

## On the Prosaic Origin of the Double Poles in the Sine-Gordon $S$ -Matrix\*

Sidney Coleman and H. J. Thun\*\*

Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

**Abstract.** We show that the apparently exotic double poles in the sine-Gordon  $S$ -matrix are nothing but ordinary anomalous thresholds.

### 1. Introduction

The most astonishing property of the sine-Gordon  $S$ -matrix (other than that it is known [1]) is the host of double poles that populate the physical sheets of forward scattering amplitudes. Such double poles have long been known to be present in analogous nonrelativistic models [2]; nevertheless, it is a surprise to find them in a relativistic field theory, where we would expect all singularities to be explicable in terms of the principles of analytic  $S$ -matrix theory (“analyticity + unitarity”) as laid down, for example, in the text of Eden et al. [3].

The purpose of this note is to explain the double poles in terms of these principles. To be more explicit, we show that the double poles are Landau singularities of a perfectly ordinary sort, just like those that occur in the real world in deuteron-deuteron scattering, for example. It is only the dimensionality of phase space that makes these singularities double poles in two dimensions and branch cuts in four dimensions. Thus the double poles have nothing to do with the special features of sine-Gordon theory (infinite number of conservation laws, factorizability, etc.) and should appear in any two-dimensional theory with appropriate mass spectrum.

In Section 2 below we explain the detailed mechanism responsible for the double poles. In Section 3 we verify our explanation by computing the locations of the double poles in the sine-Gordon  $S$ -matrix. We obtain all the double poles in the  $S$ -matrix and we obtain no double poles that do not appear in the  $S$ -matrix.

All our arguments are trivial and all our computations are pedestrian. That is the point of this note.

\* This research was supported in part by the National Science Foundation under Grant No. PHY77-22864

\*\* Max Kade Foundation Fellow. On leave of absence from the Institut für Theoretische Physik, Freie Universität Berlin, Federal Republic of Germany

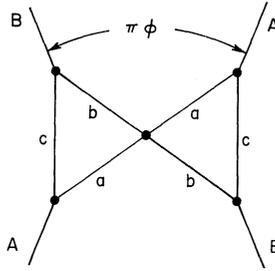


Fig. 1. The graph that produces anomalous thresholds in four dimensions and double poles in two

## 2. Anomalous Thresholds and Double Poles

For real external momenta, there is a simple rule for locating all singularities of a Green's function: Singularities occur only for those values of the momenta for which one can draw a space-time graph of a process involving classical particles all on the mass shell, all moving forward in time, and interacting only through energy- and-momentum-conserving interactions localized at a space-time point [4]. Furthermore, there is a simple set of rules, the Cutkosky rules [5], for evaluating the discontinuity associated with such a singularity: One evaluates the graph as if it were a Feynman graph, with two exceptions: the point interactions are to be replaced by actual  $S$ -matrix elements, and Feynman propagators are to be replaced by mass-shell delta-functions,  $\delta(p^2 - m^2)\theta(p^0)$ .

Unfortunately, these rules are inapplicable to singularities that lie below threshold; these occur for complex values of the external momenta. However, one can frequently push these singularities into the region of real momenta at the cost of an analytic continuation in the external masses.

As an example, let us consider the graph of Figure 1. (Ignore the angle defined in the graph; this will be explained in Section 3.) This graph produces a singularity in the forward scattering of two (unstable) particles,  $A$  and  $B$ . Time runs upward in the graph. At the beginning of the process,  $A$  and  $B$  are heading directly towards each other. Before they collide, though,  $A$  decays into a particle of type  $a$  and one of type  $c$ . The Lorentz frame is chosen such that  $c$  is at rest.  $B$  decays into a particle of type  $b$  and a second particle of type  $c$ , also at rest.  $b$  and  $a$  scatter in the forward direction (the central dot) and recombine with the stationary  $c$  particles to reform  $A$  and  $B$ . This process is obviously physically possible for classical particles with appropriate initial momenta, provided only that  $A$  is sufficiently massive to decay into  $a$  and  $c$  and  $B$  sufficiently massive to decay into  $b$  and  $c$ .

For example, if the mass of the deuteron were greater than twice the nucleon mass, this process would be possible for deuteron-deuteron forward scattering, with  $a$ ,  $b$ , and  $c$  nucleons. Of course, the real deuteron is lighter than two nucleons; however, if we continue the deuteron mass from an unphysical unstable value to its actual value, the singularity can not disappear utterly and moves below deuteron-deuteron threshold to become an anomalous singularity.

The Cutkosky rules show that in four dimensions the singularity is a branch point. We have an eight-dimensional integral (from the two loops) over six delta-

functions (from the six internal lines); thus we obtain a finite discontinuity over the cut. However, if we lived in two dimensions rather than four, we would only have a four-dimensional integral from the two loops. Thus we would have two delta-functions left over from the Cutkosky integral, and the singularity would be such as to have a double delta-function for a discontinuity; that is to say, it would be a double pole.

