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Signal Propagation in Lattice Models of Quantum Many-Body Systems

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Abstract. Upper and lower bounds are proven for the speed of propagation of general classes of physical signals in particle lattice models.

1. Introduction

Several years ago Lieb and Robinson proved a remarkable result [1] consisting of an upper bound on certain commutators of operators in spin lattice models with short range interactions. Specifically, they proved for any local operators A and B, and with space (respectively time) translations denoted σ_x (respectively τ_t) that $\|[\tau_t(A), \sigma_x(B)]\|$ is $O(\exp[-K|t|])$ as $|t| \to \infty$ for some $K > 0$ provided $|x| \geqq V|t|$, where V is some finite velocity, dependent on the interaction. (A cleaner proof, with simple formulas for V and K is contained in the recent paper [2].)

It was noted in [1] that this result implies an upper bound on the speed of propagation of "information" in such models, but no details were given on this interpretation.

As an indication that there are latent difficulties in this interpretation, consider the following general model of the propagation of "information" or "signals" in any quasi-local dynamical system. Let ϱ be a state, A and B local operators with "support" near the origin and such that $\varrho(A^*A) = 1$. We interpret

$$S(x, t) = \varrho(A^* \tau_t \sigma_x(B) A) - \varrho(\tau_t \sigma_x(B)) \tag{1}$$

as the signal, measured near the position x at time $t > 0$ by means of the observable B, due to the disturbance localized near $x = 0$ and represented by the change (effected approximately at time $t = 0$) of the state $\varrho(\cdot)$ to the state $\varrho_A(\cdot) = \varrho(A^* \cdot A)$. [The condition $\varrho(A^*A) = 1$ is just that ϱ_A be a state.] S measures the difference between the expected background, $\varrho(\tau_t \sigma_x(B))$, and the measured quantity, $\varrho(A^* \tau_t \sigma_x(B) A)$. To show that signals travel only with speed less than V it would be necessary to show that $S(x, t)$ is negligibly small for all $|x| > Vt$. (In this paper we do not analyze carefully the notion of "negligible"; for simplicity we just assume a

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signal is negligible if and only if it is exponentially damped, i.e. if and only if $S(x,t)$ is $O(\exp[-Kt])$ as $t\to\infty$ for some $K>0$.) Next consider the following specific example in a spin-$\frac{1}{2}$ lattice model. Let s_x^j be the j^{th} spin component at lattice site x and let τ_t be any Ising-type dynamics [3], so that $\tau_t(s_x^3)=s_x^3$ for all x and t. Let ϱ be the mixed ground state $(\varrho_+ + \varrho_-)/2$ where $\varrho_\pm(s_x^3)=\pm 1$ for all x, and let $A=(I+s_0^3)/2^{1/2}$, $B=s_0^3$. Then $\varrho(A^*A)=1=S(x,t)$ for all x, and all $t>0$. This would seem to correspond to infinite transmission speed.

In this paper we will elaborate on the theme of signal transmission. We will only consider signals $S(x,t)$ of the general type (1) and an effort will be made to avoid the contradictions of the above specific example.

We begin with a simple extension of the Lieb-Robinson commutator results to "particle lattice models" [4], i.e. n-body Schrödinger mechanics but with the physical space $\boldsymbol{R}^3$ replaced by the discrete space $\boldsymbol{Z}^3$. (The added degree of freedom, particle momentum, of these models over the spin lattice models will be seen to be relevant to the topic under study.) Then we consider classes of signals which are generated in a physically reasonable manner. An *upper bound* on the *maximal possible* speed of transmission of such signals is derived from the commutator results. Finally, a *lower bound* on the maximal possible speed is calculated, and compared with the upper bound.

