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Absence of Symmetry Breakdown and Uniqueness of the Vacuum for Multicomponent Field Theories

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Abstract. Correlation inequalities are used to show that the two component $\lambda(\phi^2)^2$ model (with HD, D, HP, P boundary conditions) has a unique vacuum if the field does not develop a non-zero expectation value. It follows by a generalized Coleman theorem that in two space-time dimensions the vacuum is unique for all values of the coupling constant. In three space-time dimensions the vacuum is unique below the critical coupling constant.

For the n-component $P(|\phi|^2)_2+\mu\phi_1$ model, absence of continuous symmetry breaking, as μ goes to zero, is proven for all states which are translation invariant, satisfy the spectral condition, and are weak* limit points of finite volume states satisfying N^τ_{loc} and higher order estimates.

I. Introduction

It is a general fact in statistical mechanics and quantum field theory that the appearance of multiple phases and spontaneous symmetry breaking occurs more readily as the number of space dimensions increases. In the case of the statistical mechanics of lattice systems with a continuous internal symmetry group, spontaneous symmetry breakdown can occur for three dimensional lattices [10] while for two dimensional lattices every equilibrium state is invariant under the internal symmetry [4]. For the two component rotator, absence of symmetry breakdown implies uniqueness of the phase [2].

Multicomponent quantum field theories can exhibit spontaneous symmetry breakdown in three space-time dimensions [10], while in two space-time dimensions, a general result due to Coleman [3, 11] "There are no Goldstone bosons in two space-time dimensions" shows that spontaneous breakdown of a continuous internal symmetry cannot occur, provided the symmetry is generated by a local conserved current. Given these results, we have considered two questions. The first is whether absence of symmetry breakdown implies uniqueness of the vacuum

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(clustering of the Wightman functions). The second question is whether the Coleman theorem is applicable to the models considered so far in constructive quantum field theory.

Using correlation inequalities for two-component Euclidean (quantum) fields (with HD D, HPP boundary conditions) with interaction $\lambda(\phi^2)^2$, we show that absence of a spontaneous, non-zero, field expectation implies uniqueness of the vacuum. We also prove that independence of the (standard) boundary condition in presence of a non zero external field implies independence of the standard b.c. at zero external field. Thus in three space-time dimensions the vacuum is unique for (HD HP D P) boundary conditions below the critical coupling constant.

In two space-time dimensions, we conclude that the vacuum is unique for all values of the coupling constant by answering our second question. We do not show that Coleman's theorem is applicable in its original form. We have generalized it so that we do not need to prove the existence of a conserved current but we only have to derive estimates on the time component of the current (Sects. IV and V). For a general n-component quantum field theory in two space-time dimensions, absence of spontaneous symmetry breakdown (but not uniqueness of the vacuum) follows from N_{loc}^{τ} and higher order estimates.

In the case of the plane rotator on the two dimensional lattice, perturbing the interaction enabled us, using correlation inequalities, to prove that there was only one translation invariant equilibrium state [2]. The two-component quantum field model is more singular and such perturbations cannot be applied directly. We thus have demonstrated the uniqueness of the vacuum in the sense that a certain state is *clustering* but we have not shown that there is only one Wightman state for this model.

For clarity of presentation, we elaborate the preceding outline and give here a sketch of the argument, referring to the section where a particular result can be found.

We Discuss First the Two Component Model

1. It is known that the periodic state $\langle\,\rangle_\mu$ with external field $\mu\int\phi_1 dx, (\mu>0)$ exists and is exponentially clustering [7]. Correlation inequalities show that the state $\langle\,\rangle_\mu$ converges as $\mu\downarrow 0$ to a state noted $\langle\,\rangle_+$ (Lemma III.1 and discussion preceding it). Again by correlation inequalities it follows the $\langle\,\rangle_+$ is clustering (Lemma III.3) and thus has a unique vacuum.

2. If $\langle\phi_1(x)\rangle_+=0$ then by correlation inequalities (Theorem III.5) it follows thet $\langle\,\rangle_+=\langle\,\rangle_0$, where the state $\langle\,\rangle_0$ is obtained from finite volume states with $\mu=0$ and thus by construction is invariant under the internal symmetry. Thus $\langle\,\rangle_0$ is both clustering and invariant under the symmetry.

3. In two space-time dimensions we may prove that $\langle\phi_1(x)\rangle_+=0$ and thus this theory $\langle\,\rangle_0$ exists and is clustering for all values of the coupling constants. In the analogous situation of the plane rotator on a two dimensional lattice, the absence of spontaneous magnetization at all temperatures follows from Mermin's theorem or the more general result of Dobrushin-Shlosman [4]. However in the quantum field theory case, we are dealing with a continuum quantum theory and the proof

that $\langle\phi_1(x)\rangle_+ = 0$ is based on the Goldstone [25] and Coleman [3, 11] theorems. In order to apply these methods we must establish the existence of a "current" which generates the internal symmetry and such that the vacuum Ω_+ (corresponding to the state $\langle\,\rangle_+$) is in the domain of the current. It is thus necessary to make a detailed study of the current.

4. It is straightforward to show that, for suitable ϑ,

$$j(\vartheta) = \int (\phi_2(x)\pi_1(x) - \phi_1(x)\pi_2(x))\vartheta(x)dx$$

generates the internal symmetry (Proposition IV.2). However to obtain the necessary estimates (in order to prove that Ω_+ is in the domain of the current) the local number operator estimates of Sect. IV ($j(\vartheta) \leqq \text{const}\,(H+1)$) are not sufficient: we must consider the commutator $[H_\mu, j(\vartheta)]$ where H_μ is the Hamiltonian with external field μ. Thus we define a time smoothed current $j^\mu(\eta) = \int dt e^{itH_\mu} j(\vartheta) e^{-itH_\mu}\eta(t)$. Then the estimates of Sect. IV apply to $j^\mu(\eta)$ and to $[H_\mu, j^\mu(\eta)]$. This allows us to obtain $\|j^\mu(\eta)\Omega_\mu\| \leqq C$ uniformly in $\mu(\Omega_\mu$ is the vacuum corresponding to $H_\mu)$.

5. Now $j^\mu(\eta)$ does *not* generate the internal symmetry if $\mu \neq 0$ (since H_μ does not commute with the symmetry). However we can show that for $\mu = 0$, $j(\eta)$ *does* generate the symmetry (Theorem IV.6). To extend the above estimates to $j(\eta)$ ($\|j(\eta)\Omega_+\| \leqq C$) we need a convergence argument. Correlations inequalities do *not* apply to a variable as $j(\eta)$. However using the characterisation of field theory as a state on the algebra of bounded local observables, $\mathscr{A}$, correlation inequalities and analytic continuation imply convergence of the states ω_μ (corresponding to the theory with an external field $\mu\phi_1$) on $\mathscr{A}$ as $\mu \to 0$.

This fact, the bound on $j^\mu(\eta)$ uniform in μ, and a suitable convergence of $j^\mu(\eta)$ to $j(\eta)$ as $\mu \to 0$ (Theorem IV.8) allow us to extend the above uniform bound to $j(\eta)$ (Proportion V.5). It follows that Ω_+ is in the domain of $j(\eta)$, and thus we can use the Goldstone theorem (Theorem VII.2) to show that $\langle\phi_1(x)\rangle_+ = 0$.

In the case of N-component models ($N \geqq 3$) we cannot appeal to correlation inequalities and cannot prove uniqueness of the vacuum, but we may show absence of symmetry breakdown via the Coleman theorem (Sect. 6) [3, 11]. It is however a fact that the space component of the current is more singular than the time component and thus within the constructive framework it is preferable to estimate *only* the time component. Therefore we generalize the Coleman theorem so that a minimal number of hypotheses must be verified (Sect. VII)[1].

II. Local Operator Algebras and Euclidean Fields for the N-Component Field $\phi(x)$

A) Free Fields

The Hilbert space of states for the n-component free scalar field ϕ is represented by the Fock space $\mathscr{F} = \bigotimes_{j=1}^{n} \mathscr{F}_j$ [8] where $\mathscr{F}_j$ is the usual Fock space for the time zero

1 Of course we can apply this generalized theorem to the preceeding situation $N = 2$. It is what has been done explicitly in the paper (see proof of Theorem III.7)

fields

$$\phi_j(f)=\frac{1}{\sqrt{2}}[a_j^+(\mu^{-1/2}f)+a_j(\mu^{-1/2}f)]$$

$$\pi_j(f)=\frac{i}{\sqrt{2}}[a_j^+(\mu^{1/2}f)-a_j(\mu^{1/2}f)] \qquad f\in\mathscr{S}(\mathbb{R}^s),$$

where $\mu=[-\nabla^2+m^2]^{1/2}$, ∇^2 is the Laplacian in s dimensions. Given $f=(f_1,\dots,f_n)$ we define

$$\boldsymbol{\phi}(\mathbf{f})=\sum_{j=1}^{n}\phi_j(f_j)$$

$$\boldsymbol{\pi}(\mathbf{f})=\sum_{j=1}^{n}\pi_j(f_j).$$

We denote by $\mathscr{F}_{(V)}=\bigotimes_{j=1}^{n}\mathscr{F}_{j(V)}$ the Fock space associated with the field $\boldsymbol{\phi}_{(V)}$ with periodic boundary conditions on the boundary of V. The corresponding fields $\boldsymbol{\phi}_{(V)}$ are defined by replacing μ by $\mu_V=[-\nabla_{(V)}^2+m^2]^{1/2}$ where $\nabla_{(V)}^2$ is the Laplacian for the region V with periodic boundary conditions. There is a natural embedding of $\mathscr{F}_{(V)}$ into $\mathscr{F}$, and the fields $\boldsymbol{\phi}_{(V)}$ may be considered to act on $\mathscr{F}$ [13, 17].

The free Hamiltonian is

$$H_0=d\Gamma\left(\bigoplus_{j=1}^{n}\mu\right)$$

$$H_{0(V)}=d\Gamma\left(\bigoplus_{j=1}^{n}\mu_{(V)}\right),$$

where $d\Gamma(\mu)$ denotes the second quantization of the one-particle operator μ [28].

The number operator is defined by

$$N=d\Gamma\left(\bigoplus_{j=1}^{n}1\right).$$

More generally, local number operators are defined by

$$N_{\mathrm{loc}}^{\tau}=d\Gamma\left(\bigoplus_{j=1}^{n}\zeta\mu^{\tau}\zeta\right),$$

where $0\leqq\tau\leqq1$ and $0\leqq\zeta(x)\in C_0^\infty$. A similar definition holds for periodic boundary conditions with the restriction $\operatorname{supp}\zeta\subset V$.

The local operator algebras are constructed in Fock space as follows. Let Λ denote an open bounded region of space.

$\mathscr{A}(\Lambda)=$ von Neumann algebra generated by $\{e^{i\boldsymbol{\phi}(\mathbf{f})},e^{i\boldsymbol{\pi}(\mathbf{g})}\}$, $\operatorname{supp}\mathbf{f},\mathbf{g}\subset\Lambda$. The C^*-algebra of (quasi) local observables is $\mathscr{A}=$ norm closure of $\bigcup_{\Lambda}\mathscr{A}(\Lambda)$.

Let $\Gamma_{ij}(g)$ be an $n\times n$ unitary representation of the internal symmetry group G. Then $\phi_i(x)\to\sum_{j=1}^{n}\Gamma_{ij}(g)\phi_j(x)$, $\pi_i(x)\to\sum_{j=1}^{n}\Gamma_{ij}(g)\pi_j(x)$ defines an automorphism of the C^*-algebra $\mathscr{A}$, denoted $\alpha(g)$, which is implemented in Fock space by $U(g)$.