You may find the argument of the last sentence a bit too sloppy, so here is a more careful (though longer) one: Consider an individual Feynman graph contributing to the forward scattering amplitude, with the heavy dots in Figure 1 replaced by arbitrary insertions. By momentum conservation, the two virtual  $c$  particles carry equal momenta, which we denote by  $p_c$ . Thus the Feynman integral contains two factors of  $1/(p_c^2 - m_c^2)$  and can be written as the derivative with respect to  $m_c^2$  of an integral with only one such factor. (For purposes of this differentiation, we consider the masses of any  $c$  particles hiding within the heavy dots as independent variables.) The discontinuity of this new integral can be evaluated straightforwardly (that is to say, without any doubletalk about double delta-functions) by the Cutkosky rules, and is proportional to  $\delta(p_c^2 - m_c^2)$ . Also, the Cutkosky rules fix  $p_c$  in terms of the external momenta. Thus the singularity of the new integral is proportional to  $1/(p_c^2 - m_c^2)$ , and the singularity of its derivative is a double pole.

There are other graphs that have identical kinematic structures to Figure 1: Firstly, the  $a$  and  $b$  particles can simply pass each other without interaction. Secondly, if there are particles degenerate in mass to  $a$  (or  $b$ ),  $a$  (or  $b$ ) can be transformed into one of these particles at the central dot. For brevity, we will not explicitly discuss these cases except when they lead to significant differences in the argument. In the case at hand, if there is no central interaction, then, in four dimensions, we have a four-dimensional loop integral and four delta-functions, still (barely) giving us a branch point, and, in two dimensions, we have a two-dimensional loop integral and four delta-functions, still giving us a double pole.

This is our explanation for the double poles in the sine-Gordon  $S$ -matrix. They arise from processes like those shown in Figure 1, processes identical to those that produce the anomalous thresholds in deuteron-deuteron scattering in four dimensions. In the next section we will verify this explanation by computing the locations of the singularities predicted by Figure 1 for all possible assignments of  $A$ ,  $B$ ,  $a$ ,  $b$ , and  $c$  and showing that these predict all the double poles in the sine-Gordon  $S$ -matrix.

### 3. Explicit Computations

#### 3.1. Two-Dimensional Kinematics

Because of our well-developed feeling for classical kinematics, the easiest way to understand singularities is the method of the previous section. However, for the actual computation of singularities that lie below threshold but on the real energy axis (the case of interest to us), another method is more suitable.

In this case we may always choose the time components of the external momenta to be real and the space components to be imaginary. Thus we can

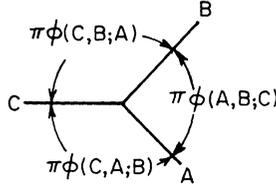


Fig. 2. Definitions of angles for a three-line vertex

associate with every momentum a real two-vector whose time component is that of the momentum and whose space component is  $i$  times that of the momentum. We call this two-vector the pseudomomentum. The rules for finding a singularity are [6]: (1) The graph must be a geometrical figure in Euclidean two-space. (2) With every internal line there must be associated a pseudomomentum of length  $m$ , the mass of the particle associated with that line. (3) The pseudomomentum must be parallel to the associated line. (4) All pseudomenta at a given vertex must sum to zero. Thus, locating a singularity becomes a problem in Euclidean plane geometry.

In particular, these rules imply that for a three-line vertex (Fig. 2) all angles are determined. For example,

$$m_A^2 + m_B^2 + 2m_A m_B \cos \pi\phi(A, B; C) = m_C^2, \quad (1)$$

with similar equations for the other angles. (The unnatural factor of  $\pi$  in the definitions of the angles is chosen to agree with conventions used in the sine-Gordon literature.) Also, by convention  $\phi(A, B; C) = \phi(B, A; C)$  is always chosen to lie between 0 and 1.

For a forward scattering process (Fig. 1), the angle  $\phi$  is related to the usual variable  $s$  by

$$s = m_A^2 + m_B^2 + 2m_A m_B \cos \pi\phi. \quad (2)$$

From this equation we see that  $\phi$  is  $\theta/i\pi$ , where  $\theta$  is the modulus of the rapidity difference of particles  $A$  and  $B$ . Thus,  $\phi=0$  is the threshold for  $AB$  scattering and  $\phi=1$  is the threshold for the crossed process,  $A\bar{B}$  scattering.

