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A Uniform Lower Bound on the Renormalized Scalar Euclidean Functional Determinant

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Abstract. Within a precise differential geometric setting we prove that the renormalized, scalar Euclidean functional determinant in an external Yang-Mills potential is bounded below by 1.

This reflects the stability of the vacuum under perturbations by external potentials. The proof is based on Kato's inequality and Seeley's analytic extension of the trace formula.

The old problem of controlling the vacuum polarization in an external field by using the functional determinant [14] has received renewed interest (see e.g. [1, 3, 5]). Within a precise differential geometric context we will prove that the renormalized scalar determinant on $\mathbb{R}^m$, given formally by

$$\Delta_{\text{ren}}(e) = \frac{\Delta(e)}{\Delta(0)} \tag{1}$$

with $\Delta(e) = \det(p - eA)^2$, satisfies the estimate

$$\Delta_{\text{ren}}(e) \geqq 1 \tag{2}$$

for all real e. A is an external Yang-Mills potential. Formally $\Delta(e)$ is the product of the eigenvalues of $(p - eA)^2$ and therefore appears in the Euclidean (functional integration) approach through the Gaussian integral over a bose field ϕ as

$$\Delta(e)^{-1} = \int \exp -1/2\phi^*(p - eA)^2 \phi d\phi^* . \tag{3}$$

Relation (2) therefore states the stability of the vacuum under a perturbation by an external Yang-Mills potential [14]. In case of a spinor field in an external electromagnetic potential the corresponding stability condition

$$\frac{\det(\not{p})^2}{\det(\not{p} - e\not{A})^2} \geqq 1 \tag{4}$$

has been proved in the relativistic context by Schwinger (see Relation 112 in [14]). Presently the methods presented here do not extend to this case (for a discussion of the Spinor Laplacian in the context of Kato's inequality, see [7]).

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Our proof will be based on Kato's inequality [8] for Laplacians in external Yang-Mills potentials [6, 7] (for the case of electromagnetic potentials see also [16, 17]). This inequality reflects the diamagnetic influence of A and its consequences have been analyzed in various aspects [2, 4, 13]. Our precise setting and generalization of estimate (2) will be as follows. First we will replace $\mathbb{R}^m$ by a compact Riemannian manifold M (with metric g) in order to be able to work with a discrete spectrum of the (elliptic) differential operators involved. To avoid zero eigenvalues, we will assume M to be a manifold with boundary ∂M and impose Dirichlet boundary conditions. $-(p-eA)^2$ will then be a special case of a so called Bochner Laplacian D_∇ given by a connection ∇ on a vector bundle V over M. The rank of V will be denoted by n (in physicists terminology the dimension of the internal symmetry space). Thus D_∇ acts on elements of $C^\infty(V)$, the set of C^∞ sections in V as

$$D_\nabla : C^\infty(V) \xrightarrow{\nabla} C^\infty(T^*M \otimes V) \xrightarrow{\nabla^1} C^\infty(T^*M \otimes T^*M \otimes V) \xrightarrow{g \otimes 1} C^\infty(V) ,$$

where ∇^1 is the connection on $T^*M \otimes V$ induced by ∇ and the Levi-Civita connection ∇_g on TM given by the metric g.

In this differential geometric context, eA corresponds to the Christoffel symbol of a connection. Now in physical applications A is hermitean. In our general setup we will therefore require V to be a hermitean vector bundle, which by definition is given by a smooth hermitean structure

$$\langle , \rangle : C^\infty(V) \times C^\infty(V) \to C^\infty(M \times \mathbb{C}) .$$

The connection ∇ is then required to respect this hermitean structure such that $D_\nabla \leqq 0$ is symmetric (for details in this context see [7]). Let furthermore $\Delta_g \leqq 0$ be the Laplace-Beltrami operator on the tangent bundle TM, defined in terms of the Levi-Civita connection ∇_g. Finally we will define the determinant of an operator with help of the analytic extension of the trace as given by Seeley [15]. Ray and Singer have used this definition to define the analytic torsion [11]. Hawking has recently used the same definition in a context similar to ours [5]. Our main result will now be given by

Theorem 1. *With the notations and conventions as above the following uniform estimate holds*

$$\frac{\det(-D_\nabla^D)}{\det(-\Delta_g^D)^n} \geqq 1 . \tag{5}$$

The superscript D refers to taking Dirichlet boundary conditions.