A similar construction holds for $(\mathscr{A}(\Lambda)_{(V)}$ with periodic boundary conditions, with the restriction $\Lambda \subset V$.

If the Wightman functions (vacuum expectation values) of $\boldsymbol{\phi}(\underline{x}, t)$ are analytically continued to purely imaginary times t, one obtains the correlation functions (Schwinger functions) of the Gaussian stochastic process $\boldsymbol{\phi}^E(x)$ (the Euclidean free field) with mean zero

$$\langle \phi_j^E(f) \rangle = 0 \qquad j = 1, \ldots, n$$

and covariance

$$\langle \phi_j^E(f) \phi_k^E(g) \rangle = \delta_{jk}(f, (-\nabla^2 + m^2)^{-1} g).$$

$f, g \in \mathscr{S}(\mathbb{R}^d)$, $+\nabla^2$ is the Laplacian in $d = s+1$ dimensions.

More generally, if ϑ denotes an open bounded region in Euclidean spacetime we may define the field $\boldsymbol{\phi}_\vartheta^E$ with covariance determined by $(f, (-\nabla_\vartheta^2 + m^2)^{-1} g)$ where ∇_ϑ^2 is the Laplacian with any of the *standard* boundary conditions on $\partial\Lambda$: Dirichlet (D), Periodic (P), Neumann (N), or Free (F)

$$(\nabla_{\vartheta(\mathrm{F})}^2 = \nabla^2) \qquad [20, 21].$$

The Schwinger functions are invariant under the transformations

$$\phi_i^E(x) \to \sum_{j=1}^n \Gamma_{ij}(g) \phi_j^E(x).$$

B) Interacting Fields

We consider free or periodic boundary conditions in two space-time dimensions.

Let $0 \leqq g(x) \leqq 1, g \in C_0^\infty, \operatorname{supp} g \subset V$. Λ_g and $\Lambda_{\operatorname{supp} g}$ are respectively defined by $g(\underline{x}) = 1$ for $\underline{x} \in \Lambda_g$ and $\operatorname{supp} g \subset \Lambda_{\operatorname{supp} g}$. Λ_t is defined as the set of points within a distance t of Λ.

The Hamiltonian $H(g)$ is defined by

$$\begin{aligned} H(g) &= H_0 + \int d\underline{x}\, g(\underline{x}) [\lambda :(\boldsymbol{\phi}(\underline{x})^2)^2 : - \sigma : \boldsymbol{\phi}(\underline{x})^2 : - \mu \phi_1(\underline{x})] \\ &= H_0 + H_I(g) \quad \text{with} \quad \lambda \geqq 0. \end{aligned}$$

$H(g)_{(V)}$ is defined analogously, with ϕ replaced by $\phi_{(V)}$ and H_0 by $H_{0(V)}$.

Higher Order Estimates [26]

For any $\kappa = 1, 2, \ldots$ there are constants a, b (depending on g but independent of V) such that

$$N^\kappa \leqq a(H(g) + b)^\kappa$$

$$N_{(V)}^\kappa \leqq a(H(g)_{(V)} + b)^\kappa.$$

N_{loc}^τ Estimates [29]

For any $\tau \leqq 1$, there are constants a, b (independent of g and V) such that

$$N_{\mathrm{loc}}^\tau \leqq a(\hat{H}(g) + b)$$

$$N_{\mathrm{loc}(V)}^\tau \leqq a(\hat{H}(g)_{(V)} + b),$$

where $\hat{H}(g) = H(g) - \inf$ spectrum $H(g)$.

Note that the proof of Spencer carries over unchanged to multicomponent and periodic fields. We sketch it in an Appendix.

Furthermore, $H(g)_{(V)} \xrightarrow[V \nearrow \mathbb{R}]{} H(g)$ strongly on $C^\infty(H_0)$ [18].

Finite Propagation Speed [27, 13]

$e^{itH(g)}\mathscr{A}(\Lambda)e^{-itH(g)} \subset \mathscr{A}(\Lambda_t)$ and the automorphism $e^{itH(g)}\mathscr{A}(\Lambda)e^{-itH(g)}$ is independent of g provided $\Lambda_t \subset \Lambda_g$ and similarly for $H(g)_{(V)}$. We may thus define the time evolution automorphism $\tau(t)$ by

$$\tau(t)\mathscr{A}(\Lambda) = e^{itH(g)}\mathscr{A}(\Lambda)e^{-itH(g)} \qquad \Lambda_t \subset \Lambda_g .$$

The Euclidean theory is constructed by replacing the free measure $d\mu_0$ by

$$d\mu_{\vartheta,b} = \frac{e^{-U_{\vartheta,b}}d\mu^0_{\vartheta,b}}{\int e^{-U_{\vartheta,b}}d\mu^0_{\vartheta,b}} \qquad U_{\vartheta,b} = \int_\vartheta [\lambda :(\boldsymbol{\phi}^{E2})^2:_{\vartheta,b} - \sigma :\boldsymbol{\phi}^{E2}:_{\vartheta,b} - \mu\phi^E_1](x)d^d(x)$$

the subcripts ϑ, b in the Wick ordering indicate that the Wick subtractions are made with respect to the measure $d\mu^0_{\vartheta,b}$ ($d\mu^0_{\vartheta,b}$ is the measure corresponding to the free theory with standard boundary conditions b on $\partial\Lambda$). Half-b boundary conditions are defined by:

$$d\mu_{\vartheta,Hb} = \frac{e^{-U_{\vartheta,F}}d\mu^0_{\vartheta,b}}{\int e^{-U_{\vartheta,F}}d\mu^0_{\vartheta,b}} .$$

One obtains the Schwinger functions

$$\langle \phi^E_{\alpha_1}(x_1)\ldots\phi^E_{\alpha_m}(x_m)\rangle_{\vartheta(H)b} = \int \phi^E_{\alpha_1}(x_1)\ldots\phi^E_{\alpha_m}(x_m)d\mu_{\vartheta(H)b}$$

$\phi^E_{\alpha_i}(x_i)$ is one component of $\boldsymbol{\phi}^E(x_i)$.

One is thus generally interested in the infinite volume limit

$$\lim_{\vartheta \nearrow \mathbb{R}^d} \langle \phi^E_{\alpha_1}(x_1)\ldots\phi^E_{\alpha_m}(x_m)\rangle_{\vartheta(H)b} .$$

III. Correlation Inequalities for Schwinger Functions and Uniqueness of the Vacuum

We consider now the *two component* $\lambda(\boldsymbol{\phi}^2)^2 - \sigma\boldsymbol{\phi}^2 - \mu\phi_1$ model, where $\lambda > 0$, $\mu \geqq 0$, σ real, in d space-time dimensions. The lattice approximation of the finite volume Euclidean Schwinger functions satisfy the following inequalities, which then carry over to the continuum finite volume and infinite volume theories. (These limits exist for $d \leqq 3$.)

Let $\mathscr{M}$ be the set of finite families of test functions [elements of $\mathscr{S}(\mathbb{R}^d)$] and $\mathscr{M}_+$ the set of finite families of non-negative test functions. For $A \in \mathscr{M}$ we define

$$\phi^E_{jA} = \prod_{f\in A} \phi^E_j(f) .$$

Let $|A|$ denote the cardinality of A. Let ϑ be an open bounded regular set in $\mathbb{R}^d$ with standard boundary conditions [28]. *The following inequalities are valid:*

a) Ginibre's Inequalities [12, 5] *if* $A, B \in \mathcal{M}_+$, $|B|$ *even, and* $\varepsilon = \pm 1$

$$\langle \phi_{1A}^E(\phi_{1B}^E + \varepsilon\phi_{2B}^E)\rangle_\vartheta \geqq \langle \phi_{1A}^E\rangle_\vartheta \langle \phi_{1B}^E + \varepsilon\phi_{2B}^E\rangle_\vartheta \geqq 0 \tag{1}$$

if $A, B \in \mathcal{M}_+$, $|A|$ *and* $|B|$ *even*, $\varepsilon_{1,2} = \pm 1$

$$\langle (\phi_{1A}^E + \varepsilon_1\phi_{2A}^E)(\phi_{1B}^E + \varepsilon_2\phi_{2B}^E)\rangle_\vartheta \geqq \langle \phi_{1A}^E + \varepsilon_1\phi_{2A}^E\rangle_\vartheta \langle \phi_{1B}^E + \varepsilon_2\phi_{2B}^E\rangle_\vartheta \geqq 0 . \tag{2}$$

b) Generalized Griffiths' Inequalities [1, 22a] *if* $A, B \in \mathcal{M}_+$

$$\langle \phi_{jA}^E \phi_{jB}^E\rangle_\vartheta \geqq \langle \phi_{jA}^E\rangle_\vartheta \langle \phi_{jB}^E\rangle_\vartheta \geqq 0 \qquad j = 1, 2, \tag{3}$$

$$\langle \phi_{1A}^E\rangle_\vartheta \langle \phi_{2B}^E\rangle_\vartheta \geqq \langle \phi_{1A}^E \phi_{2B}^E\rangle_\vartheta \geqq 0 . \tag{4}$$

Remark 1. The above inequalities together with the ϕ-bound [7] imply the existence of the theory for $d = 2$ or 3 with Dirichlet (D) or half-Dirichlet (HD) boundary conditions.

Remark 2. In the following we drop the E in ϕ^E, and we make the μ dependence in the above states explicit: $\langle\,\rangle_{\vartheta\mu b}$ denotes the Euclidean state in region ϑ with a fixed standard boundary condition b and external field $\mu \geqq 0$.

We suppose the infinite volume limit exists for some standard boundary condition b: $\langle\,\rangle_{\mu,b} = \lim_{\vartheta \nearrow \mathbb{R}^d} \langle\,\rangle_{\vartheta,\mu,b}$, for all $\mu > 0$, and we define $\langle\,\rangle_{+,b} = \lim_{\mu \searrow 0} \langle\,\rangle_{\mu,b}$. This limit exists since $\langle \phi_{1A}\rangle_{\mu,b}$ monotone decreases and $\langle \phi_{2A}\rangle_{\mu,b}$ monotone increases as $\mu \downarrow 0$ by (3) and (4). For $|B|$ *odd* $\langle \phi_{1A}\phi_{2B}\rangle_\mu = 0$ and for $|B|$ even $\langle \phi_{1A}\phi_{2B}\rangle_{\mu,b}$ converges as $\mu \downarrow 0$ by (2). In fact, we have the estimate:

III.1. Lemma. *Let* $\mu > 0$ *then:*

$$|\langle \phi_{1A}\phi_{2B}\rangle_{\mu,b} - \langle \phi_{1A}\phi_{2B}\rangle_{+,b}| \leqq \langle \phi_{1A}\phi_{1B}\rangle_{\mu,b} - \langle \phi_{1A}\phi_{1B}\rangle_{+,b} .$$

Proof. For $|B|$ odd the left hand side is zero. For $|B|$ even, by (2)

$$\langle \phi_{1A}(\phi_{1B} \pm \phi_{2B})\rangle_{\mu,b} \geqq \langle \phi_{1A}(\phi_{1B} \pm \phi_{2B})\rangle_{+,b}$$

$$\langle \phi_{1A}\phi_{1B}\rangle_{\mu,b} - \langle \phi_{1A}\phi_{1B}\rangle_{+,b} \geqq \pm[\langle \phi_{1A}\phi_{2B}\rangle_{+,b} - \langle \phi_{1A}\phi_{2B}\rangle_{\mu,b}]$$

$$|\langle \phi_{1A}\phi_{2B}\rangle_{\mu,b} - \langle \phi_{1A}\phi_{2B}\rangle_{+,b}| \leqq \langle \phi_{1A}\phi_{1B}\rangle_{\mu,b} - \langle \phi_{1A}\phi_{1B}\rangle_{+,b} . \qquad \square$$

By the same proof we have

III.2. Lemma.

$$|\langle \phi_{1A}\phi_{2B}\rangle_{\vartheta,\mu,b} - \langle \phi_{1A}\phi_{2B}\rangle_{\vartheta,0,b}| \leqq \langle \phi_{1A}\phi_{1B}\rangle_{\vartheta,\mu,b} - \langle \phi_{1A}\phi_{1B}\rangle_{\vartheta,0,b} .$$

III.3. Lemma. *If there exists a sequence* $\mu_n \to 0$ *such that* $\langle\,\rangle_{\mu_n,b}$ *has the cluster property for all n then* $\langle\,\rangle_{+,b}$ *has the cluster property.*

Proof. By a density argument and multinearity we have only to show clustering for the set $\{\phi_{1A}\phi_{2B}\}$ with $A, B \in \mathcal{M}_+$.

