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Generators of Semigroups of Completely Positive Maps

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Abstract. Any generator of a norm continuous semigroup of completely positive normal maps on a von Neumann algebra M can be decomposed into a sum of a completely positive map and a map of the form $m \to x^*m + mx$.

The present note shows that the generator L of a uniformly continuous semigroup of completely positive normal maps on a von Neumann algebra M has the form

$$L(m) = \Psi(m) + x^*m + mx, \tag{$*$}$$

where Ψ is a completely positive map of M into the algebra $B(H)$ of all bounded operators on the space where M acts, and x is an operator in $B(H)$.

The problem relates to the question of whether irreversible evolutions of a quantum system come from the restriction of a reversible evolution of a larger system. Recall that ϕ is a completely positive map of a C^*-algebra $\mathscr{A}$ into $B(H)$ when ϕ is a positive linear mapping and applying ϕ to the elements of each matrix with entries in $\mathscr{A}$ yields a positive map (for matrices of all orders). By Stinespring's generalization of a result of Neumark (on positive operator-valued spectral measures), each such ϕ when identity preserving is the composition of a $*$-representation of $\mathscr{A}$ into $B(K)$ (with K a Hilbert space containing H) followed by restriction to H (i.e. $T \to PTP$ with P the projection of K onto H). If a group of $*$-automorphisms of $\mathscr{A}$ expresses a reversible dynamics, a semigroup of completely positive maps is a restriction and possibly a framework for irreversible dynamics.

The canonical decomposition $(*)$ of the generator of norm continuous semigroups of completely positive normal maps on a von Neumann algebra was first obtained independently by Gorini, Kossakowski, and Sudarshan for finite-dimensional matrix algebras [11], and by Lindblad for approximately finite-dimensional von Neumann algebras [6]. Later Lindblad [7] showed that a decomposition is possible if certain cohomology conditions are satisfied. Evans and Lewis [4] took up Lindblad's method and showed via the result of [3], that if the algebra M is properly infinite then L has a decomposition as in $(*)$.

The proof which follows below reduces the general case to the properly infinite case by showing that a uniformly continuous semigroup Φ_t of completely positive normal maps on a von Neumann algebra M becomes a uniformly continuous semigroup $\Phi_t \otimes \mathrm{id}$ on $M \otimes B(l^2(\mathbb{N}))$ when tensored with the identity on $B(l^2(\mathbb{N}))$.

Lemma. *Let Φ be a completely positive normal map on a von Neumann algebra M acting on a Hilbert space H and let Ψ be the map $\Phi \otimes \mathrm{id}$ on $M \otimes B(l^2(\mathbb{N}))$. If $\|\Phi - \mathrm{id}\| \leqq 10^4$ then $\|\Psi - \mathrm{id}\| \leqq 10^4(\|\Phi - \mathrm{id}\|)^{1/4}$.*

Proof. The proof has three parts; the two first deal with the cases where M is either finite or properly infinite. The last part yields a reduction of the general case to the cases already considered. But before starting of we mention that by elementary use of the triangle-inequality and the Russo-Dye theorem [8, Corollary 1] we get the lemma immediate when $10^{-12} \leqq \|\Phi - id\| \leqq 10^4$.

In fact $\|\Psi - \mathrm{id}\| \leqq \|\Psi\| + 1 = \|\Phi\| + 1 \leqq 2 + \|\Phi - \mathrm{id}\|$, and it is easy to check that $(2+s) \leqq 10^4 s^{1/4}$ when $10^{-12} \leqq s \leqq 10^4$.

Let us define $s = \|\Phi - \mathrm{id}\|$ and assume that $s \leqq 10^{-12}$.

In order to use the technique from [1] we have to normalize Φ in such a way that $\tilde{\Phi}(I) = I$, we do therefore define $b = \Phi(I)$ and

$$\tilde{\Phi}(m) = b^{-1/2}\Phi(m)b^{-1/2}. \tag{1}$$

The map $\tilde{\Phi}$ satisfies $\|\tilde{\Phi} - \mathrm{id}\| \leqq 4s$. This follows from $\|I - b^{1/2}\| \leqq \|I - b\| \leqq s$; $\|b^{-1/2}\| \leqq (1-s)^{-1}$; $\|1 - b^{-1/2}\| \leqq \|b^{-1/2}\| \, \|1 - b^{1/2}\|$;
$\|\tilde{\Phi} - \mathrm{id}\| \leqq \|b^{-1/2} - I\| \, \|b^{-1/2}\| \, \|\Phi\| + \|\Phi\| \, \|I - b^{-1/2}\| + \|\Phi - \mathrm{id}\|$;

$$\|\tilde{\Phi} - \mathrm{id}\| \leqq 4s. \tag{2}$$

Independent of the particular stages of the proof we keep the same notation and we assume acording to Stinespring's theorem ([10, 1, Theorem 3.1]), that we have found a Hilbert space K containing H and a normal representation Π of M on K, such that for any m in M and the orthogonal projection p on H

