for all $P\in\mathbb{P}(\Phi,r)$.

Step 2. Assume that there exists a measure $P\in\mathbb{P}(\Phi,r)$ such that

$$(4.15)\quad P\neq Q_r,\quad \text{but}\quad V(P)=V(Q_r).$$

Without loss of generality we can assume that P is ergodic, hence we get from (4.11)–(4.13) and (4.15)

$$(4.16)\quad \bar{W}(x)=\bar{W}(\bar{x})\quad P\text{-a.e.}$$

However we will show

(4.17) $\bar{W}(x)>\bar{W}(\bar{x})$ for x in a set of strictly positive P-measure, and this contradicts (4.16).

Proof of (4.13). Computing the first and second derivatives of W_n in a way similar to (2.4), (2.5), (2.7), and using the Taylor development of W_n about $\bar{x}$ we get

$$(4.18)\quad W_n(x)=W_n(\bar{x})+\frac{1}{n}\sum_{|i|\leqq n}(x_i-\bar{x}_i)\sum_{|j|\leqq n,\,j\neq i}\Phi'(\bar{x}_i-\bar{x}_j)$$
$$+\frac{1}{4}\cdot\frac{1}{n}\sum_{\substack{|i|\leqq n,\,|j|\leqq n\\ i\neq j}}((x_i-\bar{x}_i)-(x_j-\bar{x}_j))^2\,\Phi''((\vartheta x_i+(1-\vartheta)\bar{x}_i$$
$$-(\vartheta x_j+(1-\vartheta)\bar{x}_j))$$
$$\text{for some}\quad \vartheta\in(0,1).$$

The last term is non-negative by the convexity of Φ. Using (2.33) and $\sum_{j\neq i}\Phi'(\bar{x}_i-\bar{x}_j)=0$ $(i\in\mathbb{Z})$ we see easily that

$$(4.19)\quad \sup_n\left|\sum_{|i|\leqq n}\sum_{|j|\leqq n,\,j\neq i}\Phi'(\bar{x}_i-\bar{x}_j)\right|<\infty.$$

Furthermore

$$(4.20)\quad \lim_{|i|\to\infty}\frac{1}{|i|}|x_i-\bar{x}_i|=0$$

because of $\lim_{i\to\pm\infty}\frac{x_i}{i}=\lim_{i\to\pm\infty}\frac{\bar{x}_i}{i}=r$ by (4.9).

Thus from (4.18), (4.19), (4.20) we get (4.13).

Proof of Step 2. Let P be an ergodic measure in $\mathbb{P}(\Phi,r)$ satisfying (4.15)

Case $r=R$

By $P\neq Q_R$ we have $P\{x_1<R\}>0$, hence $V(P)=E_P\left[\sum_{i\neq 0}\Phi(x_i)\right]>0$, but this contradicts $V(Q_r)=V(Q_R)=E_{Q_R}\left[\sum_{i\neq 0}\Phi(x_i)\right]=0$.

Case $r<R$

(4.2) and $P\neq Q_r$ imply that there exist a number $\eta>0$ and a set $N\subset\mathbb{Y}$ satisfying

(4.21) $P(N)>0$ and

(4.22) $\lim\limits_{n\to\infty}\frac{1}{n}\sum\limits_{-n<i\leqq n}1_{[z\in\mathbb{R}_+:z\leqq r-\eta]}(x_i-x_{i-1})>0$ for $x\in N$.

If $x_i-x_{i-1}\leqq r-\eta$, we get

(4.23) $\inf\limits_{0<\vartheta<1}\Phi''(\vartheta(x_i-x_{i-1})+(1-\vartheta)r)\geqq\inf\limits_{\delta<x\leqq r}\Phi''(x)=:\zeta>0$.