The remainder of this note will be devoted in proving this estimate. We start with a proof that D_∇^D and Δ_g^D are related by Kato's inequality in such a way, that the result of [6] may be applied. In particular this will result in domination of their semigroups and resolvents.

In the case of compact manifolds without boundary, Kato's inequality was proved in [7]. In our case with Dirichlet boundary conditions on ∂M we proceed as follows. For $\alpha \in C^\infty(V)$ and $x \in M$ set

$$|\alpha|(x) = \langle \alpha(x), \alpha(x) \rangle_x^{1/2} , \tag{6}$$

where $\langle,\rangle_x$ is the hermitean form restricted to the fibre V_x over x. This gives a map

$$|\ |: C^\infty(V) \to C^0(M \times \mathbb{R})$$

denoted by $\alpha \mapsto |\alpha|$. As in [7] the following relation holds for any $\alpha \in C^\infty(V)$

$$\operatorname{Re}\langle D_V\alpha, \alpha\rangle \leqq |\alpha|_\varepsilon \Delta_g |\alpha|_\varepsilon \tag{7}$$

with

$$0 < \varepsilon \leqq |\alpha|_\varepsilon = (|\alpha|^2 + \varepsilon^2)^{1/2} \in C^\infty(M \times \mathbb{R}) .$$

Next let $|d\mathrm{vol}|$ be the volume element in V and let $\langle\langle,\rangle\rangle$ denote the induced scalar product in $L^2(V)$ or $L^2(M \times \mathbb{R})$, the space of square integrable sections in V and $M \times \mathbb{R}$ respectively. Thus for $\alpha \in C^\infty(V)$:

$$\langle\langle \alpha, \beta\rangle\rangle = \int \langle \alpha, \beta\rangle |d\mathrm{vol}| . \tag{8}$$

By $C_0^\infty(V)$ we denote the C^∞ sections in V, which vanish on ∂M. Let $\alpha \in C_0^\infty(V)$ and $0 \leqq h \in C_0^\infty(M \times \mathbb{R})$. Performing a partial integration twice, Relation (7) gives

$$\begin{aligned} &\int \frac{1}{|\alpha|_\varepsilon} \operatorname{Re}\langle D_V\alpha, \alpha\rangle h |d\mathrm{vol}| \\ &\qquad \leqq \langle\langle \Delta_g h, |\alpha|_\varepsilon\rangle\rangle + 0(\varepsilon) . \end{aligned} \tag{9}$$

Here $O(\varepsilon)$ is a surface term resulting from the fact that $|\alpha|_\varepsilon = \varepsilon$ on ∂M. Thus in the limit $\varepsilon \to 0$

$$\operatorname{Re}\langle\langle D_V^D \alpha, \beta\rangle\rangle \leqq \langle\langle \Delta_g^D h, |\alpha|\rangle\rangle . \tag{10}$$

Here

$$\begin{aligned} D_V^D &= D_V \restriction C_0^\infty(V) \\ \Delta_g^D &= \Delta_g \restriction C_0^\infty(M \times \mathbb{R}) . \end{aligned}$$

Also

$$\beta = \operatorname{sign}_\xi \alpha h \tag{11a}$$

with

$$\operatorname{sign}_\xi \alpha = \begin{cases} \dfrac{\alpha}{|\alpha|} & \text{on } \operatorname{supp}\alpha \\ \xi & \text{else ,} \end{cases} \tag{11b}$$

where ξ is an arbitrary measureable section in the sphere bundle of V. In particular, by Relation (11) α and β are absolutely pairing in the sense of [6]. Now Δ_g^D is essentially selfadjoint. Its selfadjoint closure, the Laplace-Beltrami operator with Dirichlet boundary conditions will also be denoted by Δ_g^D. The resulting semi-group $\exp t\Delta_g^D (t \geqq 0)$ and hence the resolvent $(-\Delta_g^D + \lambda)^{-1} (\lambda \geqq 0)$ have positive kernel so they are positivity preserving. Zero is no eigenvalue of Δ_g^D. All these properties are stated in e.g. [10]. In particular (10) holds for

$$0 \leqq h = (-\Delta_g^D + \lambda)^{-1} k, \qquad \lambda \geqq 0 \tag{12}$$

with $0 \leqq k \in C^\infty(M \times \mathbb{R})$, since then $0 \leqq h \in C_0^\infty(M \times \mathbb{R})$.

