Commun. math. Phys. 64, 171–176 (1979)

Communications in Mathematical Physics
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Characterization and Uniqueness of Distinguished Self-Adjoint Extensions of Dirac Operators

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Abstract. Distinguished self-adjoint extensions of Dirac operators are characterized by Nenciu and constructed by means of cut-off potentials by Wüst. In this paper it is shown that the existence and a more explicit characterization of Nenciu's self-adjoint extensions can be obtained as a consequence from results of the cut-off method, that these extensions are the same as the extensions constructed with cut-off potentials and that they are unique in some sense.

In the Hilbert space $H := (L^2(\mathbb{R}^3))^4$ the minimal Dirac operator of a spin $\frac{1}{2}$ particle with non-zero rest mass under the influence of a potential $q : \mathbb{R}^3_+ \to \mathbb{R}$ ($\mathbb{R}^3_+ := \mathbb{R}^3 \setminus \{0\}$) q measurable, is given by

$$T := (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta + q) \upharpoonright D_0 ,$$
$$D_0 := (C_0^\infty(\mathbb{R}^3_+))^4$$

(cf. [2, 6, 10] for more details).

We consider Coulomb like potentials q, i.e. potentials q with

$$\mu := \sup_{\mathbb{R}^3_+} |xq(x)| < \infty .$$

Then T is essentially self-adjoint if $\mu < \frac{1}{2}\sqrt{3}$ (cf. [6]) and in general not essentially self-adjoint if $\mu > \frac{1}{2}\sqrt{3}$.

But as long as $\mu < 1$, physically distinguished self-adjoint extensions of T still exist:

By means of cut-off potentials we have shown in [8–10], that for q semibounded from above (or from below) and $\mu < 1$

$$\tilde{T} := T^* \upharpoonright (D(T^*) \cap D(r^{-\frac{1}{2}}))\ {}^1$$

is a self-adjoint extension of T (cf. the appendix for not semibounded potentials).

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1 For $\alpha \in \mathbb{R}$ we denote by r^α the closure of the multiplication operator

$$R_\alpha : D_0 \to H, \quad u(x) \to |x|^\alpha u(x) \quad (u \in D_0,\ x \in \mathbb{R}^3_+) .$$

The multiplication operators q, $r^\alpha q$ are defined analogously

0010-3616/79/0064/0171/$01.20

The physical interpretation of this operator $\tilde{T}$ is that all states in $D(\tilde{T})$ have finite potential energy. Obviously, $\tilde{T}$ is the unique self-adjoint extension of T with this property (cf. the introduction in [4]). For if S is a symmetric extension of T with $D(S) \subset D(r^{-1/2})$, then $S \subset \tilde{T}$ (cf. proof of the theorem).

If q is not necessarily semibound, but still $\mu < 1$ holds, Schmincke proved in [7] by another method that $\tilde{T}$ is self-adjoint (cf. Kalf [1] that $\tilde{T}$ is closed).

Finally, if

$$T_{00} := (\boldsymbol{\alpha} \cdot \mathbf{p} + \beta) \upharpoonright D_0 ,$$
$$T_0 := \overline{T_{00}}$$

denotes the operator of the free particle and $\mu < 1$ is satisfied by q, Nenciu [4] showed that there exists a unique self-adjoint extension $\tilde{\tilde{T}}$ of T such that

$$D(\tilde{\tilde{T}}) \subset D(|T_0|^{1/2})$$

where the inclusion can be interpreted as the fact that only states with finite kinetic energy are in the domain of $\tilde{\tilde{T}}$.

We show in this paper that in the case of semibounded potentials q with $\mu < 1$ the existence and uniqueness of such a self-adjoint operator $\tilde{\tilde{T}}$ is an easy consequence of the results in [10] and that $\tilde{\tilde{T}}$ is explicitly given and equal to $\tilde{T}$.

The results are still valid without the assumption of semiboundness of q as we sketch in the appendix. But, since the case where $\underline{\lim}_{x \to 0} q(x) = -\overline{\lim}_{x \to 0} q(x) = \infty$ has no physical interest we restricted ourselves in [9, 10] and in the main part of this paper on semibounded potentials.