(4.22) and (4.23) imply

$$\begin{aligned}(4.24)\quad &\inf_{0<\vartheta<1}\frac{1}{4}\cdot\frac{1}{n}\sum_{-n<i\leqq n}\Phi''(\vartheta(x_i-x_{i-1})+(1-\vartheta)r)(x_i-x_{i-1}-r)^2\\ &\geqq\inf_{0<\vartheta<1}\frac{1}{4}\cdot\frac{1}{n}\sum_{-n<i\leqq n}\Phi''(\vartheta(x_i-x_{i-1})+(1-\vartheta)r)\cdot\\ &\qquad\cdot(x_i-x_{i-1}-r)^2 1_{[z\in\mathbb{R}_+:z\leqq r-\eta]}(x_i-x_{i-1})\\ &\geqq\frac{1}{4}\zeta\eta^2\frac{1}{n}\sum_{-n<i\leqq n}1_{[z\in\mathbb{R}_+:z\leqq r-\eta]}(x_i-x_{i-1})\qquad(n\in\mathbb{N}).\end{aligned}$$

Therefore we get from (4.18), (4.24), and (4.22) that (4.17) is satisfied for all $x\in N$, so the proof of Theorem 3 is finished.

5. The Ergodic Theorem

Lemma 1. *Let $P\in\mathscr{L}$ be given*

(a) *Given $t_n\uparrow\infty$ and $\bar{P}\in\mathbb{P}$ such that $P_{t_n}\to\bar{P}$, then $I(\bar{P})=I(P_t)$ for all $t\geqq 0$ (where the intensity I is defined by* (2.27)*).*

(b) *The set $\{P_t:t\geqq 0\}$ is relatively compact in $\mathbb{P}$.*

(c) *Given $t_n\uparrow\infty$ and $\bar{P}\in\mathbb{P}$ such that $P_{t_n}\to\bar{P}$ and a continuous not necessarily bounded function $\varphi:(\mathbb{R}_+)^{\mathbb{Z}}\to\bar{\mathbb{R}}_+$. Then it is sufficient for $\lim\limits_{n\to\infty}E_{P_{t_n}}(\varphi)=E_{\bar{P}}(\varphi)$ to hold that the set of functions $\{\varphi(z(t)):t\geqq 0\}$ is uniformly integrable with respect to P. The latter condition is satisfied e.g. if there exists an $\varepsilon>0$ such that $\sup\limits_{t\geqq 0}E_{P_t}(\varphi^{1+\varepsilon})<\infty$.*

Proof. (c) Clear.

(a) Since (2.1) is invariant under translations, it follows that $I(P_t)=I(P)$ for all $t\geqq 0$, see for example [5], Lemma (4.3). Since Φ has a hard core we can apply e.g. [8], Theorem 4.6.3, so we get $I(P_t)=I(\bar{P})$ for all $t\geqq 0$.

(b) Because Φ has a hard core of length δ and hence $P_t\in\mathbb{P}$ for all $t\geqq 0$ one can apply compactness criteria from the general theory of point processes (e.g. [8], Chap. 4.3) to conclude that $\{P_t:t\geqq 0\}$ is relatively compact in the set of probability measures on $[\delta,\infty)^{\mathbb{Z}}$ [since we confine ourselves to the one-dimensional case we could alternatively apply [5], Corollary (12.4) combined with the subsequent Lemma 4(a)]. It remains to show that. $P_{t_n}\to\bar{P}$ implies $\bar{P}\in\mathbb{P}$. For the proof consider the function $\varphi(z):=\left[\sum\limits_{i\neq 0}\Phi(x_i)\right]^{1/2}$ which is continuous on the set $[\delta,\infty)^{\mathbb{Z}}$ by virtue of

$$\limsup_{n\to\infty} \sup_{x\in\mathbb{M}} \sum_{i:x_i\notin[-n,+n]} \Phi(x_i)=0$$

$\left[\text{recall } \inf_i (x_{i+1}-x_i)\geqq\delta \text{ for } x\in\mathbb{M}\right].$