2. Notation

Let $\boldsymbol{Z}^\nu$ be the ν-dimensional, infinite lattice $\{x=(x_1,\ldots,x_\nu)|x_j=0,\pm 1,\pm 2,\ldots\}$ (playing the role of physical space) with metric $|x|^2=\sum_{j=1}^{\nu} x_j^2$, and "basis vectors" $e_j=(\delta_1(j),\ldots,\delta_\nu(j))$, $j=1,2,\ldots,\nu$ where $\delta_k(\cdot)$ is a Kronecker delta function. Define $\boldsymbol{Z}_p=\{1,2,\ldots,p\}$ (as an index for spin components) and for each $j=1,2,\ldots$ let X_j be a copy of the Cartesian product $\boldsymbol{Z}^\nu\times\boldsymbol{Z}_p\equiv X$. Let $\bar{\pi}:y\in X\to\bar{y}\in\boldsymbol{Z}^\nu$ and $\hat{\pi}:y\in X\to\hat{y}\in\boldsymbol{Z}_p$ be the canonical projections, and define X^n as the Cartesian product $\mathop{\times}\limits_{j=1}^{n} X_j$, for $n=1,2,\ldots$. Let $\tilde{H}^{(n)}$ be the complex Hilbert space $\ell_2(X^n)$, and $H^{(n)}$ the subspace of functions $f(y_1,\ldots,y_n)$ antisymmetric under permutations of the y_j. Then define $H^{(0)}$ as $\boldsymbol{C}$, and the Hilbert space direct sum $H=\bigoplus_{n=0}^{\infty} H^{(n)}$. Now for each y in X we define the field operator $a(y)$ on H by

$$(a(y)f)_n(y_1,\ldots,y_n)=(n+1)^{1/2}f_{n+1}(y,y_1,\ldots,y_n),$$

where for g in H, g_n denotes the component of g in $H^{(n)}$. We note that $a(y)$ has norm one, and with its adjoint, $a^*(y)$, satisfies the following relations for all y, y' in X:

$$[a(y),a(y')]_+=0=[a^*(y),a^*(y')]_+$$
$$[a^*(y),a(y')]_+=\delta_y(y')I,$$

where $[A,B]_\pm=AB\pm BA$, and I is the identity operator on H. We define F as the family of all finite subsets of X, partially ordered by inclusion, and for each W in F, $P(W)$ denotes the operator algebra of polynomials in I, $a(y)$ and $a^*(y')$, y and y' in

W. [We make the special definition $P_0 = P(\{0\} \times \boldsymbol{Z}_p)$.] Then we define the operator algebra $P = \bigcup_{W \in F} P(W)$ and its norm closure, $\mathfrak{A}$. Space translations are effected by the *-automorphisms σ_x on $\mathfrak{A}$, defined by $\sigma_x[a(\bar{y}, \hat{y})] = a(\bar{y} + x, \hat{y})$.

Next let Φ be any real function on $X \times X$ of the form $\Phi(y, y') = \varphi(\bar{y} - \bar{y}', \hat{y}, \hat{y}')$, such that

$$\sum_{z \in \boldsymbol{Z}^\nu} |\varphi(z, \hat{y}, \hat{y}')| < \infty \quad \text{for all} \quad \hat{y}, \hat{y}' .$$

Then for m, $h > 0$ define the function E on F by:

$$\begin{aligned} E(\{y\}) &= (\nu/2mh) a^*(y) a(y) , \\ E(\{y, y'\}) &= -(1/4mh) \sum_{j=1}^{\nu} [\delta_{\bar{y}}(\bar{y}' + 2e_j) + \delta_{\bar{y}'}(y + 2e_j)] \\ &\quad \cdot [a^*(y)a(y') + a^*(y')a(y)] \\ &\quad + \tfrac{1}{2}[\Phi(y, y') + \Phi(y', y)] \\ &\quad \cdot [a^*(y)a^*(y')a(y')a(y)] , \qquad y \neq y' , \end{aligned}$$

$E(W) = 0$, when the cardinality, $|W|$, of W is 0 or $\geqq 3$.

The function E gives rise to the family of Hamiltonians, $H_V = \sum_{W \subseteq V} E(W)$ indexed by V in F, and of a limiting dynamics on $\mathfrak{A}$, defined for each A in $\mathfrak{A}$ and t in $\boldsymbol{R}$ [4] by the norm limits

$$\tau_t(A) = \lim_{V \to X} \exp(iH_V t) A \exp(-iH_V t) = \lim_{V \to X} \tau_t^V(A) . \tag{2}$$

We interpret m as the mass of the fermions and h as the unit of lattice spacing. Then on every n-particle space $H^{(n)}$, τ_t is the usual Schrödinger-Pauli dynamics for a spin dependent 2-body interaction Φ but with $\boldsymbol{Z}^\nu$ replacing $\boldsymbol{R}^3$ and thus a difference operator replacing the Laplacian in the kinetic energy, and with $h = 1$ [4].