Note that if Figure 1 is a possible graph for  $AB$  scattering, Figure 1 rotated by  $90^\circ$  is a possible graph for  $A\bar{B}$  scattering. If  $B$  and  $\bar{B}$  are identical, these are the same process, and thus we should consider singularities both from Figure 1 and from its rotation. The most efficient way of doing this (which we shall adopt) is to restrict our geometric analysis to Figure 1, and then, when we are done, deduce the singularities of the rotated graph by using crossing symmetry, that is to say, by replacing  $\phi$  by  $1 - \phi$ .

For Figure 1 to be a possible graph, the  $a$  and  $b$  lines must intersect; thus,

$$\phi(b, c; B) + \phi(a, c; A) < 1. \quad (3)$$

If this condition is satisfied, Figure 1 can be drawn. The angle  $\phi$  is then determined by elementary geometry:

$$\phi = 2 - \phi(A, c; a) - \phi(B, c; b). \quad (4)$$

These equations are all we need to locate the double poles for any forward scattering process in any theory. All we have to do is plug in the particle masses and turn the crank.

For the sine-Gordon equation, the particle spectrum consists of the following: (1) A particle called the soliton or fermion. We will denote this particle by  $f$  and choose our mass scale such that its mass is one. (2) The antisoliton or antifermion, also of mass one, which we denote by  $\bar{f}$ . (3) A sequence of bosons that can be thought of as  $f\bar{f}$  bound states. These are labelled by a positive integer  $n$ , bounded above by a real number  $\lambda$ , the only dimensionless free parameter of the theory. The masses of the bosons are given by

$$m_n = 2 \sin(\pi n / 2\lambda). \quad (5)$$

Most three-particle vertices vanish because of the conservation laws of the theory. For the non-vanishing vertices, the relevant angles may readily be computed from Equations (1) and (5) to be

$$\phi(f, \bar{f}; n) = 1 - \frac{n}{\lambda}, \quad (6a)$$

$$\phi(n, f; \bar{f}) = \phi(n, \bar{f}; f) = \frac{1}{2} \left( 1 + \frac{n}{\lambda} \right), \quad (6b)$$

$$\phi(n, m; n+m) = \frac{n+m}{2\lambda}, \quad (6c)$$

and

$$\phi(m, n+m; n) = 1 - \frac{n}{2\lambda}. \quad (6d)$$

Equations (3), (4), and (6) are what we need.

### 3.2. Fermion-Fermion Scattering

If both  $A$  and  $B$  are fermions, then either  $c$  is a fermion and  $a$  and  $b$  are bound states or vice versa. In either case, Equation (6b) implies that Equation (3) cannot be satisfied. Thus our formulas predict no double poles. This is indeed the case [1]. Identical arguments apply to fermion-antifermion scattering.

### 3.3. Fermion-Boson Scattering

$A$  is  $f$  and  $B$  is  $n$ . While the initial fermion must always decay into a fermion and a boson, the initial boson may decay into either a fermion-antifermion pair or two bosons. Thus there are several possibilities for the internal particles:

1) The initial boson decays into a fermion-antifermion pair. In equations,

$$a = m, \quad b = \bar{f}, \quad c = f. \quad (7)$$

By Equation (3),

$$2n - m > \lambda. \quad (8)$$

By Equation (4), the locations of the double poles are given by

$$\phi = \frac{1}{2} + \frac{2m-n}{2\lambda}. \quad (9)$$

It will be convenient to introduce a new variable,

$$s = n - m. \quad (10)$$

the range of  $s$  is given by

$$n > s > \lambda - n, \quad (11)$$

while

$$\phi = \frac{1}{2} + \frac{n-2s}{2\lambda}. \quad (12)$$

2) The initial boson decays into two bosons. Even after we have specified the quantum number of one decay product, there still remains a choice for the quantum number of the other. In equations, we always have

$$a = f, \quad c = s, \quad (13)$$

but we can have either

$$b = n + s, \quad (14a)$$

$$b = s - n, \quad (\text{if } s > n), \quad (14b)$$

or

$$b = n - s, \quad (\text{if } s < n). \quad (14c)$$

In both case (a) and case (b), Equation (3) becomes

$$n - s > \lambda. \quad (15)$$

This is impossible, because both  $n$  and  $s$  are less than  $\lambda$ . Thus we are left with case (c). By Equation (3),

$$s < \lambda - n. \quad (16)$$

By Equation (4),

$$\phi = \frac{1}{2} + \frac{n-2s}{2\lambda}. \quad (17)$$

Surprisingly, Equations (12) and (17) are the same equation, and the inequalities (11) and (16) describe complementary portions of the range

$$n > s > 0. \quad (18)$$

Thus, despite the fact that the double poles spring from two quite distinct families of graphs, we end up with a single sequence of double poles evenly spaced in  $\phi$ , just the right answer [1]. (Crossing symmetry gives us no new information, because the family of singularities we have found is already crossing symmetric.)