The function $t\to E_{P_t}[\varphi^2]$ is decreasing [the computation is post-poned to the subsequent Lemma 2(a)] so we can apply the criterion of Part (c) to conclude that

$$E_{\bar{P}}[\varphi]=\lim_{n\to\infty} E_{P_{t_n}}[\varphi]\leqq 1+\overline{\lim_{n\to\infty}}\, E_{P_{t_n}}[\varphi^2]\leqq 1+E_P[\varphi^2]<\infty$$

$\left(E_P[\varphi^2]=E_P\left[\sum_{i\neq 0}\Phi(x_i)\right]<\infty \text{ by } P\in\mathscr{L}\right).$

This implies $\sum_{i\neq 0}\Phi(x_i)<\infty$ $\bar{P}$-a.e., hence $z_1>\delta$ $\bar{P}$-a.e., so Lemma 1 is proved.

The proof of Proposition 2 is based on the existence of the Liapunov functions (2.13) and (2.14). Analogously we first prove [cf. (4.5)] the following

Lemma 2. *Let $P\in\mathscr{L}$.*

(a) *$t\to V(P_t)$ is decreasing.*

(b) *$t\to E_P[v_0^2(t)]=E_P\left[\left|\sum_{i\neq 0}\Phi'(x_i(t)-x_0(t))\right|^2\right]$ is decreasing.*

(c) *$t\to V(P_t)$ is convex and $\frac{d}{dt}V(P_t)=-2E_P[v_0^2(t)]$.*

Proof. Similar to the considerations in [13, p. 222] we use the shift invariance of P:

(a) When the context is clear, we omit the argument t and write E instead of E_P

$$\begin{aligned}
\frac{d}{dt}V(P_t)&=\frac{d}{dt}E\left[\sum_{i\neq 0}\Phi(x_i(t)-x_0(t))\right]=E\left[\sum_{i\neq 0}\Phi'(x_i-x_0)(v_i-v_0)\right]\\
&=-E[v_0^2]-E\left[\sum_{i\neq 0}\Phi'(x_i-x_0)\sum_{j\neq i}\Phi'(x_i-x_j)\right]\\
&=-E[v_0^2]-\sum_{i\neq 0}E\left[\Phi'(x_i-x_0)\sum_{j\neq i}\Phi'(x_i-x_j)\right]\\
&=-E[v_0^2]-\sum_{i\neq 0}E\left[\Phi'(x_0-x_{-i})\sum_{j\neq 0}\Phi'(x_0-x_j)\right]\\
&=-E[v_0^2]-E[v_0^2]=-2E[v_0^2].
\end{aligned}$$

(b)
$$\begin{aligned}
\frac{d}{dt}E[v_0^2(t)]&=2E[v_0\dot{v}_0]=-2E\left[v_0\sum_{j\neq 0}\Phi''(x_0-x_j)(v_0-v_j)\right]\\
&=-2E\left[\sum_{j\neq 0}\Phi''(x_0-x_j)v_0^2\right]+2E\left[\sum_{j\neq 0}\Phi''(x_0-x_j)v_0v_j\right]\\
&=-E\left[\sum_{j\neq 0}\Phi''(x_0-x_j)v_0^2\right]-E\left[\sum_{j\neq 0}\Phi''(x_0-x_j)v_j^2\right]\\
&\quad+2E\left[\sum_{j\neq 0}\Phi''(x_0-x_j)v_0v_j\right]\\
&=-E\left[\sum_{j\neq 0}\Phi''(x_0-x_j)(v_0-v_j)^2\right]\leqq 0
\end{aligned}$$

[where we have used $\Phi''(-x)=\Phi''(x)\geqq 0$, cf. the proof of (2.7)].

(c) Follows from (a) and (b).