In the remaining sections we will only consider those (short range) interactions Φ for which; 1) there exists $\gamma > 0$ such that

$$\sum_{\bar{\pi}(W) \ni 0} |\bar{\pi}(W)| 4^{|W|} \|\Phi(W)\| e^{\gamma D(W)} \equiv \|\Phi\|_\gamma < \infty ,$$

where $D(W) = \sup_{y, y' \in W} |\bar{\pi}(y) - \bar{\pi}(y')|$, and 2) $\Phi(X)$ is *even* for all x, i.e. $\alpha[\Phi(W)] = \Phi(W)$ for the *-automorphism α of $\mathfrak{A}$ defined by $\alpha(a(y)) = -a(y)$. For a general reference on algebraic technique see [5].

3. Commutator Results

Proposition I. *If A is in P_0 and B in $\mathfrak{A}$,*

$$\|[\tau_t(A), B]_-\| \leqq \|A\| \sum_{z \in \boldsymbol{Z}^\nu} \sup_{C \in P_0} \frac{\|[\sigma_z(C), B]_-\|}{\|C\|} \exp(-\gamma|z| + 2|t| \|\Phi\|_\gamma) .$$

Proof. The proof of (Proposition I; [2]) is very easily adapted to this situation, except for the expansion in matrix units, which can be accomplished as follows [6]. Let $\boldsymbol{N}\ni n\to y_n\in X$ be any enumeration of X, and set

$$b^n_{11}=a(y_n)a^*(y_n), \quad b^n_{12}=a(y_n),$$

$$b^n_{21}=a^*(y_n), \quad b^n_{22}=a^*(y_n)a(y_n),$$

$$V_m=\begin{cases}\prod\limits_{n=1}^{m}(b^n_{11}-b^n_{22}), & m\geqq 1\\ I, & m=0\end{cases}$$

$$e^n_{jk}=\begin{cases}b^n_{jk}, & j=k\\ V_{n-1}b^n_{jk}, & j\neq k\end{cases}, \quad n\geqq 1$$

$$E_{\underline{jk}}=\prod_{m=1}^{n_y}e^m_{j_mk_m}, \quad \underline{j},\underline{k}\in\{1,2\}^n.$$

The $E_{\underline{jk}}$ form a system of matrix units for $P(W_n)$ where

$$W_n=\{y_j|j=1,2,\ldots,n\}.$$

Furthermore each $E_{\underline{jk}}$ is of the form $\pm\prod\limits_{m=1}^{n}{}^0\, b^m_{j'_mk'_m}$, where $\prod^\circ$ is the ordered product:

$$\prod_{m=1}^{n}{}^0 c_m\equiv c_1c_2c_3\ldots c_n.$$

It now follows from the properties of matrix units that

Lemma. *Every A in $P(W_n)$ has a unique expansion*

$$A=\sum_{\underline{jk}}c_{\underline{jk}}\prod_{m=1}^{n}{}^0 b^m_{j_mk_m}$$

and the $c_{\underline{jk}}$ satisfy $|c_{\underline{jk}}|\leqq\|A\|$.

Since $\|\Phi\|_\gamma/\gamma\to\infty$ as $\gamma\to 0$ or ∞, there exists $\gamma_M>0$ such that $2\|\Phi\|_\gamma/\gamma$ attains its minimum, denoted V_Φ, when $\gamma=\gamma_M$.

Corollary. *If A and B are in P, A or B is even, and $|x|=|x|(t)\geqq V|t|$ for some fixed $V>V_\Phi$, then*

$$[\tau_t(A),\sigma_x(B)]_-\|=\mathrm{O}(\exp[-\gamma_M(V-V_\Phi)|t|]) \quad \text{as} \quad |t|\to\infty.$$

Proof. The proof follows easily using the above lemma, as in [2].

4. Signals; Upper Bounds

As noted in the introduction, we will only consider signals of the form

$$S(x,t)=\varphi(A^*\tau_t\sigma_x(B)A)-\varrho(\tau_t\sigma_x(B))$$

for A and B in P and $\varrho(A^*A)=1$.

We first note the following general result.

Proposition II. *For any signal with both A and B in P, $A^*A=I$, and either A or B even, V_Φ is an upper bound on the speed of propagation.*

Proof. Just note that if $A^*A=I$,

$$S(x,t)=\varrho([A^*,\tau_t\sigma_x(B)]_-A).$$

Since $|\varrho(CA)|\leqq\|C\|\,\|A\|$, the corollary implies the result.