Therefore the essential Kato's inequality (Theorem 2.15d in [6]) holds if we can show that the symmetric operator $D_V^D \leqq 0$ is essentially selfadjoint. For this it is sufficient to show that for some $\lambda \geqq 0$ $(-D_V+\lambda)C_0^\infty(V)$ is dense in $L^2(V)$. We will choose $\lambda=0$. For this to hold it is again sufficient to show that the differential equation.

$$-D_V u = f$$

has at least one solution $u \in C_0^\infty(V)$ for any $f \in C^\infty(V)$. By the Fredholm alternative for elliptic boundary value problems this holds if the differential equation

$$-D_V v = 0 \tag{13}$$

has no nontrivial solution $v \in C_0^\infty(V)$ (see e.g. Proposition 5.3, Chapter 2 in [9]). Now let v be a solution of this homogeneous differential equation.

Then by Kato's inequality (7)

$$-\Delta_g |v|_\varepsilon \leqq 0 \, . \tag{14}$$

Let h be as in Relation (12) with $\lambda=0$. Performing again a partial integration twice, Relation (14) gives

$$\begin{aligned}\langle\langle k, |v|_\varepsilon \rangle\rangle + O(\varepsilon) &= \int (-\Delta_g h)|v|_\varepsilon |d\mathrm{vol}| + O(\varepsilon)\\ &= \int h(-\Delta_g |v|_\varepsilon)|d\mathrm{vol}| \leqq 0 \, .\end{aligned}$$

Letting ε tend to zeros gives

$$\langle\langle k, |v| \rangle\rangle \leqq 0 \, .$$

Since $0 \leqq k \in C^\infty(V)$ is otherwise arbitrary, this gives $v=0$. Thus D_V^D is essentially selfadjoint and we denote its closure also by D_V^D.

We collect our result in

Proposition 2. *The operators $D_V^D \leqq 0$ and $\Delta_g^D \leqq 0$ are essentially selfadjoint and satisfy the essential Kato's inequality* (10).

Corollary 3. *The lowest eigenvalue of $-D_V^D$ is strictly positive. If $\| \ \|$ denotes the norm of an endomorphism from the fibre V_y to the fibre V_x, then the following estimates hold for the kernels of the semigroups and resolvents*

$$\begin{aligned}&\|\exp t D_V^D(x,y)\| \leqq \exp t\Delta_g^D(x,y)\\ &\|(-D_V^D+\lambda)^{-1}(x,y)\| \leqq (-\Delta_g^D+\lambda)^{-1}(x,y)\end{aligned} \tag{15}$$

$x, y \in M$, $t>0$, $\lambda>0$.

The first part of the corollary follows from the proof of Proposition 2 or from Corollary 2.13 in [6]. The remainder follows from Theorem 2.15d in [6] using Proposition 2.

Remark 4. Apart from the first part of Corollary 3, the statements of Proposition 2 and Corollary 3 also hold with Neumann boundary conditions. Let D_V^N and Δ_g^N denote the corresponding selfadjoint operators. Then the following interesting result is a consequence of Kato's inequality: If α satisfies

$$D_V^N \alpha = 0 \tag{16}$$

then $|\alpha|=\text{const}$, so a necessary condition for D_V^N to have a zero eigenvalue is that the sphere bundle of V admits a C^∞ section (the same argument applies to any D_V if the manifold is compact and without boundary). In fact, assume (16) holds, then by Kato's inequality

$$-\Delta_g^N|\alpha|_\varepsilon \leqq 0\,. \tag{17}$$

Now zero is a simple eigenvalue of Δ_g^N with eigenfunction space consisting of the constant functions. Let P be the projection onto this eigenspace. Then (17) gives

$$-\Delta_g^N(1-P)|\alpha|_\varepsilon \leqq 0\,.$$

Therefore, by the Lebesgue dominated converge theorem

$$(1-P)|\alpha|_\varepsilon = \lim_{\delta\to 0^+}(-\Delta_g^N+\delta)^{-1}(-\Delta_g^N)(1-P)|\alpha|_\varepsilon \leqq 0$$

since $(-\Delta_g^N+\delta)^{-1}$ is positivity preserving. This however, is only possible if $|\alpha|_\varepsilon$ and hence $|\alpha|$ is constant.