Lemma 3. *Let $P \in \mathscr{L}$, $t_n \uparrow \infty$ such that $P_{t_n} \to \bar{P}$ for some $\bar{P} \in \mathbb{P}$. Then we have*

(5.1) $E_{\bar{P}}[v_0^2]=0.$

Proof. Lemma 2(a) and (c) implies the following proposition analogous to (2.15)

(5.2) $\lim_{t \to \infty} E_{P_t}[v_0^2]=0.$

Since $\bar{P} \in \mathbb{P}$, $v_0 = \sum_{i \neq 0} \Phi'(x_i)$ is well defined $\bar{P}$-a.e. Further the function $z \mapsto \sum_{i \neq 0} \Phi'(x_i)$ is continuous on the set $(\delta, \infty)^{\mathbb{Z}}$ by virtue of $\lim_{n \to \infty} \sup_{x \in \mathbb{M}} \sum_{i: x_i \notin [-n, +n]} \Phi'(x_i)=0.$ Therefore (5.2) implies $E_{\bar{P}}[v_0^2 \wedge N]=0$ for all $N \in \mathbb{N}$, hence Lemma 3 is proved.

From the considerations so far mentioned, which as one can show are not restricted to dimension $d=1$, it follows, that for an initial distribution $P \in \mathscr{L}$ the set of ω-limit points of $\{P_t : t \geqq 0\}$ is non-empty and contained in the set of rigid states. From this and from Theorem 2 we cannot immediately deduce an ergodic theorem: the situation is different from that in (2.16), because now we know from Lemma 1(a) only $E_{\bar{P}}(z_1)=E_P(z_1)$, but not whether even the distribution of the variable $\lim_{i \to \infty} \frac{x_i}{i}$ is conserved in the limit $t \to \infty$. The following considerations are based on the fact, that, besides the potential and the kinetic energy, there exist still more Liapunov functions in the case of dimension $d=1$ (cf. [13, p. 222]).

Therefore the following reasoning is typically one-dimensional.

Lemma 4. *Let $P \in \mathscr{L}$, $k \in \mathbb{N}$ and $\varphi : \mathbb{R}_+ \to \bar{\mathbb{R}}$ be a convex function, which is differentiable on the domain $\{z \in \mathbb{R}_+ : \varphi(z) < \infty\}$. Then the function*

$$(5.3)\quad t \to E_P\left[\varphi\left(\sum_{i=1}^{k} z_i(t)\right)\right]$$

is decreasing (if the expectation exists and if it is permissible to interchange the order of differentiation and expectation, cf. (3.13)*). Particular cases are (with $\eta > 0$, $\alpha > 0$ given in* (3.12), (2.34) *resp.):*

(a) $\varphi(z)=z^{2+\eta}$,
(b) $\varphi(z)=(z-\delta)^{-2\alpha}$ $\quad (z>\delta)$,
(c) $\varphi(z)=e^{-\lambda z}$ $\quad (\lambda>0)$.

Proof. For the proof we abbreviate

(5.4) $f_i^{(n)} := \Phi'(z_i + \ldots + z_{i+n-1}) \qquad (i \in \mathbb{Z}, n \in \mathbb{N})$

using (5.4) we get from (2.1)

$$(5.5)\quad \dot{z}_i = \dot{x}_i - \dot{x}_{i-1} = \left(\sum_{n \geqq 1} f_{i+1}^{(n)} - \sum_{n \geqq 1} f_{i-n+1}^{(n)}\right) - \left(\sum_{n \geqq 1} f_i^{(n)} - \sum_{n \geqq 1} f_{i-n}^{(n)}\right)$$
$$= \left(\sum_{n \geqq 1} f_{i+1}^{(n)} - \sum_{n \geqq 1} f_i^{(n)}\right) - \left(\sum_{n \geqq 1} f_{i+1-n}^{(n)} - \sum_{n \geqq 1} f_{i-n}^{(n)}\right),$$

which implies

$$(5.6)\quad \sum_{i=1}^{k} \dot{z}_i = \sum_{n\geqq 1} (f_{k+1}^{(n)} - f_1^{(n)}) - \sum_{n\geqq 1} (f_{k+1-n}^{(n)} - f_{1-n}^{(n)})$$
$$= \sum_{n\geqq 1} [(f_{1-n}^{(n)} - f_1^{(n)}) - (f_{k+1-n}^{(n)} - f_{k+1}^{(n)})].$$

$$(5.7)\quad \frac{d}{dt} E\left[\varphi\left(\sum_{i=1}^{k} z_i\right)\right] = E\left[\varphi'\left(\sum_{i=1}^{k} z_i\right)\sum_{i=1}^{k} \dot{z}_i\right].$$