As was demonstrated in the introduction, abstract models of signals can be misleading; therefore we consider the following "physical" examples. Let

$$V=V^*\in\mathfrak{A},$$

and let the dynamics be the perturbation of some τ_t of type (2) described by

$$\tau_t^p(C)=\text{norm}\lim_{f\to\infty}\exp[i(H_f+V(t))]C\exp[-i(H_f+V(t))],$$

where $V(t)=V$ for $0\leqq t\leqq 1$ and 0 otherwise. Then by standard arguments $\tau_t^p(C)=A^*\tau_t(C)A$ for $t>1$, where A is the unitary operator

$$A=I+\sum_{n=1}^{\infty}(-i)^n\int_0^1 ds_1\ldots\int_0^{s_{n-1}} ds_n\tau_{s_1}(V)\ldots\tau_{s_n}(V). \tag{3}$$

Since A is unitary, $A^*A=I$. And although A is not in P, we have

$$\begin{aligned}\|[A^*,\tau_t\sigma_x(B)]_-\|\leqq&\sum_{n=1}^{\infty}\int_0^1 ds_1\ldots\int_0^{s_{n-1}} ds_n\sum_{j=1}^{n}\|\tau_{s_n}(V)\ldots\tau_{s_{j+1}}(V)\\&\cdot[\tau_{s_j}(V),\tau_t\sigma_x(B)]_-\tau_{s_{j-1}}(V)\ldots\tau_{s_1}(V)\|\\\leqq&\exp\|V\|\sup_{0\leqq s\leqq 1}\|[V,\tau_{t-s}\sigma_x(B)]_-\|.\end{aligned}$$

Therefore we have

Proposition III. *For any signal with A of the form* (3) *for $V=V^*$, both V and B in P and either V or B even, V_Φ is an upper bound on the speed of propagation.*

Another physical method to create signals is to inject at time $t=0$ a particle of wavefunction f into the system, i.e. take $A=a^*(f)\equiv\sum_{y\in X}a^*(y)f(y)$ for f with $\sum_{y\in X}|f(y)|^2=1$. Note that in order that $\varrho(A^*A)=1$, it is here necessary to use only such ϱ that $\varrho(AA^*)=\varrho(a^*(f)a(f))=0$, in which case the disturbance can be thought of as unitary since then $\varrho_A=\varrho_{A+A^*}$, and A^*+A is unitary. We thus have

Proposition IV. *For a signal with $A=a^*(f)$, where $\|f\|_2=1$, $\varrho(a^*(f)a(f))=0$, B is even, and both A and B are in P, V_Φ is an upper bound on the speed of propagation.*

5. Signals; Lower Bound

In this section we compute the actual speed of propagation of a certain signal (of the type of Proposition IV) for any interaction Φ, thus giving a lower bound on the maximal *attainable* propagation speed.

Let $B=a^*(0,1)a(0,1)$, $A=a^*(0,1)$, ϱ be the vacuum and Φ any interaction. Then

$$\begin{aligned}S(x,t)&=\varrho(a(0,1)\tau_t\sigma_x[a^*(0,1)a(0,1)]a^*(0,1))\\&\quad-\varrho(\tau_t\sigma_x[a^*(0,1)a(0,1)])\\&=\varrho(a(0,1)\tau_t[a^*(x,1)a(x,1)]a^*(0,1))\\&=|(\exp[i\Delta t/2m]\delta_{(0,1)})(x,1)|^2,\end{aligned}$$

where $\delta_{(0,1)}(y)=\delta_0(\bar{y})\delta_1(\hat{y})$ and Δ is the operator on $\ell_2(X)$ defined as $\sum_{j=1}^{\nu}V_j^2$, where

$$(1/i)(V_jf)(\bar{y},\hat{y})=[f(\bar{y}+e_j,\hat{y})-f(\bar{y}-e_j,\hat{y})]/2ih.$$