Note also that the dimension of the eigenspace of D_V^N with eigenvalue zero is at most n. This follows directly from the estimate

$$\operatorname{Trace}\exp tD_V^N \leqq n \operatorname{Trace}\exp t\Delta_g^N$$

in the limit $t\to\infty$ which is again a consequence of estimate (15).

We turn to a definition of the determinant. Let

$$\begin{aligned}\zeta^1(s)&=\operatorname{Trace}(-D_V^D)^{-s}\\ \zeta^2(s)&=\operatorname{Trace}(-\Delta_g^D)^{-s}\end{aligned} \tag{18}$$

be the generalized Riemann Zeta functions associated to $-D_V^D$ and $-\Delta_g^D$ respectively. They are defined for $\operatorname{Re}s$ sufficiently large and extend to meromorphic functions in the complex s-plane, analytic at $s=0$ [15]. We set

$$\begin{aligned}\det(-D_V^D)&=\exp-\frac{d}{ds}\zeta^1(s)|_{s=0}\\ \det(-\Delta_g^D)&=\exp-\frac{d}{ds}\zeta^2(s)|_{s=0}\,.\end{aligned} \tag{19}$$

Furthermore, the powers $(-D_V^D)^{-s}$ and $(-\Delta_g^D)^{-s}$ have kernels in a complex neighborhood of $s=0$ which depend analytically on s [15]. In particular we may define

$$\begin{aligned}\ln(-D_V^D)(x,y)&=-\frac{d}{ds}(-D_V^D)^{-s}(x,y)|_{s=0}\\ \ln(-\Delta_g^D)(x,y)&=-\frac{d}{ds}(-\Delta_g^D)^{-s}(x,y)|_{s=0}\,.\end{aligned} \tag{20}$$

The case $x=y$ is allowed such that

$$\begin{aligned}\det(-D_V^D)&=\exp\int \operatorname{Trace}_x \ln(-D_V^D)(x,x)|d\mathrm{vol}|(x)\\ \det(-\Delta_g^D)&=\exp\int \ln(-\Delta_g^D)(x,x)|d\mathrm{vol}|(x)\,,\end{aligned} \tag{21}$$

where Trace_x denotes the trace of endomorphisms of the fibre V_x. For arbitrary $\lambda_0 > 0$ operator calculus gives

$$\begin{aligned} & n \ln(-\Delta_g^D)(x,x) - \mathrm{Trace}_x \ln(-D_V^D)(x,x) \\ & = \int_0^{\lambda_0} \mathrm{Trace}_x(-D_V^D + \lambda)^{-1}(x,x) d\lambda - n \int_0^{\lambda_0} (-\Delta_g^D + \lambda)^{-1}(x,x) d\lambda \\ & \quad - \mathrm{Trace}_x \ln\left(1 - \frac{D^D}{\lambda_0}\right)(x,x) \\ & \quad + n \ln\left(1 - \frac{\Delta_g^D}{\lambda_0}\right)(x,x) \,. \end{aligned} \tag{22}$$

Now by estimate (15), the first two terms on the right hand side of Relation (22) combine to a quantity $\leqq 0$. Hence

$$\begin{aligned} & n \ln(-\Delta_g^D)(x,x) - \mathrm{Trace}_x \ln(-D_V^D)(x,x) \\ & \leqq \lim_{\lambda_0 \to \infty} \left\{ n \ln\left(1 - \frac{\Delta_g^D}{\lambda_0}\right)(x,x) \right. \\ & \left. - \mathrm{Trace}_x \ln\left(1 - \frac{D_V^D}{\lambda_0}\right)(x,x) \right\} = 0 \,. \end{aligned}$$

Integrating out and using Relation (21) finally proves the theorem. The cases of physical interest correspond to situations where for example

$$\sum_{\mu=1}^{m} \left(\frac{\partial}{\partial x_\mu} - ieA_\mu \right)^2$$

is considered say in the interior of a sphere with Dirichlet boundary conditions. The above discussion thus applies to cases where the A_μ are C^∞. However, it should be possible to extend the above result in two ways. First, it should be sufficient to consider piecewise smooth boundaries. Secondly, extending results of e.g. Schechter [12] for the electromagnetic case, more general A's should be allowed.

We finally remark that results analogous to estimate (5) hold with Neumann boundary conditions or on compact manifolds without boundary if to all operators involved a mass term $m^2 > 0$ is added. This is of course another way of avoiding zero modes. In the formal limit $m^2 \to 0$ Relation (5) then again holds also for these cases.

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