$$\tilde{\Phi}(m) = p\Pi(m)|H. \tag{3}$$

Let us assume M is finite, then the arguments from the proof of [2, Proposition 1.1] show, that there exists an operator r in the intersection of the commutant $\Pi(M)'$ of $\Pi(M)$ with the ultraweakly closed convex hull of the set $\{\Pi(u)p\Pi(u^*)|u$ unitary in $M\}$. If one looks into the proof of [1, Lemma 3.3] one finds, that this proof now can be transferred to our present situation, once we have estimated $\sup\|\tilde{\Phi}(u)\tilde{\Phi}(u^*) - I\|$; but the relation (2) shows that $\|\tilde{\Phi}(u)\tilde{\Phi}(u^*) - uu^*\| \leqq 8s$, so we find by the proof of [1, Lemma 3.3] that $\|r - p\| \leqq (8s)^{1/2}$. Moreover we find as in this proof already cited, that there exists a unitary v in the von Neumann algebra generated by $\Pi(M)$ and p such that v^*pv commutes with $\Pi(M)$ and

$$\|I - v\| \leqq 2^{1/2}(2(8s)^{1/2}) = 8s^{1/2}. \tag{4}$$

The map α on M given by

$$m \to \Pi(m) \to pv\Pi(m)v^*p$$

is then a star homomorphism on M for which $\|\alpha-\mathrm{id}\| \leqq \|\mathrm{id}-\tilde{\Phi}\| + \|\tilde{\Phi}-\alpha\| \leqq 4s + 16s^{1/2} \leqq 17s^{1/2}$. The Proposition 4.4 of [1] shows that there is a unitary u in M such that $\alpha(m)=umu^*$ and

$$\|I-u\| \leqq 2^{1/2}(17s^{1/2}). \tag{5}$$

The maps Φ and id on M are given by

$$\Phi: \quad m \to \Pi(m) \to b^{1/2}p\Pi(m)pb^{1/2}$$

$$\mathrm{id}: \quad m \to \Pi(m) \to pv\Pi(u^*)\Pi(m)\Pi(u)v^*p.$$

When tensoring Φ and id with id on $B(l^2(\mathbb{N}))$ we can tensor the various maps in the decomposition above, and we get

$$\Phi\otimes\mathrm{id}: \quad (m_{ij}) \to (\Pi(m_{ij})) \to (b^{1/2}p\otimes I)(\Pi(m_{ij}))(pb^{1/2}\otimes I)$$

$$\mathrm{id}\otimes\mathrm{id}: \quad (m_{ij}) \to (\Pi(m_{ij})) \to (pv\Pi(u^*)\otimes I)(\Pi(m_{ij}))(\Pi(u)v^*p\otimes I).$$

Therefore $\|\Phi\otimes\mathrm{id}-\mathrm{id}\otimes\mathrm{id}\| \leqq \|b^{1/2}p - pv\Pi(u^*)\|(1+\|b^{1/2}p\|)$,

$$\|\Phi\otimes\mathrm{id}-\mathrm{id}\otimes\mathrm{id}\| \leqq (2+s)(s+\|I-u\|+\|I-v\|) \leqq 70s^{1/2}. \tag{6}$$

In the properly infinite case we will find a type I sub factor F of M isomorphic to $B(l^2(\mathbb{N}))$ and then twist Φ a bit such that the twisted map is a multiple of the identity on F.

As above we have $\tilde{\Phi}$ on M with $\tilde{\Phi}(I)=I$ and objects (Π, K, p) such that (3) holds.

Since F has property P of Schwarts ([1, 9]) and Π is normal it is possible to prove as above, that there exists a unitary v in the von Neumann algebra generated by $\Pi(M)$ and p, such that $\|I-v\| \leqq 8s^{1/2}$ and v^*pv commutes with $\Pi(F)$.

It is also possible to repeat all the arguments between the relations (4) and (5) above, when we just refer to [1, Proposition 4.2] instead of [1, Proposition 4.4].

We can then find a unitary u in M such that $\|I-u\| \leqq 2^{1/2}(17s^{1/2})$ and the map Φ_o defined below is the identity when restricted to F.

$$\Phi_o: \quad m \to \Pi(m) \to pv\Pi(u^*)\Pi(m)\Pi(u)v^*p.$$

Again by repetition we conclude, that when tensoring with the identity on $B(l^2(\mathbb{N}))$ we get

$$\|\Phi\otimes\mathrm{id}-\Phi_o\otimes\mathrm{id}\| \leqq 70s^{1/2}. \tag{7}$$

On the other hand we claim that $\|\Phi_o\otimes\mathrm{id}-\mathrm{id}\otimes\mathrm{id}\| = \|\Phi_o-\mathrm{id}\| \leqq \|\Phi-\Phi_o\| + \|\Phi-\mathrm{id}\| \leqq 71s^{1/2}$.