### 3.4. Boson-Boson Scattering

$A$  is  $n$  and  $B$  is  $m$  with  $n \geq m$  by convention. Once again there are two possibilities to be considered:

1) Both initial bosons decay into boson pairs. We will always choose  $c=r$ , but there remain several choices for  $a$  and  $b$ :

$$a = n + r, \quad b = m + r, \quad (19a)$$

$$a = n - r, \quad b = m + r, \quad (\text{if } r < n), \quad (19b)$$

$$a = n + r, \quad b = m - r, \quad (\text{if } r < m), \quad (19c)$$

and

$$a = n - r, \quad b = m - r, \quad (\text{if } r < m). \quad (19d)$$

In case (a), by Equation (3),

$$n + m > 2\lambda. \quad (20)$$

This is impossible, since both  $m$  and  $n$  are less than  $\lambda$ . In case (b), by Equation (3),

$$m - n > 0. \quad (21)$$

This is also impossible, since, by convention,  $n \geq m$ . In case (c), by Equation (3)

$$n - m > 0. \quad (22)$$

This is true unless  $n=m$ . By Equation (4), in this case,

$$\phi = 1 - \frac{n + 2r - m}{2\lambda}. \quad (23)$$

In case (d), by Equation (3),

$$n + m < 2\lambda. \quad (24)$$

This is always true. By Equation (4), in this case,

$$\phi = \frac{m + n - 2r}{2\lambda}. \quad (25)$$

Thus, in contrast to the preceding case, we obtain two distinct families of double poles, given by Equations (23) and (25). These two families change places under crossing symmetry, so crossing gives us no new singularities, with one exception: When  $n=m$ , the direct derivation of Equation (23) breaks down, and it must be derived from crossing symmetry and Equation (25).

2) Both initial bosons decay into fermion-antifermion pairs. This case is different from all the others we have considered; there are several ways of identifying the internal particles that give graphs with identical kinematics. This is a consequence of the degeneracy in mass of fermion and anti-fermion. For example, Figure 3 shows two graphs that produce double poles in identical locations. (As usual, the directed lines represent fermions.) In addition to these, there are two other graphs, obtained from the ones shown by reversing all directed lines.

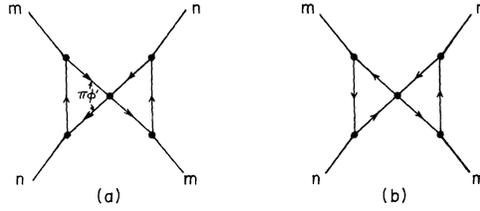


Fig. 3. Two graphs which individually produce double poles which cancel

We will now show that the double poles produced by these two graphs have opposite residues and thus cancel each other out.

The two graphs differ only in the two left-hand vertices and the scattering process at the center. The vertices in one graph are charge conjugates of the vertices in the other. It is known that the bound states have alternating charge-conjugation parities. Thus the vertices produce a relative factor of  $(-1)^{n+m}$ . As for the central scattering, if we read the graphs sidewise, in one case we have fermion-antifermion forward scattering (transmission) and in the other we have fermion-antifermion backward scattering (reflection). Thus, the sum of the two residues is proportional to

$$t(\phi') + (-1)^{n+m}r(\phi'), \quad (26)$$

where  $t$  is the transmission coefficient,  $r$  is the reflection coefficient, and  $\phi'$  is the angle defined in the figure.

From the figure and Equation (6a),

$$\phi' = 1 - \phi(f, \bar{f}; n) - \phi(f, \bar{f}; m) = \frac{n+m}{\lambda} - 1. \quad (27)$$

It is known [1] that, in general,

$$r(\phi) = \frac{\sin \pi \lambda}{\sin \pi \lambda \phi} t(\phi). \quad (28)$$

Thus, in particular,

$$r(\phi') = -(-1)^{n+m}t(\phi'), \quad (29)$$

and the expression (26) vanishes.

Thus the only double poles are those given by Equations (23) and (25). Once again, this is the right result [1].

*Acknowledgement.* One of us (H.J.T.) gratefully acknowledges the hospitality extended to him by the Lyman Laboratory of Physics.

## References

1. Zamolodchikov, A.B.: Quantum sine-Gordon model: the total S-matrix. Moscow Preprint, ITEP-12 (1977)  
Karowski, M., Thun, H.J.: Nucl. Phys. B**130**, 295 (1977)

2. Yang, C. N.: Phys. Rev. **168**, 1920 (1968)
3. Eden, R.J., Landshoff, P.V., Olive, D.I., Polkinghorne, J.C.: The analytic S-matrix. Cambridge: Cambridge University Press 1966
4. Ref. [3], p. 87
5. Ref. [3], p. 110
6. Bjorken, J.D., Drell, S.D.: Relativistic quantum fields, p. 235. New York: McGraw-Hill 1965

Communicated by A. Jaffe

Received January 25, 1978