Using Ginibre's inequality one can show the generalized Dunlop-Newman inequalities [6] $\forall A, B, C, D \in \mathcal{M}_+$

$$|\langle \phi_{1A}\phi_{2B}\phi_{1C}\phi_{2D}\rangle_{\mu,b} - \langle \phi_{1A}\phi_{2B}\rangle_{\mu,b}\langle \phi_{1C}\phi_{2D}\rangle_{\mu,b}|$$
$$\leqq \langle \phi_{1A}\phi_{1B}\phi_{1C}\phi_{1D}\rangle_{\mu,b} - \langle \phi_{1A}\phi_{1B}\rangle_{\mu,b}\langle \phi_{1C}\phi_{1D}\rangle_{\mu,b}$$

if $|B|$ even and

$$\langle \phi_{1A}\phi_{2B}\phi_{1C}\phi_{2D}\rangle^2_{\mu,b} \leqq \langle \phi_{1A}\phi_{1B}\phi_{1C}\phi_{1D}\rangle^2_{\mu,b} - \langle \phi_{1A}\phi_{1B}\rangle^2_{\mu,b}\langle \phi_{1C}\phi_{1D}\rangle^2_{\mu,b}$$

if $|B|$ odd.

These inequalities extend to the limit state $\langle\,\rangle_+$. Thus we have only to show clustering for the field ϕ_1. That is we must show

$$\lim_{|x|\to\infty} \langle \phi_{1A}\tau_x\phi_{1B}\rangle_{+,b} = \langle \phi_{1A}\rangle_{+,b}\langle \phi_{1B}\rangle_{+,b},$$

where $\tau_x\phi_{1B} = \prod_{f\in B} \phi_1(\tau_x f)$.

But

$$\langle \phi_{1A}\rangle_{+,b}\langle \phi_{1B}\rangle_{+,b} \leqq \langle \phi_{1A}\tau_x\phi_{1B}\rangle_{+,b}$$
$$\leqq \langle \phi_{1A}\tau_x\phi_{1B}\rangle_{\mu_n,b} \xrightarrow[|x|\to\infty]{} \langle \phi_{1A}\rangle_{\mu_n,b}\langle \phi_{1B}\rangle_{\mu_n,b}.$$

Since $\langle \phi_{1A}\rangle_{\mu_n,b} \searrow \langle \phi_{1A}\rangle_{+,b}$ it follows that

$$\lim_{|x|\to\infty} \langle \phi_{1A}\tau_x\phi_{1B}\rangle_{+,b} = \langle \phi_{1A}\rangle_{+,b}\langle \phi_{1B}\rangle_{+,b}.$$

This concludes the proof. [See also [28] p. 357.] □

III.4. Lemma. *If* $\langle \phi_1(f)\rangle_{+,b} = 0\ \forall f \in \mathcal{S}(\mathbb{R}^d)$ *then* $\langle \phi_{1A}\rangle_{+,b} = \langle \phi_{2A}\rangle_{+,b}\ \forall A \in \mathcal{M}_+$.

Proof. Since $\langle \phi_2(f)\rangle_{+,b} = 0$ by construction, the proof of this lemma is contained in the proof of Theorem III.4 of [2]. □

III.5. Theorem. *If* $\lim_{\mu\downarrow 0} \langle \phi_1(f)\rangle_{\mu,b} = 0$ *then* $\langle\,\rangle_{+,b} = \langle\,\rangle_{0,b}$.

Proof. By (4) and (3) and the $\phi_1 \leftrightarrow \phi_2$ symmetry of the $\mu = 0$ state we have

$$\langle \phi_{2A}\rangle_{\vartheta,\mu,b} \leqq \langle \phi_{2A}\rangle_{\vartheta,0,b} = \langle \phi_{1A}\rangle_{\vartheta,0,b} \leqq \langle \phi_{1A}\rangle_{\vartheta,\mu,b}$$

for $A \in \mathcal{M}_+$.

From Lemma III.4 it follows that $\lim_{\vartheta\nearrow\mathbb{R}^d} \langle \phi_{1A}\rangle_{\vartheta,0,b}$ exists and is equal to $\langle \phi_{1A}\rangle_{+,b}$. Similarly for $\langle \phi_{2B}\rangle_{0b}$.

From Lemma III.2 it now follows that $\lim_{\vartheta\uparrow\mathbb{R}^d} \langle \phi_{1A}\phi_{2B}\rangle_{\vartheta,0,b}$ exists and equals $\langle \phi_{1A}\phi_{2B}\rangle_{+,b}$ for all $A, B \in \mathcal{M}_+$. By multilinearity the same result holds for all $A, B \in \mathcal{M}$. □

III.6. Corollary. *If for some (standard) boundary conditions* b, b' $\langle\,\rangle_{\mu,b} = \langle\,\rangle_{\mu,b'}$ *for all* $\mu > 0$ *and* $\lim_{\mu\downarrow 0} \langle \phi_1(f)\rangle_{\mu,b} = 0$ $\forall f \in \mathcal{S}(\mathbb{R}^d)$ *then* $\langle\,\rangle_{0,b} = \langle\,\rangle_{0b'}$.

Proof. Follows directly from Theorem III.5. □

III.7. Theorem. *In two space time dimensions the infinite volume limit exists for* $b = (\mathrm{H})\mathrm{D}, (\mathrm{H})\mathrm{P}$. *The resultant theories have the cluster property and thus the associated Wightman theories have a unique vacuum; moreover* $\langle\,\rangle_{\mathrm{HP}} = \langle\,\rangle_{\mathrm{P}}$ *and* $\langle\,\rangle_{\mathrm{HD}} = \langle\,\rangle_{\mathrm{D}}$.

Proof. If $\lim_{\mu\downarrow 0} \langle \phi_1(f)\rangle_\mu = 0$ the limit $\lim_{\vartheta\uparrow\mathbb{R}^d} \langle\,\rangle_\vartheta$ exists by Theorem III.5. That $\lim_{\mu\downarrow 0} \langle \phi_1(f)\rangle_{\mu,\mathrm{P}} = 0$ follows from the existence of a local current (see Theorem VII.1). Since the Schwinger functions for periodic *bc* are continuous in σ (in fact real analytic see [7]), one can deduce that they coïncide with those obtained from H.P. boundary conditions [21].

From correlation inequalities one has:

$$\langle \phi_1(f)\rangle_{\mu(\mathrm{H})\mathrm{D}} \leqq \langle \phi_1(f)\rangle_{\mu(\mathrm{H})\mathrm{P}} \quad \text{(respectively)}.$$

So, the first part of the theorem is proven.

The second part is based on Lemma III.3 and the fact that (exponential) clustering is known for the theory with periodic *bc* and $\mu > 0$.

When we remark for $|B|$ even

$$\langle \phi_{1A}\phi_{1B}\rangle_b - \langle \phi_{1A}\rangle_b \langle \phi_{1B}\rangle_b \leqq \langle \phi_{1A}(\phi_{1B} - \phi_{2B})\rangle_b,$$
$$\langle \phi_{1A}(\phi_{1B} - \phi_{2B})\rangle_{(\mathrm{H})\mathrm{D}} \leqq \langle \phi_{1A}(\phi_{1B} - \phi_{2B})\rangle_{(\mathrm{H})\mathrm{P}},$$

for $|B|$ odd

$$\langle \phi_{1A}\phi_{1B}\rangle_{(\mathrm{H})\mathrm{D}} \leqq \langle \phi_{1A}\phi_{1B}\rangle_{(\mathrm{H})\mathrm{P}},$$

respectively, clustering is also proven for (H)D at $\mu = 0$.

Since both HD and D b.c. coïncide with weak coupling b.c. [19a] $\langle\,\rangle_{\mathrm{HD}} = \langle\,\rangle_{\mathrm{D}}$. □

Remark. Fröhlich states that for $\mu > 0$, D = P = space time averaged free [9]. It would then follow by Corollary III.6 that those theories also coïncide at $\mu = 0$.

IV. The Generator of Symmetry Transformations on the Local Algebras

The discussion of the local algebras $\mathcal{A}_\Lambda$ takes place in Fock space. The vector Ω denotes the Fock vacuum, $\mathcal{D}$ = finite particle vectors with wave functions in $\mathcal{S}$. For simplicity, we consider here the case of two component fields, although the discussion generalizes easily to n-component fields (see Sect. VI). The symmetry transformation is then given by:

$$\alpha_s \phi_1(\underline{x}) = \cos s\phi_1(\underline{x}) + \sin s\phi_2(\underline{x})$$
$$\alpha_s \phi_2(\underline{x}) = \cos s\phi_2(\underline{x}) - \sin s\phi_1(\underline{x})$$

and similarly for $\pi_{1,2}$.

We recall [13, 24] that $\mathscr{D}$ is a dense set of entire analytic vectors for $\phi_{1,2}(f)$, $\pi_{1,2}(f)$.

IV.1. Definition.

$j(\underline{x})=\phi_2(\underline{x})\pi_1(\underline{x})-\phi_1(\underline{x})\pi_2(\underline{x})$

For $\vartheta\in C_0^\infty$ $j(\vartheta)=\int d\underline{x}\,\vartheta(\underline{x})j(\underline{x})$.

We define Λ_ϑ, $\Lambda_{\operatorname{supp}\vartheta}$ such that $\vartheta(\underline{x})=1$ for $x\in\Lambda_\vartheta$ and $\operatorname{supp}\vartheta\subset\Lambda_{\operatorname{supp}\vartheta}$. We note that $j(\vartheta)$ is a well-defined operator, and indeed $\mathscr{D}$ is a dense set of analytic *(not entire)* vectors for $j(\vartheta)$. This follows by standard estimates [24]: One writes $j(\vartheta)=j_1+j_1^++j_2+j_2^+$ where "+" denotes adjoint and

$$j_1=\frac{i}{2\sqrt{2\pi}}\int dp\,dq\,\tilde{\vartheta}(p+q)\left[\sqrt{\frac{\mu(q)}{\mu(p)}}-\sqrt{\frac{\mu(p)}{\mu(q)}}\right]a_2^+(p)a_1^+(q),$$

$$j_2=\frac{i}{2\sqrt{2\pi}}\int dp\,dq\,\tilde{\vartheta}(q-p)\left[\sqrt{\frac{\mu(q)}{\mu(p)}}+\sqrt{\frac{\mu(p)}{\mu(q)}}\right]a_1^+(q)a_2(p).$$

Then

$$\begin{aligned}\|(N+1)^{-1/2}j_1(N+1)^{-1/2}\| &\leqq \frac{1}{2\sqrt{2\pi}}\left\|\tilde{\vartheta}(p+q)\left[\sqrt{\frac{\mu(q)}{\mu(p)}}-\sqrt{\frac{\mu(p)}{\mu(q)}}\right]\right\|_2\\ &\leqq \frac{1}{2\sqrt{2\pi}}\left\|\tilde{\vartheta}(p+q)(p+q)\left[\frac{1}{\mu(p)}+\frac{1}{\mu(q)}\right]\right\|_2\\ &=\frac{1}{\sqrt{2\pi}}\|\tilde{\vartheta}(p)p\|_2\left\|\frac{1}{\mu(p)}\right\|_2\\ &<\infty.\end{aligned}\tag{5}$$

$j_2=\dfrac{i}{\sqrt{2\pi}}d\Gamma(K)$, where K is the operator on the one particle space with kernel.

$$K(q,p)=\tilde{\vartheta}(q-p)\left[\sqrt{\frac{\mu(q)}{\mu(p)}}+\sqrt{\frac{\mu(p)}{\mu(q)}}\right]\frac{1}{2}$$

(and mapping particle 2 into particle 1).