The non-trivial part here is the first equality, but this is immediate once one remarks, that when Φ_o is the identity on F, Φ_o has the form $\Psi\otimes\mathrm{id}$ on $F^c\otimes F$ where F^c is the relative commutant of F. It follows automatically from the Stinespring decomposition of Φ_o, that when Φ_o is the identity on F we get that Φ_o is a F module map in the sence, that for m in M and f in F $\Phi_o(fm)=f\Phi_o(m)$ and $\Phi_o(mf)=\Phi_o(m)f$. Now it is easy to verify, that Φ_o maps F^c into F^c and then has the form $\Psi\otimes\mathrm{id}$. All together we get in the properly infinite case

$$\|\Phi\otimes\mathrm{id}-\mathrm{id}\otimes\mathrm{id}\|\leqq 141 s^{1/2}.$$

The general case is based upon similar arguments. Since the center has property P, we can copy the arguments given in the infinite case and we find, that there exists a completely positive map Φ_o on M which is identity map on the center of M and

$$\|\Phi\otimes\mathrm{id}-\Phi_o\otimes\mathrm{id}\|\leqq 70 s^{1/2} \tag{8}$$

$$\|\Phi_o-\mathrm{id}\|\leqq 71 s^{1/2}. \tag{9}$$

The two previous results applied to Φ_o then yields together with (8)

$$\|\Phi\otimes\mathrm{id}-\mathrm{id}\otimes\mathrm{id}\|\leqq 70 s^{1/2}+141(71 s^{1/2})^{1/2}\leqq 10^4 s^{1/4}$$

and the lemma follows.

Corollary. *Let Φ be a completely positive map on a C^*-algebra $\mathfrak{A}$ and let $\mathfrak{B}$ be a C^*-algebra. If $\|\Phi-\mathrm{id}\|\leqq 10^4$ then $\Phi\otimes\mathrm{id}$ on the minimal C^*-tensorproduct of $\mathfrak{A}$ with $\mathfrak{B}$ satisfies*

$$\|\Phi\otimes\mathrm{id}-\mathrm{id}\|\leqq 10^4(\|\Phi-\mathrm{id}\|)^{1/4}.$$

Proof. Transposition of Φ to be second dual yields the result.

Theorem. *Let $(T_t)_{t\geqq 0}$ be a uniformly continuous semigroup of completely positive normal maps on a von Neumann algebra M acting on a Hilbert space H, then there exist an x in $B(H)$ and a completely positive normal map Ψ of M into $B(H)$, such that the generator L of $(T_t)_{t\geqq 0}$ has the form*

$$L(m)=\Psi(m)+x^*m+mx. \tag{$*$}$$

Proof. We remind the reader, that the introduction tells, that the theorem is true when M is properly infinite.

The lemma proved above shows, that the semigroup $(S_t)_{t\geqq 0}=(T_t\otimes\mathrm{id})_{t\geqq 0}$ on $M\otimes B(l^2(\mathbb{N}))$ is also uniformly continuous. This can in fact be proved as follows: For all $s\in\mathbb{R}_+$ for which $\|T_s-\mathrm{id}\|\leqq 10^4$ and any t in $\mathbb{R}_+$ we get $\|S_{t+s}-S_t\|\leqq\|S_t\|\,\|S_s-\mathrm{id}\|\leqq 10^4\|T_t\|(\|T_s-\mathrm{id}\|)^{1/4}$. We can now deduce that the generator $\tilde{L}$ of $(S_t)_{t\geqq 0}$ is an ultraweakly continuous bounded map of the form

$$\tilde{L}(a)=\tilde{\Psi}(a)+y^*a+ay.$$

In order to show that L has the stated form we consider the restriction of $\tilde{L}$ to an algebra $M\otimes e$ where e is a minimal projection in $B(l^2(\mathbb{N}))$.

It is rather obvious that $\tilde{L}(m\otimes e)=L(m)\otimes e$, since $S_t(m\otimes e)=T_t(m)\otimes e$.

This relation in particular shows that $0=(I\otimes(I-e))[\tilde{\Psi}(m\otimes e)+y^*(m\otimes e)+(m\otimes e)y](I\otimes(I-e))$ and

$$0=(I\otimes(I-e))\tilde{\Psi}(m\otimes e)(I\otimes(I-e)).$$

Suppose m is positive then $\tilde{\Psi}(m\otimes e)$ is positive, hence the equality yields that

$$0=(I\otimes(I-e))\tilde{\Psi}(m\otimes e)=\tilde{\Psi}(m\otimes e)(I\otimes(I-e)).$$

It is now possible to define a completely positive normal map Ψ of M into $B(H)$ by $\Psi(m)\otimes e = \tilde{\Psi}(m\otimes e)$.

Since both $\tilde{L}$ and $\tilde{\Psi}$ maps $M\otimes e$ into $B(H)\otimes e$, the map $m\otimes e \to y^*(m\otimes e) + (m\otimes e)y$ maps $M\otimes e$ into $B(H)\otimes e$, and we get, when we define x in $B(H)$ by $x\otimes e = (I\otimes e)y(I\otimes e)$ that

$$L(m)\otimes e = \tilde{L}(m\otimes e) = (\Psi(m) + x^*m + mx)\otimes e$$

or

$$L(m) = \Psi(m) + x^*m + mx\,.$$

The theorem follows.

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