(In other words we inject a particle at the origin in a given spin state, and look for it at position x at time t.) Since

$$(1/i)V_j\exp(ir\cdot\bar{y})f(\hat{y})=[(\sin r_j)/h]\exp(ir\cdot\bar{y})f(\hat{y})$$

and

$$\delta_{(0,1)}(\bar{y},\hat{y})=[1/(2\pi)^\nu]\int_{-\pi}^{\pi}dr\exp(ir\cdot\bar{y})\delta_1(\hat{y}),$$

we have

$$\begin{aligned}&[\exp(i\Delta t/2m)\delta_{(0,1)}](\bar{y},\hat{y})\\&\quad=[1/(2\pi)^\nu]\int_{-\pi}^{\pi}dr\exp\left[-i(t/2m)\sum_{j=1}^{\nu}(\sin^2 r_j)/h^2+ir\cdot\bar{y}\right]\delta_1(\hat{y})\\&\quad=\delta_1(\hat{y})\prod_{j=1}^{\nu}\exp(it/4h^2m)I_j/2\pi,\end{aligned}$$

where

$$I_j=\int_{-\pi}^{\pi}ds\exp ih(s),\qquad h(s)=(t/4h^2m)\cos 2s+\bar{y}_js.$$

We distinguish two cases.

Case I. Assume

$$\bar{y}_jh\geqq ct\geqq 0\quad\text{for some fixed}\quad c>1/2mh \tag{A$_\mathrm{I}$}$$

and let $s=s_R+is_I$, s_R and s_I real. Extending h to an entire function of s, we have for $s_I\neq 0$,

$$\begin{aligned}(1/s_I)\operatorname{Im}h(s)&=\bar{y}_j-(t/2h^2m)\sin(2s_R)[\sinh(2s_I)]/2s_I\\&\geqq(t/h)[c-(1/2mh)\sin(2s_R)[\sinh(2s_I)]/2s_I].\end{aligned}$$

Therefore if $0\neq s_I\simeq 0$, $(1/s_I)\operatorname{Im}h(s)>0$. We may then choose $0\neq\delta\gtrsim 0$ such that for some $K>0$, $\operatorname{Im}h(s_R+i\delta)\geqq Kt$ for all s_R. By Cauchy's theorem, we may deform the path of integration of I_j to the union of Γ_1, Γ_2, and Γ_3 where

Γ_1 is parametrized as $s=-\pi+is_I$, $0\leqq s_I\leqq\delta$,
Γ_2 is parametrized as $s=s_R+i\delta$, $0\leqq s_R\leqq\pi$,
Γ_3 is parametrized as $s=\pi-is_I$, $-\delta\leqq s_I\leqq 0$.

Now $\left|\int_{\Gamma_2} ds \exp ih(s)\right| \leqq 2\pi \exp - Kt$, and

$$\begin{aligned}\int_{\Gamma_1 \cup \Gamma_3} ds \exp ih(s) &= \int_0^\delta idw\ \{\exp i[(-\pi + iw)\bar{y}_j + (t/4h^2m)\cos 2iw] \\ &\quad - \exp i[(\pi + iw)\bar{y}_j + (t/4h^2m)\cos 2iw]\} \\ &= 2\sin(\bar{y}_j\pi)\int_0^\delta idw\ \exp[-\bar{y}_j w + (it/4h^2m)\cos 2iw] \\ &= 0\end{aligned}$$

since $y_j \in \boldsymbol{Z}$. In summary, if (A_I) holds, then $|I_j| \leqq 2\pi \exp - Kt$ for some $K > 0$, and so

$$S(x,t) = |[\exp(i\Delta t/2m)\delta_{(0,1)}](x,1)|^2 = \mathrm{O}(\exp - Kt) \quad \text{as} \quad t \to \infty .$$

Case II. Now assume

$$0 < t = h\bar{y}_j/c \quad \text{for some fixed} \quad c < 1/2mh. \tag{A$_{\mathrm{II}}$}$$

Then there are four values of s in $[-\pi, \pi]$ such that $h'(s) = 0$, i.e. such that $\sin 2s = c/2mh$; we label them

$$-\pi < s_1 < -3\pi/4 < s_2 < -\pi/2 < 0 < s_3 < \pi/4 < s_4 < \pi/2 .$$

Extending $h(s)$ to an entire function of $s = s_R + is_I$, and using Cauchy's theorem, we deform the path of integration of I_j into the union of eleven line segments as follows. First we define

$$s_0^- = -\pi , \qquad s_0^+ = -\pi + i\delta ,$$

$$s_5^- = \pi + i\delta , \qquad s_5^+ = \pi$$

$$s_k^- = s_k + \delta \exp i\theta_k , \qquad s_k^+ = s_k - \delta \exp i\theta_k , \qquad k = 1,2,3,4 ,$$

where $\theta_1 = \theta_3 = 3\pi/4$ and $\theta_2 = \theta_4 = 5\pi/4$. Here δ is such that when we parametrize the curves $\Gamma_k, \tilde{\Gamma}_k$ by

$$\Gamma_k : s = s_k^- + a(s_k^+ - s_k^-), \quad 0 \leqq a \leqq 1, \quad k = 0,1,2,3,4,5 ,$$

$$\tilde{\Gamma}_k : s = s_k^+ + a(s_{k+1}^- - s_k^+), \quad 0 \leqq a \leqq 1, \quad k = 0,1,2,3,4$$

then $\operatorname{Im} h(s) \geqq K\bar{y}_j$ on all $\tilde{\Gamma}_k$, for some $K > 0$.