We write

$$\begin{aligned}K(q,p)&=\tilde{\vartheta}(q-p)+\tilde{\vartheta}(q-p)\frac{1}{2}\left[\left(\sqrt{\frac{\mu(q)}{\mu(p)}}-1\right)+\left(\sqrt{\frac{\mu(p)}{\mu(q)}}-1\right)\right]\\ &=\tilde{\vartheta}(q-p)+K_1(q,p).\end{aligned}$$

Note that K_1 is a Hilbert-Schmidt operator, since

$$|K_1(q,p)|\leqq\left|\tilde{\vartheta}(q-p)\frac{1}{2}|q-p|\left[\frac{1}{\mu(p)}+\frac{1}{\mu(q)}\right]\right|\in L^2(dq\,dp).$$

Thus $K = M_\vartheta + K_1$, where M_ϑ denotes the operator multiplication by $\vartheta(\underline{x})$. Thus:

$$\|(N+1)^{-1/2} j_2 (N+1)^{-1/2}\| \leqq \frac{1}{\sqrt{2\pi}} \|K\| \leqq \frac{1}{\sqrt{2\pi}} \{\|\vartheta\|_\infty + \|K_1\|_{\mathrm{HS}}\} < \infty .$$

We have thus shown:

Number Operator Estimate

$$\|(N+1)^{-1/2} j(\vartheta)(N+1)^{-1/2}\| < \infty .$$

Equivalently,

$$\|j(\vartheta)(N+1)^{-1}\| < \infty .$$

Since by higher order estimates, $N^\kappa \leqq a(H(g)+b)^\kappa$, the number operator estimate is sufficient when uniformity in g is not required. However, we will also need an estimate uniform in g, and for this we need a local number operator estimate. For this, one restricts the $\underline{x}$-space kernels to finite regions. For j_2, we consider K in $\underline{x}$-space. Since $\vartheta(\underline{x})$ has compact support $d\Gamma(M_\vartheta)$ is directly estimated by a local number operator N_{loc}. To estimate $d\Gamma(K_1)$, we label unit intervals such that $\underline{x} \in \Delta_i \Rightarrow |\underline{x}| \geqq |i|$.
Consider $d\Gamma(\chi_{\Delta_i} K_1 \chi_{\Delta_j})$ where χ_{Δ_i} is the characteristic function of Δ_i.
If $N_{ij} = d\Gamma(\chi'_{\Delta_i} \oplus \chi'_{\Delta_j})$ with χ'_{Δ_i} of compact support and $\chi'_{\Delta_i} = 1$ on Δ_i, an estimate similar to (5) gives

$$\|(N_{ij}+1)^{-1/2} d\Gamma(\chi_{\Delta_i} K_1 \chi_{\Delta_j})(N_{ij}+1)^{-1/2}\| \leqq \|\chi_{\Delta_i} K_1 \chi_{\Delta_j}\| .$$

We then remark that

$$\begin{aligned}\sum_{ij} \|\chi_{\Delta_i} K_1 \chi_{\Delta_j}\|_{\mathrm{HS}} &= \sum_{ij} (1+|i|^2+|j|^2)^{-1} \|(1+|i|^2+|j|^2)\chi_{\Delta_i} K_1 \chi_{\Delta_j}\|_{\mathrm{HS}} \\ &\leqq \left(\sum_{ij} \frac{1}{(1+|i|^2+|j|^2)^2}\right)^{1/2} \left(\sum_{ij} \|(1+|i|^2+|j|^2)\chi_{\Delta_i} K_1 \chi_{\Delta_j}\|_{\mathrm{HS}}^2\right)^{1/2} \\ &\leqq C\|(1+\underline{x}^2+\underline{y}^2)K_1\|_{\mathrm{HS}} = C\|[1-\partial_p^2-\partial_q^2]K_1(p,q)\|_2 .\end{aligned}$$

Since differentations only improve the behavior of $K_1(p,q)$ we see that

$$\sum_{ij} \|\chi_{\Delta_i} K_1 \chi_{\Delta_j}\|_{\mathrm{HS}} < \infty .$$

Using the N_{loc}^τ estimate, we obtain

$$\begin{aligned} d\Gamma(M_\vartheta + K_1) &\leqq \|\vartheta\|_\infty a(\hat{H}(g)+b) + \sum_{ij} \|\chi_{\Delta_i} K_1 \chi_{\Delta_j}\|_{\mathrm{HS}} a(\hat{H}(g)+b) \\ &\leqq C(\hat{H}(g)+b) .\end{aligned}$$

The operator j_1 is estimated in the same way as $d\Gamma(K_1)$. We thus have.

Local Number Operator Estimate

$$\|(\hat{H}(g)+b)^{-1/2} j(\vartheta)(\hat{H}(g)+b)^{-1/2}\| \leqq C \quad \textit{(uniform in } g\textit{)} .$$

We note that the periodic current $j(\vartheta)_{(V)}$ is defined in the same way as $j(\vartheta)$ with ϕ, π replaced by the periodic fields $\phi_{(V)}$, $\pi_{(V)}$.

The number operator estimate goes through as for $j(\vartheta)$

Number Operator Estimate

$$\|(N_{(V)}+1)^{-1/2}j(\vartheta)_{(V)}(N_{(V)}+1)^{-1/2}\| \leqq C.$$

Equivalently.

$$\|j(\vartheta)_{(V)}(N_{(V)}+1)^{-1}\| \leqq C' \qquad \textit{(uniformly in } V\textit{)}.$$

The local number operator estimate also goes through for $j(\vartheta)$. We remark, for example, that $\int dq\,dp\,K_1(q,p)a^+(q)a(p)$ is replaced by

$$\sum_k \Delta \sum_l \Delta K_1(q_k, p_l)a_V^+(q_k)a_V(p_l)$$

where $\Delta = \frac{2\pi}{V}$ and q_k, $p_k = \frac{2\pi}{V}k$.

This can be written in $\underline{x}$-space as

$$\int_{-V/2}^{V/2} d\underline{x}\,d\underline{y}\,K_1(\underline{x},\underline{y})_{(V)}a_V^+(\underline{x})a_V(\underline{y}),$$

where $K_1(\underline{x},\underline{y})_{(V)} = \sum_{m,n} K_1(\underline{x}+nV, \underline{y}+mV)$.

Then

$$\sum_{i,j\in V} \|\chi_{\Delta_i}K_{1(V)}\chi_{\Delta_j}\| \leqq \sum_{i,j} \|\chi_{\Delta_i}K_1\chi_{\Delta_j}\| \leqq C.$$

We thus obtain

Local Number Operator Estimate

$$\|(\hat{H}(g)_{(V)}+b)^{-1/2}j(\vartheta)_{(V)}(\hat{H}(g)+b)^{-1/2}\| \leqq C \qquad \textit{(uniformly in } V \textit{ and } g\textit{)}.$$

We now begin a detailed discussion of the current and associated symmetry transformations. We will write $H = \hat{H}(g)_{\mu=0}$, $H_\mu = H - \mu\phi_1(g)$. We suppose $\Lambda_g \supset \Lambda_{\mathrm{supp}\,\vartheta}$.

IV.2. Proposition [13, 8]. *$\phi_j(f)$, $\pi_j(f)$, $j(\vartheta)$ are essentially self-adjoint on $\mathscr{D}$.*

If supp $\vartheta \subset \Lambda$ *then* $e^{isj(\vartheta)} \in \mathscr{A}_\Lambda$ $\forall s \in \mathbb{R}$. *Also,* $e^{isj(\vartheta)}\mathscr{A}_{\Lambda_\vartheta}e^{-isj(\vartheta)} = \alpha_s\mathscr{A}_{\Lambda_\vartheta}$.

Proof. Essential self-adjointness follows since $\mathscr{D}$ is a dense set of analytic vectors for ϕ, π, j. For s sufficiently small we may calculate $(\psi_1, e^{i\phi(f)}e^{isj(\vartheta)}\psi_2)$ for $\psi_1, \psi_2 \in \mathscr{D}$ by expanding the exponentials. If supp f is contained in the complement of Λ we conclude that $e^{isj(\vartheta)}$ commutes with $e^{i\phi(f)}$. Similarly for $e^{i\pi(f)}$. By duality it follows that $e^{isj(\vartheta)} \in \mathscr{A}'_{\Lambda_c} = \mathscr{A}_\Lambda$. It then follows for all $s \in \mathbb{R}$ by the group property.

Again by expanding exponentials, and using $\vartheta(\underline{x}) = 1$ for $\underline{x} \in \Lambda_\vartheta$ it follows that $e^{isj(\vartheta)}$ implements the automorphism α_s on $\mathscr{A}_{\Lambda_\vartheta}$ for small s. The result then follows for all s by the group property. $\square$

In order to guarantee that the infinite volume vacuum is in the domain of the current we must average $j(\vartheta)$ in *time*. That is, we must consider

$$j(\eta)=\int dt\,\eta(t)e^{itH}j(\vartheta)e^{-itH}.$$

The following propositions are used to show that $j(\eta)$ also generates the symmetry automorphism (Theorem IV.6).

By a higher order estimate and the number operator estimate we have $\|j(\vartheta)(H+1)^{-1}\|\leqq\|j(\vartheta)(N+1)^{-1}\|\,\|(N+1)(H+1)^{1}\|<\infty$.

The estimate $\|j(\vartheta)(N+1)^{-1}\|<\infty$ implies that $j(\vartheta)$ may be extended from finite particle vectors so that $\mathscr{D}(j(\vartheta))\supset\mathscr{D}(N)$. This means that $\mathscr{D}(j(\vartheta))\supset\mathscr{D}(H_0)$. Since H is essentially self-adjoint on $\mathscr{D}(H_0)\cap\mathscr{D}(H_I(g))$ it follows from $\|j(\vartheta)(H+1)^{-1}\|<\infty$ that the domain of essential self adjointness of $j(\vartheta)$ can be extended to include $\mathscr{D}(H)$.

So $j_t=e^{itH}j(\vartheta)e^{-itH}$ is well defined and in fact essentially self adjoint on $\mathscr{D}(H)$, the domain of H, since $j(\vartheta)$ is. This last estimate also implies that $\varepsilon j(\vartheta)$ is a Kato small perturbation of H for ε small enough. Then $H+\varepsilon j(\vartheta)$ is self-adjoint on $\mathscr{D}(H)$.

In the same way, $H+\varepsilon j_t$ is self adjoint on $\mathscr{D}(H)$.