Theorem. *Let* $q : \mathbb{R}^3_+ \to \mathbb{R}$ *be a measurable function, semibounded from above or from below and*

$$\mu := \sup_{\mathbb{R}^3_+} |xq(x)| < 1 .$$

Denote

$$\tilde{T} := T^* \upharpoonright (D(T^*) \cap D(r^{-1/2}))$$
$$\tilde{\tilde{T}} := T^* \upharpoonright (D(T^*) \cap D(|T_0|^{1/2}))$$

(T, T_0 as above).
Then $\tilde{T}$, $\tilde{\tilde{T}}$ *are self-adjoint and*

$$\tilde{T} = \tilde{\tilde{T}} . \tag{1}$$

Moreover, $\tilde{T}$ *and* $\tilde{\tilde{T}}$ *are the unique self-adjoint extensions of* T *with domain contained in* $D(r^{-1/2})$ *or* $D(|T_0|^{1/2})$, *resp.*

First, we prove the following

Proposition. *Under the assumptions and with the notations of the theorem*

$$D(\tilde{T}) \subset D(|T_0|^{1/2}) \tag{2}$$

holds.

Proof. Let $\{q_t\}_{t>0}$ be a family of bounded cut-off potentials as in [10, Theorem 4] and

$$T_t := \overline{T_{00} + q_t} \qquad (t>0)\,.$$

Then $\{T_t\}_{t>0}$ has a strong resolvent limit which is self-adjoint and equal to $\tilde{T}$ [8, 10].

The multiplication operators rq_t $(t>0)$ are uniformly bounded and

$$s - \lim_{t\to\infty} \overline{rq_t} = \overline{rq}\,. \tag{3}$$

$r^{-1/2}$ and r^{-1} are relatively bounded with respect to T_0 and so with respect to $|T_0|$ (cf. [9, Lemma 3] with $q=0$, or [10, Lemma 3] and [2, § V.5.4]).

$r^{-1/2}$ is also relatively bounded with respect to $\tilde{T}$. For, if $u \in D(\tilde{T})$ then there exists a family $\{u_t\}_{t>0}$ with $\lim_{t\to\infty} u_t = u$ and $\lim_{t\to\infty} T_t u_t = \tilde{T}u$. But then $\{r^{-1/2}u_t\}_{t>0}$ is weakly convergent to $r^{-1/2}u$ (see the 2nd step in the proof of the theorem in [9]), thus $\|r^{-1/2}u\| \leqq \underline{\lim}\, \|r^{-1/2}u_t\|$. By Lemma 3 in [9] we have

$$(1-\mu)\|r^{-1/2}u_t\| \leqq \|T_t u_t\| + 2\|u_t\| \qquad (t>0)$$

and so

$$(1-\mu)\|r^{-1/2}u\| \leqq \|\tilde{T}u\| + 2\|u\| \qquad (u \in D(\tilde{T}))\,. \tag{4}$$

Moreover, 0 is in the resolvent set of T_0, T_t $(t>0)$ and $\tilde{T}$ (cf. [10]). Therefore the following operators are everywhere defined and bounded:

$$\begin{aligned}
&r^{-1}T_0^{-1}, \quad (r^{-1}T_0^{-1})^* = \overline{T_0^{-1}r^{-1}}\,,\\
&r^{-1}|T_0|^{-1}\,,\\
&r^{-1/2}|T_0|^{-1/2}, \quad (r^{-1/2}|T_0|^{-1/2})^* = \overline{|T_0|^{-1/2}r^{-1/2}}\,,\\
&\text{(cf. [5, Theorem X.18])}\\
&r^{-1/2}T_0^{-1}, \quad (r^{-1/2}T_0^{-1})^* = \overline{T_0^{-1}r^{-1/2}}\,,\\
&r^{-1/2}\tilde{T}^{-1}\,.
\end{aligned} \tag{5}$$

Using (3), (5), the strong resolvent convergence of $\{T_t\}_{t>0}$ and the second resolvent equation we have for $u \in H$

$$\begin{aligned}
&\|(\tilde{T}^{-1} - T_0^{-1} + \overline{T_0^{-1}r^{-1}}\,\overline{rq}\tilde{T}^{-1})u\|\\
&\quad = \lim_{t\to\infty} \|(-T_0^{-1}q_tT_t^{-1} + \overline{T_0^{-1}r^{-1}}\,\overline{rq}\tilde{T}^{-1})u\| =\\
&\quad \leqq \overline{\lim_{t\to\infty}}\, \|\overline{T_0^{-1}r^{-1}}\|\,(\|\overline{rq_t}\|\,\|T_t^{-1}u - \tilde{T}^{-1}u\|\\
&\qquad + \|(\overline{rq_t} - \overline{rq})\tilde{T}^{-1}u\|)\\
&\quad = 0\,,
\end{aligned}$$

which gives the representation

$$\tilde{T}^{-1} = T_0^{-1} - \overline{T_0^{-1}r^{-1}}\,\overline{rq}\,\tilde{T}^{-1}\,. \tag{6}$$