To prove that this expression is $\leqq 0$, by (5.6) it is sufficient to show:

$$(5.8)\quad E\left[\varphi'\left(\sum_{i=1}^{k} z_i\right)[(f_{1-n}^{(n)} - f_1^{(n)}) - (f_{k+1-n}^{(n)} - f_{k+1}^{(n)})]\right] \leqq 0 \qquad (n \in \mathbb{N}).$$

Using again the shift invariance of P in a way similar to the proof of Lemma 2, it follows that for all $n \in \mathbb{N}$:

$$(5.9)\quad E\left[\varphi'\left(\sum_{i=1}^{k} z_i\right)(f_{1-n}^{(n)} - f_1^{(n)})\right] = E\left[\left(\varphi'\left(\sum_{i=n+1}^{n+k} z_i\right) - \varphi'\left(\sum_{i=1}^{k} z_i\right)\right) f_1^{(n)}\right].$$

$$(5.10)\quad E\left[\varphi'\left(\sum_{i=1}^{k} z_i\right)(f_{k+1-n}^{(n)} - f_{k+1}^{(n)})\right] = E\left[\left(\varphi'\left(\sum_{i=n+1}^{n+k} z_i\right) - \varphi'\left(\sum_{i=1}^{k} z_i\right)\right) f_{k+1}^{(n)}\right].$$

Substituting (5.9) and (5.10) in (5.8) and using (5.4) we get

$$(5.11)\quad E\left[\varphi'\left(\sum_{i=1}^{k} z_i\right)[(f_{1-n}^{(n)} - f_1^{(n)}) - (f_{k+1-n}^{(n)} - f_{k+1}^{(n)})]\right]$$
$$= -E\left[\left(\varphi'\left(\sum_{i=n+1}^{n+k} z_i\right) - \varphi'\left(\sum_{i=1}^{k} z_i\right)\right)\left(\Phi'\left(\sum_{i=k+1}^{k+n} z_i\right) - \Phi'\left(\sum_{i=1}^{n} z_i\right)\right)\right],$$

but this is $\leqq 0$, because $\sum_{i=n+1}^{n+k} z_i \geqq \sum_{i=1}^{k} z_i$ if and only if $\sum_{i=k+1}^{k+n} z_i \geqq \sum_{i=1}^{n} z_i$ and because φ' and Φ' are both increasing functions. This proves Lemma 4.

This is the right place to supplement the proof of Theorem 2.

Proof of Theorem 2. Given $r \leqq R$ and $P \in \mathbb{P}$ satisfying (4.1)–(4.3) we have to show that

$$(5.12)\quad P\{z_1 = r\} = 1.$$

From (4.1) we get

$$(5.13)\quad E_P\left[z_1\left(-\sum_{j\neq 1} \Phi'(x_1 - x_j) + \sum_{j\neq 0} \Phi'(x_0 - x_j)\right)\right] = 0$$

(the integrability of the integrand follows easily from (2.24), (4.3), (4.1) and $\sum_{n\geqq 2} |\Phi'(n\delta)| < \infty$).