IV.3. Proposition. $e^{-isH}e^{is(H+\varepsilon j_t)}$ *implements* $\alpha_{s\varepsilon}$ *on* $\mathscr{A}(\Lambda)$ *if* $\Lambda_{s+t}\subset\Lambda_\vartheta$ *(We recall that* Λ_d *is the set of points within a distance d of* Λ *and* Λ_ϑ *is the set on which* $\vartheta=1$.*)*

Proof. Since $H+\varepsilon j_t$ is self adjoint on $\mathscr{D}(H)$ the Trotter product formula holds and

$$e^{is(H+\varepsilon j_t)}=\text{strong}\lim_{n\to\infty}(e^{i(s/n)H}e^{i(s\varepsilon/n)j_t})^n.$$

Then for $A\in\mathscr{A}_\Lambda$

$$e^{is(H+\varepsilon j_t)}Ae^{-is(H+\varepsilon j_t)}=\text{strong}\lim_{n\to\infty}(e^{i(s/n)H}e^{i(s\varepsilon/n)j_t})^nA(e^{-i(s\varepsilon/n)j_t}e^{-i(s/n)H}).$$

Now since $e^{isj(\vartheta)}$ implements α_s and e^{itH} implements τ_t (which *commutes* with α_s) it follows that e^{isj_t} implements α_s and therefore

$$e^{is(H+\varepsilon j_t)}Ae^{-is(H+\varepsilon j_t)}=\text{strong}\lim_{n\to\infty}(\tau_{s/n}\alpha_{s\varepsilon/n})^nA=\tau_s\alpha_{s\varepsilon}A.\quad\square$$

IV.4. Proposition. $e^{-isH}e^{is(H+\varepsilon\sum_K\eta_{t_K}j_{t_K})}$ *implements* $\alpha_{s\varepsilon}$ *on* $\mathscr{A}_\Lambda$ *if* $\Lambda_{s+t_{\max}}\subset\Lambda_\vartheta$ *and*

$$\sum_K\eta_{t_K}=1\ \ \eta_{t_K}\geqq0\forall K.$$

Proof. By induction based on

$$\begin{aligned}&e^{is(H+\varepsilon\{\eta_1j_{t_1}+\eta_2j_{t_2}\})}\\&=e^{is(\eta_1\{H+\varepsilon j_{t_1}\}+\eta_2\{H+\varepsilon j_{t_2}\})}\\&=\lim_{n\to\infty}[e^{i(s/n)\eta_1\{H+\varepsilon j_{t_1}\}}e^{i(s/n)\eta_2\{H+\varepsilon j_{t_2}\}}]^n.\end{aligned}$$

By Proposition IV.3 this implements

$$\lim_{n\to\infty}(\alpha_{(s/n)\eta_1}\alpha_{(s/n)\eta_2})^n=\alpha_{s(\eta_1+\eta_2)}=\alpha_s\quad\text{if}\quad\eta_1+\eta_2=1.\quad\square$$

We note that $j_t(H+1)^{-1}=e^{itH}j(\vartheta)(H+1)^{-1}e^{-itH}$ is strongly continuous in t. Thus $\sum \Delta\eta(t_K)j_{t_K} \xrightarrow[\text{strong}]{} \int dt\,\eta(t)j(t)$ on $\mathscr{D}(H)$.

We define $j(\eta)=\int dt\,\eta(t)j(t)$ for $\eta\in C_0^\infty$.

Thus $\|j(\eta)(H+1)^{-1}\|<\infty$.

Since $\pm i[H,j(\eta)]=\pm j(\dot\eta)$ where $\dot\eta=\frac{d}{dt}\eta(t)$ and $\|j(\dot\eta)(H+1)^{-1}\|<\infty$, it is a standard result that $j(\eta)$ is essentially self-adjoint on any core for H [24].

IV.5. Proposition. $e^{is(H+\varepsilon\sum\Delta\eta(t_K)j_{t_K})} \xrightarrow[\text{strong}]{} e^{is(H+\varepsilon j(\eta))}$.

Proof. $H+\varepsilon\sum\Delta\eta(t_K)j_{t_K} \xrightarrow[\text{strong}]{} H+\varepsilon j(\eta)$ on $\mathscr{D}(H)$ and $H+\varepsilon j(\eta)$ is self-adjoint on $\mathscr{D}(H)$.

It is a standard result [23] that these properties imply the desired strong convergence. □

We thus conclude that $e^{-isH}e^{is(H+\varepsilon j(\eta))}$ implements $\alpha_{s\varepsilon}$ on $\mathscr{A}_\Lambda$ if $\Lambda_{s+T}\subset\Lambda_\vartheta$ where $\operatorname{supp}\eta\subset[-T,T]$, $\eta\geqq 0$.

IV.6. Theorem. *Let* $\operatorname{supp}\eta\subset[-T,T]$, $\eta\geqq 0$. *Let* $\tilde\Lambda=$ *set of points within a distance* $T+1$ *of* $\operatorname{supp}\vartheta$ *and let* Λ *be such that* $\Lambda_T\subset\Lambda_\vartheta$, *then* $e^{isj(\eta)}\in\mathscr{A}_{\tilde\Lambda}$ *and* $e^{isj(\eta)}$ *implements* α_s *on* $\mathscr{A}_\Lambda$.

Proof. Take s small then use the group property. $\varepsilon j(\eta)=[H+\varepsilon j(\eta)]-H$ on $\mathscr{D}(H)$, and since $\varepsilon j(\eta)$ is essentially self-adjoint on $\mathscr{D}(H)$ the Trotter product formula holds, and so $e^{is\varepsilon j(\eta)}=\operatorname{strong}\lim_{n\to\infty}[e^{-i(s/n)H}e^{i(s/n)(H+\varepsilon j(\eta))}]^n$.

The operator in brackets implements $\alpha_{(s/n)\varepsilon}$ on $\mathscr{A}(\Lambda)$. Thus $e^{is\varepsilon j(\eta)}$ implements $\alpha_{s\varepsilon}$ on $\mathscr{A}_\Lambda$.

To show $e^{isj(\eta)}\in\mathscr{A}_{\tilde\Lambda}$ we use duality. Let $A\in\mathscr{A}_{\Lambda'}$ where $\Lambda'\subset\tilde\Lambda_c$. Then since $e^{isj(\vartheta)}$ implements the *identity automorphism* on $\mathscr{A}_{\Lambda'}$ the same argument as above shows that $e^{-isH}e^{is(H+\varepsilon j(\eta))}$ implements the identity automorphism, as does $e^{isj(\eta)}$. That is, $e^{isj(\eta)}Ae^{-isj(\eta)}=A$. Thus $e^{isj(\eta)}\in\mathscr{A}'_{\tilde\Lambda_c}=\mathscr{A}_{\tilde\Lambda}$. □

We have thus attained our first main result, that $j(\eta)$ is the local generator of the automorphism α_s.

We now proceed to investigate the case of a non-zero external field $\mu\phi_1(g)$; $H_\mu=H+\mu\phi_1(g)$.

Since $\|\phi_1(g)(N+1)^{-1}\|<\infty$, it follows by a higher order estimate that $\|\mu\phi_1(g)(H+1)^{-1}\|<1$ for μ sufficiently small.

Thus H_μ is self-adjoint on $\mathscr{D}(H)$.

Define

$$j_t^\mu=e^{itH_\mu}j(\vartheta)e^{-itH_\mu},$$

$$j^\mu(\eta)=\int dt\,\eta(t)j_t^\mu.$$

As before, $j^\mu(\eta)$ is essentially self-adjoint on any core for H_μ, in particular on $\mathscr{D}(H)$.

IV.7. Proposition. $j^\mu(\eta)\xrightarrow[\text{strong}]{\mu\to 0} j(\eta)$ *on* $\mathscr{D}(H)$.

Proof. Since $H_\mu \xrightarrow[\mu\to 0]{} H$ on $\mathscr{D}(H)$ it follows that $e^{itH_\mu}\xrightarrow[\text{strong}]{\mu\to 0} e^{itH}$. Also, $He^{-itH_\mu} = e^{-itH_\mu}H_\mu - \mu\phi e^{-itH_\mu}$. Since $\|\phi e^{-itH_\mu}f\| \leqq \|\phi(H_\mu + C)^{-1}\|\ \|(H_\mu + C)f\| \leqq C$ uniformly in μ as $\mu\to 0$, it follows that $He^{-itH_\mu}\to He^{-itH}$ on $\mathscr{D}(H)$.
It follows that $j_t^\mu = e^{itH_\mu}[j(\vartheta)(H+1)^{-1}(H+1)e^{-itH_\mu}$ converges to j_t on $\mathscr{D}(H)$ as $\mu\to 0$.

Also, $\|j_t^\mu f\| \leqq \|j(\vartheta)(H_\mu + C)^{-1}\|\ \|(H_\mu + C)f\| \leqq C'$ uniformly in μ and t.
Finally, $\|j^\mu(\eta)f - j(\eta)f\| \leqq \int dt|\eta(t)|\ \|j_t^\mu f - j_t f\|$. Since $\|j_t^\mu f - j_t f\| \to 0$ pointwise in t as $\mu\to 0$ and is uniformly bounded, it follows by dominated convergence that $\int dt|\eta(t)|\ \|j_t^\mu f - j_t f\| \to 0$. □

IV.8. Theorem. $e^{isj^\mu(\eta)}\xrightarrow[\text{strong}]{\mu\to 0} e^{isj(\eta)}$ *and* $e^{isj^\mu(\eta)}\in\mathscr{A}_{\tilde{\Lambda}}$ *(where* $\tilde{\Lambda}$ *is defined as in Theorem* IV.6).

Proof. The strong convergence of $e^{isj^\mu(\eta)}$ follows from Proposition IV.7 and the fact that $j(\eta)$ is essentially self-adjoint on $\mathscr{D}(H)$. That $e^{isj^\mu(\eta)}\in\mathscr{A}_{\tilde{\Lambda}}$ follows by duality as in Theorem IV.6, since $j(\vartheta)$ implements the identity automorphism on $\Lambda_{\operatorname{supp}\vartheta C}$. □

We have thus attained our second main result, that $e^{isj^\mu(\eta)}$ converges strongly to $e^{isj(\eta)}$. Finally, we must investigate the periodic current.

IV.9. Theorem. $e^{ij_V^\mu(f)}\xrightarrow[\text{strong}]{V\to\infty} e^{ij^\mu(f)}$.

Proof. It is sufficient to show $j_V^\mu(f)\to j^\mu(f)$ on a core for $j^\mu(f)$. One takes $C^\infty(H)$ and the proof follows Lemma III.4. of [17]. □

V. Convergence of States on $\mathscr{A}$ and Absence of Spontaneous Symmetry Breakdown

Before discussing the convergence of periodic states (based on Euclidean methods) we recall the general results concerning local norm compactness [15, 17]. The algebra $\mathring{\mathscr{A}}(\Lambda)$ is defined as the C^*-algebra generated by $e^{i\{\boldsymbol{\Phi}(\mathbf{f})+\boldsymbol{\pi}(\mathbf{g})\}}$ $\operatorname{supp}\mathbf{f}, \operatorname{supp}\mathbf{g}\subset\Lambda$ and $\mathring{\mathscr{A}}$ = norm closure of $\bigcup_\Lambda \mathring{\mathscr{A}}(\Lambda)$, and similarly for $\mathring{\mathscr{A}}(\Lambda)_{(V)}$. The isomorphism p_V of $\mathring{\mathscr{A}}(\Lambda)$ on to $\mathring{\mathscr{A}}(\Lambda)_{(V)}$ is determined by

$$p_V e^{i\{\boldsymbol{\phi}(\mathbf{f})+\boldsymbol{\pi}(\mathbf{g})\}} = e^{i\{\boldsymbol{\phi}(\mathbf{f})_{(V)}+\boldsymbol{\pi}(\mathbf{g})_{(V)}\}},$$

where $\Lambda\subset V$.