For $u\in H$ and $v\in D_0$

$$\begin{aligned}&(\overline{T_0^{-1}r^{-1}}\,\overline{rq}\,\tilde{T}^{-1}u,\, T_{00}v)-(\overline{T_0^{-1}r^{-1/2}}\,\overline{rq}r^{-1/2}\tilde{T}^{-1}u,\, T_{00}v)\\ &\quad=(\overline{rq}\,\tilde{T}^{-1}u, r^{-1}v)-(r^{-1/2}\overline{rq}\,\tilde{T}^{-1}u, r^{-1/2}v)\\ &\quad=0\,.\end{aligned}$$

Since $T_{00}D_0$ is dense in H,

$$\overline{T_0^{-1}r^{-1}}\,\overline{rq}\,\tilde{T}^{-1}=\overline{T_0^{-1}r^{-1/2}}\,\overline{rq}r^{-1/2}\tilde{T}^{-1}\,. \tag{7}$$

The Eqs. (6) and (7) together with the inclusion

$$R(T_0^{-1})=D(T_0)=D(|T_0|)\subset D(|T_0|^{1/2})$$

show that it is sufficient for (2) to prove

$$R(\overline{T_0^{-1}r^{-1/2}})\subset D(|T_0|^{1/2})\,. \tag{8}$$

But by the functional calculus for self-adjoint operators (cf. [5, VII])

$$|T_0|^{1/2}T_0^{-1}|T_0|^{1/2}$$

is a densely defined and bounded operator, therefore

$$\begin{aligned}T_0^{-1}&=|T_0|^{-1/2}|T_0|^{1/2}T_0^{-1}|T_0|^{1/2}|T_0|^{-1/2}\\ &=|T_0|^{-1/2}\overline{|T_0|^{1/2}T_0^{-1}|T_0|^{1/2}}|T_0|^{-1/2}\end{aligned}$$

and with (5)

$$\overline{T_0^{-1}r^{-1/2}}=|T_0|^{-1/2}\overline{|T_0|^{1/2}T_0^{-1}|T_0|^{1/2}}\,\overline{|T_0|^{-1/2}r^{-1/2}}\,,$$

which proves (8).

Proof of the Theorem. $r^{-1/2}$ is relatively bounded with respect to $|T_0|^{1/2}$, therefore

$$D(|T_0|^{1/2})\subset D(r^{-1/2})\,.$$

Together with the proposition this inclusion gives

$$\begin{aligned}D(\tilde{\tilde{T}})&=(D(T^*)\cap D(|T_0|^{1/2}))\subset(D(T^*)\cap D(r^{-1/2}))\\ &=D(\tilde{T})=(D(T^*)\cap D(\tilde{T}))\subset(D(T^*)\cap D(|T_0|^{1/2}))=D(\tilde{\tilde{T}})\,,\end{aligned}$$

which proves (1).

The self-adjointness of $\tilde{T}$ is proved in [10]. Finally, since every symmetric extension of T is a restriction of T^* and therefore every symmetric extension S of T with $D(S)\subset D(r^{-1/2})$ or $D(S)\subset D(|T_0|^{1/2})$ has to be a restriction of $\tilde{T}$ or $\tilde{\tilde{T}}$ resp., $\tilde{T}$ and $\tilde{\tilde{T}}$ are the unique self-adjoint extensions with these properties.

Appendix

We sketch the way how the results can be proved in the case of a non-semibounded potential q by means of a double cut-off procedure.