By the same calculation as in Case I, we see that the integrals over Γ_0 and Γ_5 cancel, and that the integrals over all $\tilde{\Gamma}_k$ are $\mathrm{O}(\exp - K\bar{y}_j)$. For s near s_i, $h(s) \simeq h(s_i) + h''(s_i)(s - s_i)^2/2$. We define $\tilde{c} = m\cos(2s_1)/4h$, and note that $\cos(2s_1) = \cos(2s_3) = -\cos(2s_2) = -\cos(2s_4)$. In this notation, $h(s_i) = \bar{y}_j(s_i + \tilde{c}/c)$, and $h''(s_i) = \pm 4\tilde{c}\bar{y}_j/c$. Elementary saddle point theory shows that

$$\begin{aligned}\int_{\Gamma_k} ds \exp ih(s) &= -2\exp(i\theta_k)\exp[i\bar{y}_j(s_k + \tilde{c}/c)] \\ &\quad \cdot [\pi c/8\tilde{c}\bar{y}_j]^{1/2} + \mathrm{O}(\bar{y}_j^{-3/2}) .\end{aligned}$$

Since $s_3 = s_1 + \pi$, $s_4 = s_2 + \pi$, if $\bar{y}_j$ is an odd (respectively even) integer, the integrals over Γ_1 and Γ_3 cancel (respectively are equal) and over Γ_2 and Γ_4 cancel (respectively are equal). We assume then that $\bar{y}_j$ is an even integer, and find that

$$I_j = -8\sin[\bar{y}_j(s_2 - s_1)]\exp(i\tilde{c}\bar{y}_j/c)[\pi c/8\tilde{c}\bar{y}_j]^{1/2} + \mathrm{O}(\bar{y}_j^{-3/2}) .$$

(Note that since $0<s_2-s_1<\pi/2$, $\sin[\bar{y}_j(s_2-s_1)]\not\to 0$ as $\bar{y}_j\to\infty$.) Therefore if (A_{II}) holds, $\bar{y}_j^{1/2}I_j\not\to 0$ as $\bar{y}_j\to\infty$ and so if (A_{II}) holds for all j, and all x_j are even integers, $t^\nu S(x,t)\not\to 0$ as $t\to\infty$.

In summary, the given signal travels at the speed $1/2mh$.

6. Summary and Conclusion

In summary, we find that independently of the interaction, a signal can be transmitted with speed $1/2mh$. On the other hand we note that the upper bound V_Φ computed in §4 is always at least as large as that of a free system, which is of the order $100/mh$. It would be of interest to know what a maximal attainable signal speed is, even for a free system. As the operator representing particle velocity along an axis has norm $1/mh$ this would seem a natural limit. Surprisingly, this cannot be attained by a signal with $B=a^*(0)a(0)$ and a local A of the form $a^*(f)$, since such signals are merely finite sums of (finite translates of) the type computed in §4. The "reason" for this is that the group velocity, for a packet of waves of the form $\int dq g(q)\exp i(q\cdot x-E(q)t)$ for g supported mainly near $q=q_0$, is given by $\left.\frac{dE}{dq}\right|_{q=q_0}$. In the situation of §4, $\frac{dE}{dq}=(\sin 2q)/2mh$, which has a maximum of $1/2mh$ as computed.

Another question raised by the above analysis is the special role played, in disturbances of the form $\varrho\to\varrho_A$, by the condition $A^*A=I$. In particular, can a "physical" signal be generated with $A^*A\neq I$, in such a way as to violate the above upper bound on signal speed? (The example in the introduction violates the upper bound but does not seem physical, while Proposition IV concerns physical signals with $A^*A\neq I$ but respecting the upper bound.) Disturbances of the type $\varrho\to\varrho_A$ in quasi-local models have been routinely considered to be "local" (see e.g. [7]), so it would certainly be of interest to further clarify their physical significance.

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