The Hamiltonian $H(g)_{(V)}$, with $g=\chi_V$ has a unique ground state Ω_V in $\mathscr{F}_V$. The state ω_V is defined on $\mathscr{B}(\mathscr{F})$ – the bounded operators on $\mathscr{F}$ – by $\omega_V(\,)=(\Omega_V, \cdot\,\Omega_V)$. (We use the embedding $\mathscr{F}_V\subset\mathscr{F}$ to consider $\Omega_V\in\mathscr{F}$.) The state $\tilde{\omega}_V$ is defined on $\mathring{\mathscr{A}}(\Lambda)$ if $\Lambda\subset V$ by $\tilde{\omega}_V = \omega_V\cdot p_V$.
As $V\to\infty$ the states $\tilde{\omega}_V$ are defined eventually on a dense subalgebra of $\mathring{\mathscr{A}}$.

V.1. Theorem [17]. *Let* ω *be a state on* $\mathring{\mathscr{A}}$ *which is a weak* limit of the* $\tilde{\omega}_V$. *Then* ω *is locally Fock ie for each bounded region* $\Lambda\subseteqq\mathbb{R}^2$ $\omega\restriction\mathscr{A}(\Lambda)$ *is normal.* □

With techniques used in Theorem V.1 one can deduce the following result.

V.2. Proposition [17]. *Let* ω *be a weak* limit of the* $\tilde{\omega}_V$. *Suppose* $A_V \in \mathscr{A}_{(V)}(\Lambda)$ *converges weakly to* $A \in \mathscr{A}(\Lambda)$ *as* $V \to \infty$ *and also*

$$\|A_V\| \leqq C,$$

$$\|A\| \leqq C.$$

Then there exists a sequence $(V_k)_{k\in\mathbb{N}}$ *such that* $\omega_{V_k}(A_{V_k}) \to \omega(A)$.

There is one other useful result:

V.3. Proposition. *Suppose* $\{\omega_\mu\}$ *is a set of states on* $\mathscr{A}$ *such that* $\{\omega_\mu\} \upharpoonright \mathscr{A}(\Lambda)$ *lies in a norm compact subset of the dual of* $\mathscr{A}(\Lambda)$. *If* ω *is a state on* $\mathscr{A}$ *which is a weak* limit of* $\{\omega_\mu\}$ *then there is a subsequence* $\omega_{\mu_n} \xrightarrow[\text{norm}]{} \omega$ *on* $\mathscr{A}(\Lambda)$. *If* ω *is the unique weak* limit point of* $\{\omega_\mu\}$ *then* $\omega_\mu \xrightarrow[\text{norm}]{} \omega$ *on* $\mathscr{A}(\Lambda)$.

The above proposition follows from the general fact that if a set is compact with respect to one topology then it is compact and therefore closed with respect to a weaker Haussdorf topology. Thus the norm closure is equal to the weak* closure [on $\mathscr{A}(\Lambda)$] and any weak* limit point is a norm limit of a subsequence. □

Using Euclidean methods [7] one knows that the periodic Schwinger functions converge as $V \to \infty$ for external field $\mu > 0$. By analytic continuation [18] and using the methods of Proposition I.1 of [16] it follows that $\tilde{\omega}_{\mu V}(e^{i\phi(f_1)} \dots e^{i\phi(f_n)})$ converges for $f_i \in \mathscr{S}(\mathbb{R}^2)$. Let $\omega_{\mu_1}, \omega_{\mu_2}$ be weak* limit points of $\tilde{\omega}_{\mu V}$. Then $\omega_{\mu_1}, \omega_{\mu_2}$ agree on operators of the form $e^{i\phi(f_1)} \dots e^{i\phi(f_n)}$. Since $\omega_{\mu_1}, \omega_{\mu_2}$ are locally normal and $\{e^{i\phi(f_1)} \dots e^{i\phi(f_n)}\}$ is strongly dense in $\mathscr{A}(\Lambda)$ it follows that $\omega_{\mu_1}, \omega_{\mu_2}$ agree on $\mathscr{A}(\Lambda)$. It follows that $\tilde{\omega}_{\mu V}$ has a unique weak* limit point.

Using similar methods as in Theorem V.1 one can deduce that for Λ a bounded region of $\mathbb{R}^2$, $\{\omega_\mu \upharpoonright \mathscr{A}(\Lambda)\}$ lies in a norm compact subset of the dual $\mathscr{A}(\Lambda)^*$.

V.4. Proposition. *If* ω_+ *is a weak* limit of* ω_μ *as* $\mu \to 0$ *then there exists a subsequence* $\{\mu_n\}_{n\in\mathbb{N}}$ *such that*

$$\lim_{n\to\infty} \omega_{\mu_n}(e^{ij_{\mu_n}(f)}) = \omega_+(e^{ij_{\mu=0}(f)}).$$

Proof. By Proposition V.3 there exists a subsequence ω_{μ_n} which is norm convergent on $\mathscr{A}(\Lambda)$. Since each $\omega_{\mu_n} \upharpoonright \mathscr{A}(\Lambda)$ is normal, the norm limit ω_+ is normal on $\mathscr{A}(\Lambda)$.

$$\begin{aligned} &\omega_+(e^{ij(f)}) - \omega_{\mu_n}(e^{ij_{\mu_n}(f)}) \\ &\quad = \omega_+(e^{ij(f)} - e^{ij_{\mu_n}(f)}) + (\omega_+ - \omega_{\mu_n})(e^{ij_{\mu_n}(f)}). \end{aligned}$$

The first term in the r.h.s. converges to zero because $e^{ij_\mu(f)}$ converges ultrastrongly to $e^{ij(f)}$ and $\omega_+ \upharpoonright \mathscr{A}(\Lambda)$ is normal. From the norm convergence of $\omega_{\mu_n} \upharpoonright \mathscr{A}(\Lambda)$ we see that the second term also converges to zero. □

Remark. for $N = 2\omega_\mu$ is weak* convergent to ω_+ and subsequences are not needed.

V.5. Proposition. *Let* Ω_+ *be the vector associated with* ω_+ *by the* G.N.S. *construction and* π_+ *the corresponding representation of* $\mathscr{A}$, *then*, *for* $f \in C_0^\infty(\mathbb{R}^2)$, Ω_+ *is in the domain of* $\pi_+(j(f))$.

Proof. The periodic local number operator estimate gives

$$\|j_V^\mu(f)\Omega_{\mu V}\| = \left\|\frac{1}{\hat{H}_{\mu(V)}+b}(\hat{H}_{\mu(V)}+b)j_V^\mu(f)\Omega_{\mu V}\right\|$$

$$= \left\|\frac{1}{\hat{H}_{\mu(V)}+b}[j_V^\mu(i\dot{f})+bj_V^\mu(f)]\frac{b}{\hat{H}_{\mu(V)}+b}\Omega_{\mu V}\right\|$$

$$\leqq \left\|\frac{1}{(\hat{H}_{\mu(V)}+b)^{\frac{1}{2}}}[j_V^\mu(i\dot{f})+bj_V^\mu(f)]\frac{1}{(\hat{H}_{\mu(V)}+b)^{\frac{1}{2}}}\right\| \leqq C \quad \text{uniformly in } V.$$

Thus

$$\omega_V^\mu \frac{(|e^{isj^\mu(f)}-1|^2)}{s^2} = \omega_V^\mu\left(\frac{2-e^{isj_V^\mu}-e^{isj_V^\mu}}{s^2}\right) \leqq C^2 \forall s\,.$$

Therefore from Proposition V.2, $\omega^\mu\left(\dfrac{2-e^{isj^\mu}-e^{-isj^\mu}}{s^2}\right) \leqq C^2\,\forall s$ which together with Proposition V.4 implies: $\omega_+\left(\dfrac{2-e^{isj}-e^{-isj}}{s^2}\right) \leqq C^2\,\forall s.$ □

Since by Theorem IV.6 $j(\eta)$ generates the automorphism α_s on $\mathscr{A}(\Lambda)$ (for suitable η) we have, for $A \in \mathscr{A}(\Lambda)$,

$$\omega_+(\alpha_s A) = \omega_+(e^{isj(\eta)}Ae^{-isj(\eta)}) = (\Omega_+, e^{is\pi_+(j(\eta))}\pi_+(A)e^{-is\pi_+(j(\eta))}\Omega_+)\,.$$

Thus $\left.\dfrac{d}{ds}\right|_{s=0} \omega_+(\alpha_s A) = i(\pi_+(j(\eta))\Omega_+, \pi_+(A)\Omega_+) - i(\Omega_+, \pi_+(A)\pi_+(j(\eta))\Omega_+)$. We then have

V.6. Theorem. ω_+ is invariant under the automorphism group. In particular $\omega_+(\phi_1(f))=0$.

Proof. The constant C in Proposition V.5 is in fact a norm $|||f|||$ on f which is continuous in the topology of compact support test functions, as follows from the discussion of the number operator estimates. We obtain $\|j(f)\Omega\| \leqq |||f|||$, which shows that

$$\int d\underline{x}h(\underline{x})U(\underline{x})j(\vartheta\eta)\Omega = j((h*\vartheta)\eta)\Omega = \int d\underline{x}\vartheta(\underline{x})U(\underline{x})j(h\eta)\Omega \quad \text{if} \quad \int d\underline{x}h(\underline{x})=1\,.$$

Thus $j'(x) = U(x)j(h\eta)U^{-1}(x)$ plays the role of $j(x)$ in the generalized Coleman Theorem VII.1. That theorem then implies that ω_+ is invariant under α_s and so, in particular $(\Omega_+, \phi_1(f)\Omega_+)=0$.

By Theorem III.7 we have existence of HD, D, HP, P boundary conditions for the $\mu=0$ theory, as well as clustering.

VI. Absence of Spontaneous Symmetry Breaking for the N-Component $P(|\phi|^2)$ Model

We consider the n-component $P(|\phi|^2)+\mu\phi_1$ model, and use the notations of the preceeding sections.

VI.I. Theorem. *If ω_μ is translation invariant, satisfies the spectral condition, and is a weak* limit point of finite volume states satisfying higher order and N^τ_{loc} estimates, then all the weak* limit points of ω_μ as $\mu \to 0$ are* SO(n) *invariant.*

Proof. Let us denote by $g_k(s)$ the rotation of angle s in the plane (x_{k+1}, x_k), the orientation being such that the rotation of e_{k+1} to e_k is positive ($\{e_k\}_{k=1}^n$ is a basis of $\mathbb{R}^n$ and e_k is along x_k). Then by a theorem of group theory [31], we know that each g of SO(n) may be written as $g = g^{(n-1)} \dots g^{(1)}$ where $g^{(k)} = g_1(s_1^k) \dots g_k(s_k^k)$.

$$\left.\begin{aligned} 0 &\leqq s_1^k < 2\pi \\ 0 &\leqq s_j^k < \pi \end{aligned}\right\} 1 < j \leqq k$$

are called the Euler angles of the rotation g.