Let $q:\mathbb{R}^3_+ \to \mathbb{R}$ be a measurable function and

$$\mu := \sup_{\mathbb{R}^3_+} |xq(x)| < 1 \ .$$

Denote

$$\begin{array}{lll} q^{(\tau)}(x) := \min\{\tau, q(x)\} & & (x \in \mathbb{R}^3_+, \tau > 0) \\ q_t^{(\tau)}(x) := \max\{-t, q^{(\tau)}(x)\} & & (x \in \mathbb{R}^3_+, \ \tau > 0, \ t > 0) \\ T_t^{(\tau)} \ := T_0 + q_t^{(\tau)} & & (t, \tau > 0) \\ T^{(\tau)} \ := T_{00} + q^{(\tau)} & & (\tau > 0) \\ T \ := T_{00} + q \ . & & \end{array}$$

For each $\tau > 0$ we can apply Theorem 4 in [10], which shows that the strong graph limit $\tilde{T}^{(\tau)}$ of the family $\{T_t^{(\tau)}\}_{t>0}$ exists, is equal to the strong resolvent limit of $\{T_t^{(\tau)}\}_{t>0}$ [8], that

$$\tilde{T}^{(\tau)} = T^{(\tau)*} \restriction (D(T^{(\tau)*}) \cap D(r^{-1/2}))$$

and $\tilde{T}^{(\tau)}$ is a self-adjoint extension of $T^{(\tau)}$.

Moreover, we have

$$D(\tilde{T}^{(\tau)}) = \text{constant} \quad (\tau > 0) \ ,$$

$$\overline{\tilde{T}^{(\tau)} - \tilde{T}^{(\tau')}} = \overline{q^{(\tau)} - q^{(\tau')}} \qquad (\tau, \tau' > 0)$$

and

$$\|\tilde{T}^{(\tau)} u\| \geqq \frac{1}{1+\mu} \sqrt{1-\mu^2}\, \|u\| \qquad (u \in D(\tilde{T}^{(\tau)}), \tau > 0)$$

(cf. [10, Theorem 5]).

This allows us to apply the convergence theorem [8], and [10, Theorem 1] a second time, now with respect to the family $\{\tilde{T}^{(\tau)}\}_{\tau>0}$. Thus, the strong graph limit

$$\hat{T} := g - \lim T^{(\tau)}$$

exists, is a self-adjoint extension of T and has at least the spectral gap $\left(-\frac{1}{1+\mu}\sqrt{1-\mu^2}, \frac{1}{1+\mu}\sqrt{1-\mu^2}\right)$ (see [3] for a best possible result). Since (4) holds for $\tilde{T}^{(\tau)}$ uniformly in $\tau > 0$, the argument in the second step of the proof of the theorem in [9] shows

$$D(\hat{T}) \subset D(r^{-1/2}) \ . \qquad (9)$$

By definition of $\hat{T}$, for every $u \in D(\hat{T})$

$$u = \lim_{\tau\to\infty} \lim_{t\to\infty} \lim_{n\to\infty} u_{t,n}^{(\tau)} ,$$

$$\hat{T}u = \lim_{\tau\to\infty} \lim_{t\to\infty} \lim_{n\to\infty} T_t^{(\tau)} u_{t,n}^{(\tau)}$$

hold with a suitable family $\{u_{t,n}^{(\tau)}\}_{\substack{\tau>0 \\ t>0 \\ n\in\mathbb{N}}}$ in D_0.

Then the method in the first step of the proof of the theorem in [9] allows us to show that

$$D(T^*)\cap D(r^{-1/2})\subset D(\hat{T}) .$$

From this and (9)

$$\hat{T}=T^*\upharpoonright(D(T^*)\cap D(r^{-1/2}))=:\tilde{T}$$

follows.

To get the results of the theorem in this paper for a nonsemibounded potential q, we only need to show that (4) and (6) are still true for the above operator $\tilde{T}$. (4) can be proved in the same way, because it is true for $\tilde{T}^{(\tau)}$, uniformly in $\tau>0$. (6) follows immediately from the fact that the multiplication operators $rq_t^{(\tau)}$ $(t>0,\ \tau>0)$ are uniformly bounded in t, $\tau>0$ that

$$s\text{-}\lim_{\tau\to\infty} s\text{-}\lim_{t\to\infty} \overline{rq_t^{(\tau)}}=\overline{rq}$$

and

$$s\text{-}\lim_{\tau\to\infty} s\text{-}\lim_{t\to\infty} T_t^{(\tau)^{-1}}=s\text{-}\lim_{\tau\to\infty} \tilde{T}^{(\tau)^{-1}}=\tilde{T}^{-1} .$$

Acknowledgements. The authors would like to thank B. Simon for stimulating discussions and Princeton University for her kind hospitality.

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Communicated by J. Ginibre

Received August 21, 1978