Using the notation (5.4) and $\sum_{n\geqq 2} |\Phi'(n\delta)| < \infty$ we can write (5.13) in the form

$$(5.14)\quad 0 = \sum_{n\geqq 1} E_P[z_1[(f_{1-n}^{(n)} - f_1^{(n)}) - (f_{2-n}^{(n)} - f_2^{(n)})]]$$

and applying (5.8) and (5.11) with $\varphi(z) = z^2$ and $k = n = 1$ we get from (5.14)

$$(5.15)\quad (z_{i+1} - z_i)(\Phi'(z_{i+1}) - \Phi'(z_i)) = 0 \quad \text{for all} \quad i \in \mathbb{Z} \ \ P\text{-a.e.}$$

It is easy to see that (5.15) and (4.2) imply (5.12).

Lemma 5. *Let $P \in \mathscr{L}$ be an ergodic measure satisfying $E_P(z_1) = : r \leqq R \leqq \infty$, where R is the range of Φ (cf. (2.31)). Given a sequence $t_n \uparrow \infty$ and $\bar{P} \in \mathbb{P}$ such that $P_{t_n} \to \bar{P}$, we have $(z_2(t_n) - z_1(t_n)) \xrightarrow{L^2(P)} 0$ for $n \to \infty$.*

Proof. Lemmas 4(a) and 1(c) imply

(5.16) $\lim_{n\to\infty} E_{P_{t_n}}[(z_1 + \ldots + z_k)^2] = E_{\bar{P}}[(z_1 + \ldots + z_k)^2] < \infty \qquad (k \in \mathbb{N}),$

using

(5.17) $$E_P[(z_2(t) - z_1(t))^2] = -E_P[(z_1(t) + z_2(t))^2] + 4E_P[z_1^2(t)]$$
$$= -E_{P_t}[(z_1 + z_2)^2] + 4E_{P_t}[z_1^2] \qquad (t \geqq 0)$$

and applying (5.23) we get

(5.18) $\lim_{n\to\infty} E_P[(z_2(t_n) - z_1(t_n))^2] = E_{\bar{P}}[(z_2 - z_1)^2]$;

to prove the proposition it is therefore sufficient to show that $\bar{P}$ is a mixture of measures Q_r $(0 < r \leqq R)$.

Case I: $R = \infty$

Without loss of generality we can assume that $\bar{P}$ is ergodic (decompose $\bar{P}$ into its ergodic components) and we define $\bar{r} := E_{\bar{P}}(z_1) < \infty [E_{\bar{P}}(z_1) < \infty$ by (5.16)]. Then $\bar{P}$ satisfies the assumptions (4.1)–(4.3) of Theorem 2: (4.1) follows from Lemma 3, (4.2) is satisfied, because $\bar{P}$ is ergodic with $E_{\bar{P}}(z_1) = \bar{r} < \infty$, and (4.3) follows from $E_{\bar{P}}[(z_1 - \delta)^{-\alpha}] < \infty$ $\left(\text{use Lemma 1(c) and } \sup_{t \geqq 0} E_{P_t}[(z_1(t) - \delta)^{-2\alpha}] < \infty \text{ according to Lemma 4(b)}\right)$ and from (2.34). Since $\bar{r} < R = \infty$, the proposition follows from Theorem 2.

Case II: $R < \infty$

It is proved by Nguyen Xuan Xanh. With his kind permission we reproduce his proof:

If we show

(5.19) $\bar{P}\{z_1 \leqq R\} = 1,$

then we can proceed in the same way as in the case $R = \infty$: using (5.19) we can assume without loss of generality that $\bar{P}$ is ergodic with $\bar{r} := E_{\bar{P}}(z_1) \leqq R$, and so we can apply Theorem 2 again. Assume that (5.19) is wrong, i.e. that there exists $\varepsilon > 0$ such that

(5.20) $\overline{\lim_{n\to\infty}} P\{z_1(t_n) \geqq R + \varepsilon\} > 0.$

To deduce a contradiction we observe firstly that the functions

(5.21) $t \mapsto z_i(t) \vee R \qquad (i \in \mathbb{Z}) \qquad$ are decreasing

$\left(\text{for } x_i - x_{i-1} \geqq R \text{ we get } \dot{x}_i = \sum_{j>i} \Phi'(x_j - x_i) \leqq 0 \text{ and } \dot{x}_{i-1} = -\sum_{j<i-1} \Phi'(x_{i-1} - x_j) \geqq 0, \text{ hence } \dot{z}_i \leqq 0\right).$