One can construct n operators $j_1(\vartheta) \dots j_n(\vartheta)$ which are the time components of the "currents" associated with the rotation subgroups $g_1, \dots, g_n$:

$$j_k(\vartheta) = \int d\underline{x}\, \vartheta(\underline{x}) \left[\phi_k(\underline{x}) \pi_{k+1}(\underline{x}) - \phi_{k+1}(\underline{x}) \pi_k(\underline{x})\right], \qquad \vartheta \in C_0^\infty(\mathbb{R}).$$

By N-estimates those $j_k(\vartheta)$ are self-adjoint, and by N^τ_{loc} estimates they obey local number operator estimates $j_k(\vartheta) \leqq C(H(g) + b)$ [$H(g)$ is the Hamiltonian associated with the finite volume state]. Using once more the N^τ_{loc} estimate one proves the local Fock property for infinite volume state ω_μ. If ω_+ is one weak* limit point of the ω_μ as $\mu \to 0$, π^+ the corresponding representation of $\mathscr{A}$ associated with ω_+ by the G.N.S. construction and Ω_+ the associated vacuum, one can prove as in the preceeding section (in particular by using higher order estimates):

$$\frac{d}{ds} (\Omega_+ \alpha_s^i \pi^+(A) \Omega_+) = (\Omega_+ [\pi^+(j_i(\eta)), \pi^+(A)] \Omega_+),$$

where α_s^i is the one parameter group of automorphisms corresponding to rotation subgroup $g_i(s)$. By the generalized Coleman theorem $(\Omega_+ \alpha_s^i \pi^+(A) \Omega_+)$ $= (\Omega_+ \pi^+(A) \Omega_+)$.

The proof is then completed by the fact that each rotation g of SO(n) can be written as a product $g = g^{(n-1)} \dots g^{(1)}$.

Remark. A typical application of this theorem is the case $P = \lambda : (\phi\phi)^2 : n = 3$ and periodic boundary conditions. For this case the infinite volume state ω_μ may be constructed by the method of [7].

One can also consider the general case: P a semi-bounded polynomial, no restriction on n and periodic boundary conditions, since the spectral condition has recently been proven for this theory [22].

VII. Generalized Coleman Theorem

We consider the following general framework for a field theory in two space-time dimensions, with an internal symmetry group.

1. $\mathscr{H}$ is a Hilbert space carrying a unitary representation $U(a)$ of the space-time translation group, $a \in \mathbb{R}^2$, satisfying the spectral condition $H^2 - P^2 \geqq 0$, $H \geqq 0$. There is a (not necessarily unique) vacuum vector Ω.

2. $\mathscr{A}$ is an algebra of operators (taken to be bounded local observables or to be unbounded fields). The vector Ω is in the domain of all $A \in \mathscr{A}$.

3. There is a one-parameter group α_s of transformations of $\mathscr{A}$, which commute with the space-time translations.

4. $j(x)$ is a (not necessarily tempered) operator valued distribution. We define the smoothed out current

$$j(x) = \int dx' h(x-x') J(x'),$$

where $h \in C_0^\infty$. $j(x)$ is assumed to have the following properties:

a) translation covariance: for any space-time translation a,

$$U(a) j(x) U(a)^{-1} = j(x+a);$$

b) the domain of $j(x)$ contains the vacuum Ω and $j(x)\Omega = j^+(x)\Omega$;

c) relative locality: for each $A \in \mathscr{A}$ there exists a diamond $D \subset \mathbb{R}^2$ such that $(j(x)\Omega, A\Omega) - (A^+\Omega, j(x)\Omega) = 0$ if $x \in D^c =$ set of points space-like to all points in D. This also holds with A replaced by $j(y)$;

d) for all $A \in \mathscr{A}$,

$$\left.\frac{d}{ds}\right|_{s=0} (\Omega, \alpha_s A\Omega) = \int d\underline{x} [(j(\underline{x}, t)\Omega, A\Omega) - (A^+\Omega, j(\underline{x}, t)\Omega)].$$

Within this framework we prove the theorem

VII.1. Theorem. *For all $s \in \mathbb{R}$ and all $A \in \mathscr{A}$, $(\Omega, \alpha_s A\Omega) = (\Omega, A\Omega)$.*

Thus the state $(\Omega, \cdot\,\Omega)$ on $\mathscr{A}$ is invariant under the symmetry transformations. This is a generalization of the Coleman theorem "There are no Goldstone bosons in two space-time dimensions" (see [11]). Notice that Lorentz invariance plays no role (except in the form of the spectral condition) and indeed we do not suppose j is the time-component of a conserved current (the "space component" plays no role). The assumption that α_s commutes with spacetime translations suffices, instead. This is particularly useful for models in two space-time dimensions, since the space-component of the current is more singular than the time component. Furthermore, we do not need uniqueness of the vacuum, nor its cyclicity with respect to $\mathscr{A}$.

For the application to Theorems III.5 and III.7 we need only $\langle \phi_1(f) \rangle_+ = 0$ and for this the "Goldstone Theorem" VII.2 suffices. However, for general n-component models the full Theorem VII.1 is required to show the absence of symmetry breakdown.

VII.2. Goldstone Theorem. *Let E_0 be the spectral projection onto vectors with $P^2 = 0$. Then*

$$\left.\frac{d}{ds}\right|_{s=0} (\Omega, \alpha_s A\Omega) = \int d\underline{x} [(E_0 j(\underline{x}, t)\Omega, A\Omega) - (A^+\Omega, E_0 j(\underline{x}, t)\Omega)].$$

Proof. The proof is based on the Jost-Lehman-Dyson representation and is standard [25]. We remark that in place of current conservation we use the commutation of α_s with time translations. □

We note that E_0 may be replaced by $E_0' = E_0 - E_1$, where E_1 is the projection onto vectors invariant under space-time translations (vacua). This follows from the following lemma, taking $f(x) = (j(x)\Omega, A\Omega) - (A^+\Omega, j(x)\Omega)$ [in this case $F(\underline{x})$ in the lemma has compact support].

VII.3. Lemma. *Let $f(x) = \int d\mu(p) e^{ipx}$ where μ is a finite complex measure. Suppose that for each $g(t) \in \mathscr{D}$, $F(\underline{x}) \equiv \int dt g(t) f(t, \underline{x})$ is integrable in $\underline{x}$. Then μ has no support at $f = 0$ (and in particular no support at $p = 0$).*

Proof. Write $\mu = v(p_0)\delta(f) + \mu'(p)$ where v is a measure on p_0 and μ' has no support at $f = 0$. Then $F(\underline{x})$ integrable implies $\tilde{F}(f) = v(\tilde{g})\delta(f) + \int \tilde{g}(p_0) d\mu'(p)$ is bounded and continuous, and thus as a measure cannot have support at $f = 0$. Thus $v = 0$. □

Notation. We introduce the function $\vartheta_r(\underline{x}) = \vartheta\left(\frac{\underline{x}}{r}\right)$ with $\vartheta(\underline{x}) \in C_0^\infty$, $\operatorname{supp} \vartheta \subset [-2, 2]$ and $\vartheta(\underline{x}) = 1$ for $\underline{x} \in [-1, 1]$; $r \in \mathbb{R}^+$.

Then we may write, using Theorem VII.2 and Lemma VII.3

$$\left.\frac{d}{ds}\right|_{s=0} (\Omega, \alpha_s A\Omega) = \lim_{r \to \infty} \{(E_0' j(\vartheta_r)\Omega, A\Omega) - (A^+\Omega, E_0' j(\vartheta_r)\Omega)\}.$$

VII.4. Proposition. *If there exists a constant $C < \infty$ such that, for all $r \in \mathbb{R}^+$,*

$$\|E_0' j(\vartheta_r)\Omega\| \leqq C$$

then for all s

$$(\Omega, \alpha_s A\Omega) = (\Omega, A\Omega).$$

Proof. (See also [11].)

By weak compactness there exists a sequence $E_0' j(\vartheta_{r_n})\Omega \equiv \psi_n$ which converges weakly to a vector ψ. Then

$$\left.\frac{d}{ds}\right|_{s=0} (\Omega, \alpha_s A\Omega) = \lim_{n \to \infty} \{(\psi_n, A\Omega) - (A^+\Omega, \psi_n)\} = (\psi, A\Omega) - (A^+\Omega, \psi).$$

Since α_s commutes with space-time translations, we can replace A by $U(a)AU(a)^{-1}$, $a \in \mathbb{R}^2$ and we obtain

$$(\psi, A\Omega) - (A^+\Omega, \psi) = (\psi, U(a)A\Omega) - (A^+\Omega, U(-a)\psi).$$

Comparing the support of the Fourier transform with respect to a on both sides, and using $E_1\psi = 0$ we conclude

$$(\psi, A\Omega) - (A^+\Omega, \psi) = 0.$$

Thus $\left.\frac{d}{ds}\right|_{s=0} (\Omega, \alpha_s A\Omega) = 0.$

By the group property it follows that $(\Omega, \alpha_s A\Omega) = (\Omega, A\Omega)$. □

VII.5. Theorem. *In two space-time dimensions, there exists a $C < \infty$ such that $\|E_0' j(\vartheta_r)\Omega\| \leqq C$ for all $r \in \mathbb{R}^+$.*

The proof of this theorem, which in turn demonstrates the Theorem VII.1, is based on the fact that $\delta(p^2)$ is too singular to be a distribution in two dimensions [3]. We need

VII.6. Lemma. *Let $F(x)$ be defined by*

$$F(x-y)=(j(x)\Omega, E_0'j(y)\Omega).$$

Then $F(x)$ has the representation

$$F(x)=\int_0^\infty e^{i\alpha(t-x)}P_1(\alpha)d\alpha+\int_0^\infty e^{i\alpha(t+x)}P_2(\alpha)d\alpha,$$

where P_1 and P_2 are analytic, in $\mathscr{S}$, and odd.

Proof. Because of the positivity of the Hilbert space metric, $\tilde{F}(p)$ is a positive measure on $\{p^2=0, p_0>0\}$, the forward light cone. Introcuding the variables

$$u=t+\underline{x}, \mathrm{v}=t-\underline{x}, p_u=\frac{p_0-p_1}{2}, p_\mathrm{v}=\frac{p_0+p_1}{2}, p_u u+p_\mathrm{v}\mathrm{v}=p\cdot x,$$

the forward light cone becomes $p_u=0$, $p_\mathrm{v}>0$ and $p_\mathrm{v}=0$, $p_u>0$. Then

$$F=\int e^{i(p_u u+p_\mathrm{v}\mathrm{v})}[\delta(p_u)d\nu_1(p_\mathrm{v})+\delta(p_\mathrm{v})d\nu_2(p_u)],$$

where $\operatorname{supp}\nu_i\subseteqq\mathbb{R}^+-0$

$$F(x)-F(-x)=\int_{-\infty}^{+\infty} e^{i\alpha\mathrm{v}}d\mu_1(\alpha)+\int_{-\infty}^{+\infty} e^{i\alpha u}d\mu_2(\alpha)=G(\mathrm{v})-H(u).$$

The signed measures μ_i are defined by

$$\begin{aligned} d\mu_i(\alpha)&=d\nu_i(\alpha), \quad &\alpha>0\\ &=-d\nu_i(\alpha), \quad &\alpha<0. \end{aligned}$$

By locality (Property 4c) $F(x)-F(-x)=0$ for $x\in D^c$ which becomes

$$G(\mathrm{v})+H(u)=0 \quad \text{if} \quad \begin{matrix} u>d \\ \mathrm{v}<-d \end{matrix} \quad \text{or} \quad \begin{matrix} \mathrm{v}>d \\ u<-d \end{matrix} \quad \text{for some} \quad d\in\mathbb{R}^+.$$

We conclude that

$$\begin{aligned} &H(u)=C_1 \quad \text{if} \quad u>d \quad \text{and} \quad H(u)=C_2 \quad \text{if} \quad u<-d\\ &G(\mathrm{v})=-C_1 \quad \text{if} \quad \mathrm{v}<-\mathrm{d} \quad \text{and} \quad G(\mathrm{v})=-C_2 \quad \text{if} \quad \mathrm{v}>d. \end{aligned}$$

By the antisymmetry of $F(x)-F(-x)$ we have $C_1=-C_2$. Then we can write $G(\mathrm{v})=C_1\varepsilon(\mathrm{v})+\hat{G}(\mathrm{v})$, where $\hat{G}$ has compact support and

$$\begin{aligned} \varepsilon(\mathrm{v})&=1 \quad &\text{for} \quad \mathrm{v}>0\\ &=-1 \quad &\text{for} \quad \mathrm{v}<0. \end{aligned}$$

The Fourier transform of G is μ_1 but the Fourier transform of $\hat{G}$ is analytic while the fourier transform of ε(v) equals $\lim_{\delta\to 0} 2i\frac{p}{p^2+\delta^2}$ which is too singular to be a measure.[2]

We conclude that $C_1=0$. Thus G has compact support, and being C^∞ by the smoothness of j, we conclude that the fourier transform of G is analytic, in $\mathscr{S}$, and odd. Similarly for $H(u)$. □

Proof of Theorem VI.5. From Lemma VII.6 we have

$$\begin{aligned}\|E'_0 j(\vartheta_r)\Omega\|^2 &= \int d\underline{x}d\underline{y}\vartheta_r(\underline{x})\vartheta_r(\underline{y})F(\underline{x}-\underline{y})\\ &= \int_0^\infty d\alpha|\tilde{\vartheta}_r(\alpha)|^2 P_1(\alpha) + \int_0^\infty d\alpha|\tilde{\vartheta}_r(-\alpha)|^2 P_2(\alpha)\\ &= \int_0^\infty d\alpha|\tilde{\vartheta}(\alpha)|^2 rP_1\left(\frac{\alpha}{r}\right) + \int_0^\infty d\alpha|\tilde{\vartheta}(\alpha)|^2 rP_2\left(\frac{\alpha}{r}\right)\end{aligned}$$

which converges as $r\to\infty$ to $P'_1(0)\int_0^\infty d\alpha|\tilde{\vartheta}(\alpha)|^2\alpha + P'_2(0)\int_0^\infty d\alpha|\tilde{\vartheta}(-\alpha)|^2\alpha$ (by dominated convergence) which is $<\infty$. □

Appendix: N^τ_{loc} Estimates

For any $\tau\leqq 1$, there are constants a,b (independant of g and V and depending on ζ only through diam supp ζ) *such that:*

$$N^\tau_{\mathrm{loc}} \leqq a(\hat{H}(g)+b), \tag{A1}$$

$$N^\tau_{\mathrm{loc}V} \leqq a(\hat{H}(g)_V+b), \tag{A2}$$

where $\hat{H}(g)=H(g)-\inf\operatorname{spectrum} H(g)$.

Proof. We only sketch the proof following [29]. Let $P(\xi_1,\xi_2)$ be defined by $P(\phi_1,\phi_2)=\lambda:((\phi)^2)^2:-\sigma:(\phi)^2:-\mu\phi_1$. Let h be a function of compact support satisfying $-1\leqq h\leqq 1$ and suppose: $0\leqq P(\xi_1,\xi_2)g(\underline{x})+P_1(\xi_1,\xi_2)h(\underline{x})$, $P_1(\xi_1,\xi_2)$ is a polynomial (we only consider $\deg P \neq \deg P_1$).

$$E_V(g,h)=\inf\operatorname{spectrum} H(g)_V + \int :P_1(\phi_{1V},\phi_{2V}):(\underline{x})h(\underline{x})d\underline{x},$$

$$H_{1V}(g)=H_{0V}+N^\tau_{\mathrm{loc}\,V}+\int :P_1(\phi_{1V},\phi_{2V}):(\underline{x})h(\underline{x})d\underline{x}.$$

$E_{1V}(g,h)$ and $E_{1V}(g)$ are defined as above.

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2 Defining $T_\delta=2i\frac{p}{p^2+\delta^2}$, take $h_n(p)$ a sequence of antisymmetric functions in $\mathscr{S}$ with

$$h_n=1 \quad\text{for}\quad \frac{1}{n}\leqq p\leqq 1, \qquad 0\leqq h_n\leqq 1 \quad\text{for}\quad p\geqq 0,$$

$$h_n=0 \quad\text{for}\quad p\geqq 2, \qquad 0\leqq p<\frac{1}{2n}.$$

Then $|T_\delta(h_n)|\geqq \ln\frac{1+\delta^2}{\frac{1}{n^2}+\delta^2}$. Since $T_\delta \xrightarrow[\delta\to 0]{} \tilde{\varepsilon}(p)$ it follows that $|\tilde{\varepsilon}(h_n)|\geqq 2\ln n \xrightarrow[n\to\infty]{} \infty$. But for a measure $|T(h_n)|\leqq 2|\mu|([0,2])<\infty$

By using methods of [16] and of [29] we have: (ϕ^j bounds)

$$|E_V(g,h)-E_V(g)| \leqq MD \tag{A3}$$

M is a constant, $D=(\operatorname{diam}\operatorname{supp} h)+1$

$$|E_{1V}(g,h)-E_{1V}(g)| \leqq MD. \tag{A4}$$

(A2) becomes:

$$|E_{1V}(g)-E_V(g)| \leqq C,$$

where C is independent of V and g and depends only on ζ through diam supp ζ.

$E_V(g)-E_{1V}(g)$ is estimed by application of the Duhamel formula:

$$e^{-(2T+1)H_{1V}} = e^{-TH_{1V}}e^{-H_V}e^{-TH_{1V}} - \int_0^1 e^{-TH_{1V}}e^{-sH_V}(N^\tau_{\text{loc}})e^{-(1-s)H_{1V}}e^{-TH_V}ds$$

the a^+_{Vj} and a_{Vj} of $N^\tau_{\text{loc}\,V}$ are then pulled through.

The terms produced are estimated by using cutoff-dependent bound (for instance Higher order estimates) and with the "ϕ^j bounds" (A3) and (A4).

The remaining estimates on kernels carries over for periodic boundary condition: they are based on Lemma 4.1 of [29] which becomes:

$$\|\mu^p_V \eta_i C_{f_V} \eta_j \mu^p_V\| \leqq C_{np}(|i-j|+1)^{-n},$$

$$\eta_i\eta_j = 0 \quad \text{and} \quad \operatorname{supp}\eta_i, \operatorname{supp}\eta_j \subseteqq \left[-\frac{V}{2}, \frac{V}{2}\right].$$

f is the same as in Lemma 4.1 of [29] and $f_V(x) = \sum_{n=\infty}^{+\infty} f(x+nV)$. □

References

1. Bricmont, J.: Correlation inequalities for two component fields. Ann. Soc. Sc. Brux. **90**, 245–252 (1976)
2. Bricmont, J., Fontaine, J.-R., Landau, L.J.: On the uniqueness of the equilibrium state for plane rotators. Commun. math. Phys. **56**, 281–296 (1977)
3. Coleman, S.: There are no Goldstone bosons in two dimensions. Commun. math. Phys. **31**, 259–264 (1973)
4. Dobrushin, R.L., Shlosman, S.B.: Absence of breakdown of continuous symmetry in two-dimensional models of statistical physics. Commun. math. Phys. **42**, 31–40 (1975)
5. Dunlop, F., Newman, C.: Multicomponent field theories and classical rotators. Commun. math. Phys. **44**, 223–235 (1975)
6. Dunlop, F.: Correlation inequalities for multicomponent rotators. Commun. math. Phys. **49**, 247–256 (1976)
7. Fröhlich, J.: Marseille conference: Poetic phenomena in (2-dim) quantum field theory: non uniqueness of the vacuum, the solitons and all that, pp. 112–130 (1975)
8. Fröhlich, J.: New super-selection sectors "soliton-state" in two dimensional bose quantum field models. Commun. math. Phys. **47**, 269–310 (1976)
9. Fröhlich, J.: Private communication
10. Fröhlich, J., Simon, B., Spencer, T.: Infrared bounds, phase transitions, and continuous symmetry breaking. Commun. math. Phys. **50**, 78–85 (1976)
11. Gal-Ezer, E.: Spontaneous breakdown in two dimensional space-time. Commun math. Phys. **44**, 191–195 (1975)

12. Ginibre, J.: General formulation of Griffith's inequalities. Commun. math. Phys. **16**, 310–328 (1970)
13. Glimm, J., Jaffe, A.: A $\lambda\phi^4$ quantum field theory without cutoff. I. Phys. Rev. **176**, 1945–1951 (1968)
14. Glimm, J., Jaffe, A.: The $\lambda\phi^4$ quantum field theory without cutoffs. II. The field operators and the approximate vacuum. Ann. Math. **91**, 362–401 (1970)
15. Glimm, J., Jaffe, A.: The $\lambda\phi^4$ quantum field theory without cutoffs. III. The physical vacuum. Acta Math. **125**, 203–261 (1970)
16. Glimm, J., Jaffe, A.: The $\lambda(\phi^4)_2$ quantum field theory without cutoffs. IV. Perturbations of the hamiltonian. J. Math. Phys. **13**, 1568–1584 (1972)
17. Glimm, J., Jaffe, A.: The energy momentum spectrum and vacuum expectation values in quantum field theories. II. Commun. math. Phys. **22**, 1–22 (1971)
18. Glimm, J., Jaffe, A.: Quantum field theory models in statistical mechanics and quantum field theory. Dewitt, C., Stora, R. (eds.). New York: Gordon and Breach 1971
19. Glimm, J., Spencer, T.: The Wightman axioms and the mass gap for the $P(\phi)_2$ quantum field theory (preprint)
19a. Glimm, J., Jaffe, A.: Ann. Inst. H. Poincaré Sect. A **22**, 109 (1975)
20. Guerra, F., Rosen, L., Simon, B.: The $P(\phi)_2$ euclidean quantum field theory as classical statistical mechanics. Ann. Math. **101**, 111–259 (1975)
21. Guerra, F., Rosen, L., Simon, B.: Boundary conditions for the $P(\phi)_2$ euclidean field theory. Ann. inst. H. Poincaré Sect. A**25**, 231–334 (1976)
22. Heifets, E.P., Osipov, E.P.: The energy momentum spectrum in the $P(\phi)_2$ quantum field theory. Commun. math. Phys. **56**, 161–172 (1977)
22a. Kunz, H., Pfister, Ch. Ed, Vuillermot, P. A.: Inequalities for some classical spin vector models. J. Ph. A Math. and Gen. **9**, 1673 (1976)
23. Reed, M., Simon, B.: Methods of modern mathematical physics. I. Functional analysis. New York, London: Academic Press 1972
24. Reed, M., Simon, B.: Methods of modern mathematical physics. II. Fourier analysis self adjointness. New York, London: Academic Press 1975
25. Reeh, H.: Symmetry operations and spontaneously broken symmetries in relativistic quantum field. Fortschr. Physik **16**, 687–706 (1968)
26. Rosen, L.: A $\lambda\phi^{2n}$ field theory: Higher order estimates. Com. Pure Appl. Math. **24**, 417–457 (1971)
27. Segal, I.: Notes torward the construction of non linear relativistic quantum fields. Proc. Nat. Acad. Sci. USA **37**, 1178 (1967)
28. Simon, B.: The $P(\phi)_2$ euclidean (quantum) field theory. Princeton: Princeton University Press 1974
29. Spencer, T.: Perturbations of the $P(\phi)_2$ quantum field Hamiltonian. J. Math. Phys. **14**, 823–828 (1973)
30. Spencer, T.: The mass gap for the $P(\phi)_2$ quantum field model with a strong external field. Commun. math. Phys. **39**, 63–76 (1974)
31. Vilenkin, N. Ja: Fonctions spéciales et théorie de la représentation des groupes. Paris: Dunod 1969

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