Therefore the events $\{z_1(t) \geqq R+\varepsilon\}$ are decreasing in t, so we get from (5.20) the relation

(5.22) $$P\Big(\bigcap_{n\geqq 1} \{z_1(t_n) \geqq R+\varepsilon\}\Big) > 0.$$

The ergodic theorem and (5.22) imply

(5.23) $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} 1_{\left[\bigcap_{n\geqq 1} \{z_i(t_n) \geqq R+\varepsilon\}\right]}(x) > 0$$

for points $x \in \mathbb{M}$ which are elements of a set of positive P-measure.

Since P is ergodic and $E_P(z_1) = r \leqq R$ we can find a point $x \in \mathbb{M}$ satisfying (5.23) and furthermore the condition

(5.24) $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} z_i(0) = r \leqq R.$$

So let x be such a point. Define

(5.25) $$I := \{i \in \mathbb{N} : z_i(t_n) \geqq R+\varepsilon \text{ for all } n \in \mathbb{N}\} = \{i_1, i_2, \dots\}, \text{ where } i_1 < i_2 < \dots.$$

The points $\{x_j : i_k \leqq j < i_{k+1}\}$ form clusters not interacting with any other particle for any $t \geqq 0$, because the "barriers" $z_{i_k}(t)$ and $z_{i_{k+1}}(t)$ remain $\geqq R+\varepsilon$ for all $t \geqq 0$ [cf. (5.25) and (5.21)]. Hence the clusters $\{x_j(t) : i_k \leqq j < i_{k+1}\}$ are finite systems without boundary condition. Considerations similar to those in Sect. 2 show that $\lim_{t\to\infty} z_j(t) \geqq R$ $(i_k < j < i_{k+1})$. Using (5.21) we deduce that the spacing available to the particles $\{x_j : i_k - 1 \leqq j \leqq i_{k+1}\}$ at time $t=0$ must be $\geqq (i_{k+1} - (i_k - 1)) \cdot R + 2\varepsilon$, and more generally

(5.26) $$x_{i_n} - x_{i_1 - 1} \geqq (i_n - (i_1 - 1)) \cdot R + n \cdot \varepsilon \qquad (n \in \mathbb{N}).$$

This implies

(5.27) $$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} z_i(0) = \lim_{n\to\infty} \frac{x_{i_n}}{i_n} \geqq R + \varepsilon \cdot \lim_{n\to\infty} \frac{n}{i_n} > R$$

[by (5.25) and (5.23)].

But (5.27) contradicts (5.24).

So Case II is handled and hence Lemma 5 is proved.

We are now ready to prove the ergodic theorem for infinite systems, which is analogous to Proposition 2:

Theorem 4. *Let Φ be a potential with range $R \leqq \infty$ and satisfying the properties (2.30)–(2.34). Let $P \in \mathscr{L}$ be ergodic with $E_P(z_1) =: r \leqq R$. Then*

(5.28) $$\lim_{t\to\infty} E_P[(z_1(t) - r)^2] = 0.$$

Proof (Kesten, Spitzer). Let $t_n \uparrow \infty$; from Lemma 1(b) it follows that there exist a subsequence of $(t_n)_{n\in\mathbb{N}}$, which without loss of generality we will again denote by $(t_n)_{n\in\mathbb{N}}$, and some $\bar{P} \in \mathbb{P}$ such that

(5.29) $$P_{t_n} \to \bar{P}.$$

Since by Lemma 1(a) $r \equiv E_P(z_1(t))$ for all $t \geqq 0$, we have to show:

$$\text{(5.30)} \quad \overline{\lim_{n\to\infty}}\, E_P[z_1^2(t_n)] \leqq r^2 .$$

To use the ergodicity of P for this purpose (if P is ergodic, P_t is ergodic, too, but this will not be used in the actual performance of the proof) it is plausible to approximate r^2 by the averages given in the L^2-ergodic theorem:

$$\text{(5.31)} \quad E_P\left[\left(\frac{1}{k}\sum_{i=1}^{k} z_i(t)\right)^2\right] \leqq r^2 + \varepsilon \quad \text{for all} \quad k \geqq k(t).$$

Lemma 4 is the essential tool to get rid of the time dependence of $k(t)$:

$$\text{(5.32)} \quad E_P\left[\left(\frac{1}{k}\sum_{i=1}^{k} z_i(t)\right)^2\right] \leqq E_P\left[\left(\frac{1}{k}\sum_{i=1}^{k} z_i(0)\right)^2\right] \quad \text{for all} \quad t \geqq 0 \qquad (k \in \mathbb{N}).$$

Next we have to find a connection between the left-hand sides of (5.30) and (5.31), but this is given by Lemma 5:

$$\text{(5.33)} \quad \lim_{n\to\infty} \left| \mathrm{E}_P\left[\left(\frac{1}{k}\sum_{i=1}^{k} z_i(t_n)\right)^2\right] - E_P[z_1^2(t_n)]\right| = 0 \qquad (k \in \mathbb{N}).$$

These considerations [we postpone the verification of (5.33) to the end of the proof] are already sufficient to prove (5.30): Given $\varepsilon > 0$ choose $k \geqq k(0)$ so large that (5.31) holds for $t=0$; using (5.33) with this k we find an index n_0 such that for all $n \geqq n_0$ we have [taking into account (5.32)]:

$$\text{(5.34)} \quad E_P[z_1^2(t_n)] \leqq E_P\left[\left(\frac{1}{k}\sum_{i=1}^{k} z_i(t_n)\right)^2\right] + \varepsilon$$

$$\leqq E_P\left[\left(\frac{1}{k}\sum_{i=1}^{k} z_i(0)\right)^2\right] + \varepsilon \leqq r^2 + 2\varepsilon .$$

Proof of (5.33). By Lemma 4, we know that $\sup_{t\geqq 0} E_P\left[\left(\sum_{i=1}^{k} z_i(t)\right)^2\right] < \infty$ $(k \in \mathbb{N})$, so it is sufficient to prove $\left(\sum_{i=1}^{k} z_i(t_n) - k z_1(t_n)\right) \xrightarrow{L^2(P)} 0$. But this holds, because from (5.29) and Lemma 5 it follows that $L^2(P) - \lim_{n\to\infty} (z_i(t_n) - z_1(t_n)) = 0$ for all $i \in \mathbb{N}$.

Remark 5.1. Let Φ be convex, not necessarily strictly convex on (δ, R), and let $P \in \mathscr{L}$. Then it is clear, that Theorem 4 does not hold in general. But we can still deduce the following far weaker statement:

Given indices $i, j \in \mathbb{Z}$ (fixed), the spacings $(x_i(t) - x_j(t))$ converge in distribution as $t \to \infty$.

Proof. By Lemma 1(b) the set of the distributions of the variables $\{(z_1(t) + \ldots + z_k(t)) : t \geqq 0\}$ $(k \in \mathbb{N}$ fixed) is relatively compact. But the Laplace transforms $E_P[\exp(-\lambda(z_1(t) + \ldots + z_k(t))]$ are decreasing in t because of Lemma 4(c). and this implies that they have the same limit for all sequences $t_n \uparrow \infty$.

Remark 5.2. Using the correspondence between measures $P \in \mathbb{P}$ and translation invariant point processes on $\mathbb{R}^1$ (cf. [8, 11]) we deduce easily from Theorem 4 the ergodic theorem mentioned in the introduction.

Remark 5.3. The proof of Theorem 4 is based on the assumptions (1.8) and (1.9) mentioned in the introduction. Interesting open problems arise if one drops one of these conditions.

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